A special case of the XYZ model with boundaries

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Outline

Preliminaries

In 1972 R. Baxter noticed that the ground state energy of the periodic XYZ hamiltonian

$$H_{XYZ} = -\sum_{i=1}^{N}(J_x\sigma_i^x \otimes \sigma_{i+1}^x + J_y\sigma_i^y \otimes \sigma_{i+1}^y + J_z\sigma_i^z \otimes \sigma_{i+1}^z)$$

has the simple value

$$\lim_{N\to\infty} \frac{E}{N} = -J_x + J_y + J_z, \quad \text{if} \quad J_xJ_y + J_xJ_z + J_yJ_z = 0.$$

In the XXZ case it corresponds to $J_x = J_y = 1$, $J_z = \Delta = -1/2$.

In 2000 Stroganov noticed that this statement holds for finite **odd** $N = 2n + 1$ and the ground state wavefunction possesses some remarkable combinatorial properties. For example, the properly normalized ground state wavefunction has all integer coefficients with the largest component given by

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Preliminaries

Many people have been working on different extensions of these ideas for the XXZ case (connections to alternating sign matrices, loop models, Temperley-Lieb processes, lattice sypersymmetry, etc):
Batchelor, De Gier, Di Francesco, Fendley, Hagendorf, Ikhlef, Jacobsen, Nienhuis, Mitra, Motegi, Pasquier, Pearce, Ponsaing, Pyatov, Razumov, Rittenberg, Saleur, Stroganov, Zinn-Justin, Zuber, ...

These ideas have also been extended to the periodic XYZ spin chain at odd number of sites (connections to the three-coloring problem, lattice sypersymmetry, Painlevé equations, etc):
Bazhanov, Fendley, Hagendorf, VM, Rosengren,...

The XXZ model: different scenarios

There are three different cases to consider:

1. Periodic spin chain, odd number of sites $N = 2n + 1$.

The XXZ hamiltonian commutes with the 6-vertex model transfer-matrix with $\Delta = -1/2$ (disordered regime).
The Baxter's TQ-relation

$$T(u)Q(u) = \sin^N(u + \eta/2)Q(u - \eta) + \sin^N(u - \eta/2)Q(u + \eta).$$

The ground state eigenvalue $T(u) = (a + b)^N = \sin(u)^N$, $N = 2n + 1$, $\eta = 2\pi/3$.
For $f(u) = \sin^N(u)Q(u)$ we obtain the functional equation

$$f(u) + f(u + \frac{2\pi}{3}) + f(u + \frac{4\pi}{3}) = 0$$

which fixes (+periodicity conditions) the trigonometric polynomial $Q(u)$ uniquely (Stroganov, 2000).

The XXZ model: different scenarios

2. Twisted boundary conditions, even number of sites $N = 2n$.

$$H_{XYZ} = -\sum_{i=1}^{N}(\sigma_i^x \otimes \sigma_{i+1}^x + \sigma_i^y \otimes \sigma_{i+1}^y - \frac{1}{2}\sigma_i^z \otimes \sigma_{i+1}^z)$$

$$\sigma_{N+1}^z = \sigma_N^z, \quad \sigma_{N+1}^{\pm} = e^{i\phi}\sigma_1^{\pm}, \quad \phi = \frac{2\pi}{3}, \quad \sigma^{\pm} = \sigma^x \pm i\sigma^y.$$

The hamiltonian is invariant under left-right reflection + complex conjugation, the ground state energy is again

$$E_0 = -3N/2, \quad \text{for even} \quad N = 2n.$$

2. Open boundary conditions, any number of sites. $U_q(sl(2))$-invariant

hamiltonian (Pasquier, Saleur, 1990)

