Integrability, Solvability and Enumeration.

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INTRODUCTION

- Many 2d combinatorial or lattice models are solvable for some properties and/or lattices but not others.
- Why this is so is not fully understood.
- Various numerical techniques, magically, seem to be exact for the solvable situations and not for the others.
- This is even less well understood!
- Four such methods will be discussed.

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THE TWO-DIMENSIONAL ISING MODEL. ENTING AND GUTTMANN

- Take $t_1 = \tanh(J_x/kT)$ and $t_2 = \tanh(J_y/kT)$.
- The log of the reduced p.f. is

$$\log \Lambda(t_1, t_2) = \sum_{n,m} a_{n,m} t_1^{2m} t_2^{2n} = \sum_n R_n(t_1^2) t_2^{2n}.$$

- Baxter showed $R_n(t_1^2) = P_{2n-1}(t_1^2)/(1-t_1^2)^{2n-1}$.
- R_n is rational, with num. and den. pols of degree $2n-1$,
- In the complex t_1^2 plane, only singularity is at $t_1^2 = 1$.
- Maillard found an *inversion relation* for the p.f.,

$$\log \Lambda(t_1, t_2) + \log \Lambda(1/t_1, -t_2) = \log(1-t_2^2).$$

- Also an obvious symmetry relation

$$\Lambda(t_1, t_2) = \Lambda(t_2, t_1).$$

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2D Ising free-energy

- Remarkably, these two relations, plus the structure of R_n determines, order by order, the numerator polynomials.
- Alternatively, the two functional relations, and the structure of R_n implicitly gives the Onsager solution.
- A mere 70 years after Onsager, we could *conjecture* the exact solution from simple calculations—that of the first few R_ns.
- An attempt to do the same for the susceptibility fails because the structure of the R_ns is not so simple.
- In general we find R_n for unsolved models have denominators containing cyclotomic polynomials of all degrees.

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2D Ising susceptibility

- $\chi(t_1, t_2) = \sum_{n,m} c_{n,m} t_1^m t_2^n = \sum_n H_n(t_1) t_2^n.$
- The inversion and symm. relations are

$$\chi(t_1, t_2) + \chi(1/t_1, -t_2) = 0, \quad \chi(t_1, t_2) = \chi(t_2, t_1).$$

- The first few denominators of $H_n(t_1)$ are:

$$\begin{aligned}
D_0(x) &= (1 - t_1) \\
D_1(x) &= (1 - t_1)^2 \\
D_2(x) &= (1 - t_1)^3(1 + t_1) \\
D_3(x) &= (1 - t_1)^4 \\
D_4(x) &= (1 - t_1)^4(1 + t_1)^3(1 - t_1^3) \\
D_5(x) &= (1 - t_1)^6(1 + t_1)^2 \\
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2D Ising susceptibility

- The numerators and denominators are the same degree, and are symmetric, unimodal with positive coefficients.
- But the degree of the polynomials increases non-linearly.
- The functional relations are insufficient to determine the numerator.
- In Wu, McCoy, Tracy and Barouch, $\chi(t) = \sum \chi^{(2n+1)}(t)$, where $\chi^{(2n+1)}(t) = O(t^{(2n+1)^2-1})$.
- $H_4(t)$ sees the first denominator occurrence of $(1-t^3)$, reflecting $\chi^{(3)} = O(t^8)$.
- Similarly, $H_{12}(t)$ sees the first occurrence of $(1-t^5)$, reflecting $\chi^{(5)} = O(t^{24})$.

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2D Ising susceptibility structure

- $H_n(t)$ is rational, with poles on the unit circle in the t-plane.
- These become dense as $n \to \infty$.
- Then (barring miraculous cancellation) $\chi(t_1, t_2)$ as a function of t_1 for t_2 fixed (a) has a natural boundary, and (b) is neither algebraic nor D-finite, despite the fact that $H_n(t_1)$ is rational.
- For some models this argument can be refined into a proof (absence of cancellations).
- For Ising χ, we could prove positivity and unimodality, that would do. (No cancellations then possible).
- Andrew Rechnitzer did this for SAPs, bond animals, bond trees.
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CONNECTION WITH NATURAL BOUNDARIES

- Subsequently Nickel showed, conjecturally, that the *isotropic* Ising susceptibility has a natural boundary on the unit circle in the $s = \sinh(2K)$ plane.
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POTTS AND PERCOLATION – JACOBSEN AND SCULLARD

- 2d Ising model: K_c and f.e. known on all Archimedean lattices.
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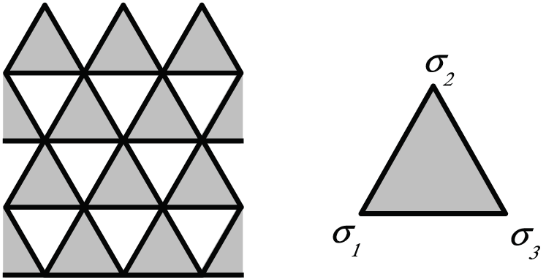
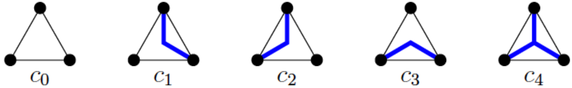
Three-terminal lattices: sq, tri and hex.

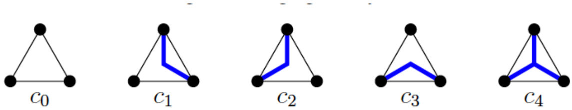


(a) (b)

(Fig. from Jac-Scull). All interactions in up-pointing triangles.



THREE-TERMINAL LATTICES – CONTINUED.



(Fig. from Jac-Scull). All possible interactions between spins in triangles.

Boltzmann weight

$$w_{123} = c_0 + c_1\delta_{23} + c_2\delta_{13} + c_3\delta_{12} + c_4\delta_{123}.$$

Proceeding via the F-K representation, let $G_A = (V, A)$ be a sub-graph of G, $|A|$ is # of edges in A, and $k(A)$ is the # of conn. comps. of G_A.

$$Z = \sum_{A \subseteq E} q^{k(A)} \prod_{p=0}^{4} (c_p)^{N_p},$$

where N_p is the # of up-triangles of type c_p.

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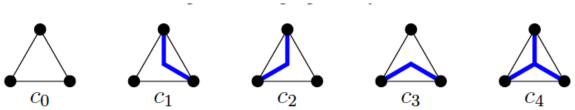

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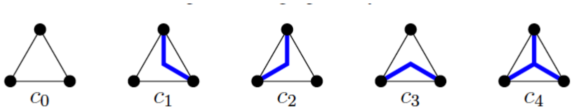

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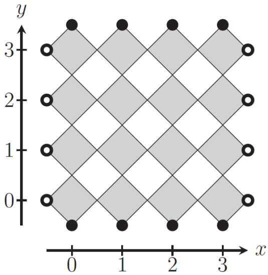
FOUR-TERMINAL LATTICES: KAGOME AND OTHERS



(Fig. from Jac-Scull). A 4×4 square basis.

For the kagome and other unsolved cases a four-terminal lattice is needed.

The calculation of $P_B(q, v)$ is much more complicated.

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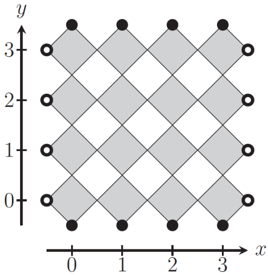

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FOUR-TERMINAL LATTICES: CALCULATION OF $P_B(