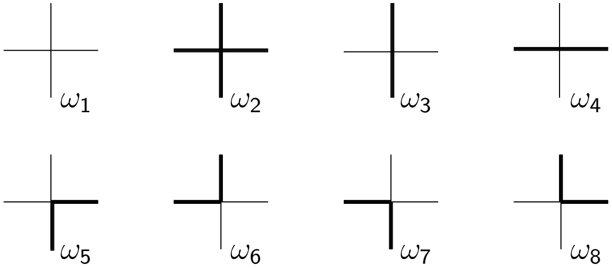
$$H_{XYZ} = -\left[\sum_{i=1}^{N-1}(\sigma_i^x \otimes \sigma_{i+1}^x + \sigma_i^y \otimes \sigma_{i+1}^y + \frac{q+q^{-1}}{2}\sigma_i^z \otimes \sigma_{i+1}^z) + \frac{q-q^{-1}}{2}(\sigma_1^z - \sigma_N^z)\right]$$

This hamiltonian can be rewritten in terms of the generators of Temperley-Lieb algebra. For $q = e^{i\pi/3}$ the ground state energy is

$$E_0 = -\frac{3}{2}(N-1)$$

Eight-vertex model and TQ-relation

Zero-field symmetric eight-vertex model (R. Baxter, 1972)



$$\omega_1 = \omega_2 = a, \qquad \omega_3 = \omega_4 = b, \qquad \omega_5 = \omega_6 = c, \qquad \omega_7 = \omega_8 = d\ .$$

Transfer-matrix and partition function

$$[\mathbf{T}(u)]_{i_1 \ldots i_N}^{j_1 \ldots j_N} = \mathbf{Tr} \prod_{k=1}^{N} W(i_k, j_k), \qquad Z = \mathbf{Tr}\left[\mathbf{T}(u)^M\right]$$

Weights

Baxter's parameterization of the weights

$$
\begin{aligned}
a &= \rho\ \vartheta_4(2\eta\,|\,q^2)\ \vartheta_4(u-\eta\,|\,q^2)\ \vartheta_1(u+\eta\,|\,q^2),\\
b &= \rho\ \vartheta_4(2\eta\,|\,q^2)\ \vartheta_1(u-\eta\,|\,q^2)\ \vartheta_4(u+\eta\,|\,q^2),\\
c &= \rho\ \vartheta_1(2\eta\,|\,q^2)\ \vartheta_4(u-\eta\,|\,q^2)\ \vartheta_4(u+\eta\,|\,q^2),\\
d &= \rho\ \vartheta_1(2\eta\,|\,q^2)\ \vartheta_1(u-\eta\,|\,q^2)\ \vartheta_1(u+\eta\,|\,q^2),
\end{aligned}
$$

and the normalization factor ρ

$$
\rho = 2\ \ \vartheta_2(0\,|\,q)^{-1}\ \vartheta_4(0\,|\,q^2)^{-1}.
$$

One can introduce a **Q**-operator commuting with $T(u)$

$$[\mathbf{T}(u), \mathbf{Q}(v)] = 0, \quad \forall u, v$$

TQ-relation for eigenvalues of $\mathbf{T}(u)$ and $\mathbf{Q}(u)$

$$\boxed{T(u)\, Q(u) = \phi(u-\eta)\, Q(u+2\eta) + \phi(u+\eta)\, Q(u-2\eta)}$$

$$\phi(u) = \vartheta_1^N(u\,|\,q)$$

Periodicity conditions

$$Q_\pm(u+\pi) = \pm Q_\pm(u), \quad Q_\pm(u+\pi\tau) = q^{-N/2}\, e^{-iNu}\, Q_\mp(u)$$

Bethe-ansatz equations $\quad \dfrac{Q(u_i+2\eta)}{Q(u_i-2\eta)} = -\dfrac{\phi(u_i+\eta)}{\phi(u_i-\eta)}$

Quantum Wronskian

TQ-relation is a second order difference equation

$$T(u)\,Q_{\pm}(u) = \phi(u-\eta)\,Q_{\pm}(u+2\eta) + \phi(u+\eta)\,Q_{\pm}(u-2\eta)$$

Quantum Wronskian relation

$$\boxed{Q_{+}(u+\eta)Q_{-}(u-\eta) - Q_{+}(u-\eta)Q_{-}(u+\eta) = \phi(u)W(q,\eta)}$$

For odd values of $N = 2n+1$ we don't need the external field and all states are double-degenerate.

Further we are interested in a disordered regime:

$$0 < \eta < \pi/2, \quad \eta < u < \pi - \eta$$

Ground state

$$\text{For} \quad N = 2n+1, \quad \eta = \frac{\pi}{3}, \quad \text{the ground state eigenvalue}$$

$$T(u) = (a+b)^N = \phi(u) = \vartheta_1^N(u\,|\,q)$$

$$\Psi_\pm(u) \equiv \Psi_\pm(u,q,n) = \frac{\vartheta_1^{2n+1}(u\,|\,q)}{\vartheta_1^n(3u\,|\,q^3)} Q_\pm(u,q,n),$$

TQ-relation becomes

$$\Psi_\pm(u+\frac{2\pi}{3}) + \Psi_\pm(u+\frac{4\pi}{3}) = -\Psi_\pm(u)$$

There are exactly two solutions which satisfy the following PDE

$$\boxed{6\,q\frac{\partial}{\partial q}\Psi(u,q,n) = \Big\{-\frac{\partial^2}{\partial u^2} + 9\,n\,(n+1)\,\wp(3u\,|\,q^3) + c(q,n)\Big\}\Psi(u,q,n)}$$

Hamiltonian

Parameters

$$\gamma = \frac{(a-b+c-d)(a-b-c+d)}{(a+b+c+d)(a+b-c-d)} = -\left[\frac{\vartheta_1(\pi/3\,|\,q^{1/2})}{\vartheta_2(\pi/3\,|\,q^{1/2})}\right]^2$$

$$\zeta = \frac{cd}{ab} = \frac{\gamma+3}{\gamma-1}, \quad \zeta = 2\xi+1$$

Hamiltonian

$$H = -\sum_{i=1}^{N}\left[\sigma_x \otimes \sigma_x - \frac{\xi}{\xi+1}\sigma_y \otimes \sigma_y + \xi \sigma_z \otimes \sigma_z\right]$$

Ground state energy $E_0 = -N\frac{\xi^2+\xi+1}{\xi+1}, \quad N = 2n+1$

Trigonometric limit $\gamma \to -3, \quad \xi \to -\frac{1}{2}$ corresponds to $\Delta = -1/2$
6-vertex model

Polynomial eigenstate

There are two linearly independent solutions

$$Q_{1,2}(u) = \frac{1}{2}[Q_+(u) \pm Q_-(u)], \quad Q_{1,2}(u+\pi) = (-1)^n Q_{2,1}(u)$$

$$x = \gamma \frac{\overline{\vartheta}_3^{\,2}(u)}{\overline{\vartheta}_4^{\,2}(u)}, \qquad \overline{\vartheta}_{3,4}(u) = \vartheta_{3,4}(\frac{u}{2} \,|\, q^{1/2})$$

$$Q_1(u) = \overline{\vartheta}_4(u)\, \overline{\vartheta}_3^{\,2n}(u)\, \mathcal{P}_n(x,z), \qquad z = \gamma^{-2}$$

$$Q_2(u) = \overline{\vartheta}_4(u)\, \overline{\vartheta}_3^{\,2n}(u)\, \mathcal{P}_n(\frac{1}{xz},z), \qquad z = \gamma^{-2}$$

$$\mathcal{P}_n(x,z) = \sum_{k=0}^{n} r_k^{(n)}(z)\, x^k, \quad \bar{s}_n(z) = r_0^{(n)}(z), \quad s_n(z) = r_n^{(n)}(z)$$

Ground state eigenvectors

The spectrum of the transfer-matrix for odd N is double degenerate and there are two ground state eigenvectors $\Psi_\pm$ corresponding to **different** eigenvalues of the Q-operator.

$$\mathbf{T}(u)\Psi_\pm = \phi(u)\Psi_\pm, \quad \mathbf{S}\Psi_\pm = \pm\Psi_\pm, \quad \mathbf{S} = \prod_{i=1}^{N} \sigma_z,$$

$$\mathbf{Q}(u)\Psi_\pm = Q_\pm(u,q,n)\Psi_\pm, \quad \Psi_+ = \mathbf{R}\Psi_-, \; \mathbf{R} = \prod_{i=1}^{N} \sigma_x$$

$$\mathcal{P}_1(x,z) = x + 3$$

$S_z = 1/2$	$S_z = -3/2$
$\psi_{001} = 1$	$\psi_{111} = \zeta$

Table: Components of the Ψ_- for $N = 3$.

$$\mathcal{P}_2(x,z) = x^2(1+z) + 5x(1+3z) + 10$$

$S_z = 3/2$	$S_z = -1/2$	$S_z = -5/2$
$\psi_{00001} = 2\zeta$	$\psi_{01011} = 2$	$\psi_{11111} = \zeta(1+\lambda)$
	$\psi_{00111} = 1+\lambda$	

Table: Components of the Ψ_- for $N = 5$.

$$\mathcal{P}_3(x,z) = x^3(1+3z+4z^2) + 7x^2(1+5z+18z^2) + 7x(3+19z+18z^2) + 35 + 21z$$

$S_z = 5/2$	$S_z = 1/2$	$S_z = -3/2$	$S_z = -7/2$
$\psi_{0000001} = \zeta\,\alpha_1$	$\psi_{0001011} = \alpha_1$	$\psi_{0101111} = \zeta\,\alpha_3$	$\psi_{1111111} = \zeta^2\,\alpha_4$
	$\psi_{0000111} = \alpha_2$	$\psi_{0110111} = \zeta\,\alpha_3$	
	$\psi_{0010101} = \alpha_3$	$\psi_{0011111} = \zeta\,\alpha_4$	
	$\psi_{0010011} = \alpha_4$		

Table: Components of the Ψ_- for $N = 7$.

$$\alpha_1 = 3+5\lambda, \quad \alpha_2 = 1+5\lambda+2\lambda^2, \quad \alpha_3 = 7+\lambda, \quad \alpha_4 = 4+3\lambda+\lambda^2$$

$$\mathcal{P}_n(x,z) = \bar{s}_n(z) + \ldots + s_n(z)x^n$$

Now a number of **conjectures**: (checked up to $N = 25$)

1. The component of the eigenvector Ψ_- with one arrow down is given by

$$\Psi_{0\ldots001} = \frac{1}{N}\zeta^{[\frac{n}{2}]}\lambda^{([\frac{n}{2}][\frac{n-1}{2}])}\bar{s}_n(\lambda^{-1}), \quad \zeta = \frac{cd}{ab}, \quad \lambda = \zeta^2$$

2. The component of the vector Ψ_- with all arrows down is given by

$$\Psi_{11\ldots11} = \zeta^{[\frac{n+1}{2}]}\lambda^{([\frac{n}{2}][\frac{n+1}{2}])}s_n(\lambda^{-1}).$$

3. The norm of the vector Ψ_-

$$|\Psi(\lambda)|^2 = \sum_{i_1\ldots i_N}\Psi^2_{i_1\ldots i_N} = (4/3)^n\,\lambda^{n(n+1)/2}\,s_n(\lambda^{-1})\,s_{-n-1}(\lambda^{-1})$$

4. Introduce the component with alternating arrows

$$A_n(\lambda) = \Psi_{00101\ldots01}, \; n \text{ odd}, \quad A_n(\lambda) = \Psi_{0101\ldots011}, \; n \text{ even}$$

In the trigonometric limit $\lambda \to 0$ it gives the number of alternating sign matrices.

$$A_{2k}(\lambda) = 2p_{1,k-1}(\lambda)p_{2,k-1}(\lambda), \quad A_{2k+1}(\lambda) = p_{1,k}(\lambda)\,p_{2,k-1}(\lambda)$$

$$p_{1,k}(\xi^2) = \frac{(\xi+3)^{k(k+1)}}{2^{k^2}}\tau_{k+1,k}\left[\frac{1-\xi}{3+\xi}\right], \quad p_{2,k}(\xi^2) = \frac{(\xi+3)^{k(k+1)}}{2^{k(k+1)}}\tau_{k+1,k+1}\left[\frac{1-\xi}{3+\xi}\right]$$

$$s_{2k+1}(y^2) = \tau_{k,k-1}(y)\tau_{k,k-1}(-y), \quad s_{2k}(y^2) = \tau_{k-1,k-1}(y)\tilde{\tau}_{k,k}(y), \quad z = y^2$$

τ-functions for Painlevé VI

Modified hamiltonian in Painlevé VI theory $h(t)$ (Okamoto, 1987) satisfies $\mathbf{E}_{VI}$ equation

$$h'(t)\Big[t(1-t)h''(t)\Big]^2+\Big[h'(t)[2h(t)-(2t-1)h'(t)]+b_1b_2b_3b_4\Big]^2=\prod_{k=1}^{4}\Big(h'(t)+b_k^2\Big)$$

Starting with a solution $h_0(t)=h(b_1,b_2,b_3,b_4;t)$ one can construct a series $h_n(t)=h(b_1,b_2,b_3-n,b_4;t)$ applying a sequence Backlund transformations. Introduce a family of tau-functions

$$\tau_n(z)=\exp\{\int\tilde{h}_n(z)dz\}$$

They satisfy 'Toda' relations (Okamoto, 1987)

$$\boxed{\frac{\tau_{n+1}(z)\tau_{n-1}(z)}{\tau_n^2(z)}+\nu_2(z,n)[\log\tau_n(z)]''_z+\nu_1(z,n)[\log\tau_n(z)]'_z+\nu_0(z,n)=0}$$

The XYZ chain with open boundaries

In 1994 Inami and Konno constructed a general solution of the Sklyanin's reflection equation for the 8-vertex model. The corresponding hamiltonian

$$H_{XYZ} = -\sum_{i=1}^{N-1}(J_x\sigma_i^x \otimes \sigma_{i+1}^x + J_y\sigma_i^y \otimes \sigma_{i+1}^y + J_z\sigma_i^z \otimes \sigma_{i+1}^z) + \sum_{\alpha=x,y,z}(\phi_\alpha^-\sigma_1^\alpha - \phi_\alpha^+\sigma_N^\alpha)$$

where

$$J_x = 1, \quad J_y = \mathrm{dn}\left(\frac{4K}{\pi}\eta\right), \quad J_z = \mathrm{cn}\left(\frac{4K}{\pi}\eta\right)$$

and $\phi^\pm$ are 6 arbitrary parameters. There is a natural elliptic parameterization

$$\phi_1^\pm = k^2\mathrm{sn}\left(\frac{4K\eta}{\pi}\right)\prod_{i=1}^{3}\frac{\mathrm{cn}(\alpha_i^\pm)}{\mathrm{dn}(\alpha_i^\pm)}$$

$$\phi_2^\pm = k^2k'^2\mathrm{sn}\left(\frac{4K\eta}{\pi}\right)\prod_{i=1}^{3}\frac{\mathrm{sn}(\alpha_i^\pm)}{\mathrm{dn}(\alpha_i^\pm)}$$

$$\phi_3^\pm = ik'^2\mathrm{sn}\left(\frac{4K\eta}{\pi}\right)\prod_{i=1}^{3}\frac{1}{\mathrm{dn}(\alpha_i^\pm)}$$

The XYZ chain with open boundaries

Now we choose $\eta = \pi/3$. We expect that

$$\lim_{N\to\infty} \frac{E_0}{N} = -\frac{\xi^2+\xi+1}{\xi+1}.$$

Similarly to the 6-vertex model we conjecture

$$E_0 = -(N-1)\frac{\xi^2+\xi+1}{\xi+1}.$$

Let us choose $\phi_i^- = \phi_i^+ = \phi_i$ and impose $\phi_2 = 0$.
There exists a unique solution for $N = 2, 3$

$$\phi_1 = \frac{2\xi+1}{\sqrt{1-\xi^2}}, \quad \phi_3 = i\frac{\xi(\xi+2)}{\sqrt{1-\xi^2}}.$$

With such a choice of parameters E_0 is the ground state energy for all $N = 2, 3, 4, 5, 6, 7$!
For $N = 2$ the ground state eigenvector

$$v_{\pm,\pm} = \pm(1+2\xi), \quad v_{\pm,\mp} = -i(2+\xi) \pm \sqrt{1-\xi^2}.$$

The XYZ chain with open boundaries

Once we know the ground state energy, we can try to choose $\phi_2 \neq 0$.

For $N = 2$ there is a one-parametric family of eigenvectors corresponding to the same eigenvalue E_0. It obtained by

$$\alpha_1^{\pm} = 2K\left(\phi + \frac{1}{3}\right), \alpha_2^{\pm} = 2K\left(\phi - \frac{1}{3}\right), \alpha_3^{\pm} = 2K\phi.$$

The eigenvector is highly nontrivial

$$v_{\pm,\pm} = i\frac{2^{1/3}(1+\xi)\sqrt{1-\xi^2}\,k\,\theta_2}{(e^{2\pi i\tau}\theta_2\theta_3\theta_4)^{1/3}}\theta_1\left(3\phi \pm \frac{\tau}{2}\Big|q^6\right)\theta_4\left(3\phi \mp \frac{\tau}{2}\Big|q^6\right)$$

$$v_{\mp,\pm} = \pm\theta_2\left(3\phi + \frac{3\tau}{2}\Big|q^6\right)\theta_2\left(3\phi - \frac{3\tau}{2}\Big|q^6\right) +$$

$$+i\frac{\theta_1\left(\frac{\pi}{3}\Big|q\right)}{\theta_2\left(\frac{\pi}{3}\Big|q\right)}\theta_1\left(3\phi + \frac{3\tau}{2}\Big|q^6\right)\theta_1\left(3\phi - \frac{3\tau}{2}\Big|q^6\right)$$

The XYZ chain with open boundaries

Now let us look at the eigenvalues of the transfer-matrix. For convenience we shift $u \to u + \frac{\pi}{3}$ in a, b, c, d.
The transfer-matrix is defined by

$$t(u) = \mathrm{tr}(K_0^+(u) T_0(u) K_0^-(u) \hat{T}_0(u))$$

where we choose

$$K^+(u) = K^-(-u-2\eta), \quad T_0(u) = R_{0,N}(u) ... R_{0,1}(u), \quad \hat{T}_0(u) = R_{1,0}(u) \ldots R_{N,0}(u),$$

and

$$K^-(u) = \frac{\theta_1(2u|q)}{\theta_1(u|q)} \left(I + \frac{\mathrm{sn}\left(\frac{2Ku}{\pi}\right)}{\mathrm{sn}\left(\frac{4K}{3}\right)} \left[\phi_x \sigma_x + \phi_y \sigma_y + \phi_z \sigma_z \right] \right)$$

This double transfer-matrix commutes with the H_{XYZ} with boundary conditions described above.

The XYZ chain with open boundaries

For $N = 2$ the ground state eigenvalue of the transfer-matrix $T(u)$ is

$$T(u) = \frac{\theta_3(3(u+\phi)|q^6)\theta_3(3(u-\phi)|q^6)}{\theta_3(3\phi|q^6)^2}\,\theta_1\left(u+\frac{\pi}{3}\middle|q\right)^4.$$

Conjecture

For any $N = 2, 3, \ldots$ the ground state eigenvalue of $T(u)$ is

$$T(u) = \frac{\theta_3(3(u+\phi)|q^6)\theta_3(3(u-\phi)|q^6)}{\theta_3(3\phi|q^6)^2}\,\theta_1\left(u+\frac{\pi}{3}\middle|q\right)^{2N}.$$

(checked numerically up to $N = 7$).

Summary and outlook

- We found the open XYZ spin chain where the ground state energy is preserved by a hidden supersymmetry.
- In the trigonometric limit it degenerates into the $U_q(sl(2))$-invariant spin hamiltonian at $\eta = \pi/3$
- The ground state eigenvectors nontrivially depend on the extra boundary parameter.
- Is their a hidden algebraic structure which generalizes the Temperley-Lieb algebra ? There is no difference between odd and even values of N.
- Is it possible to generalize the twisted case of the XXZ model ?
- The special case $\phi_2 = 0$ is "almost" polynomial. Can it be treated similarly to the periodic case ?

Thank you for your attention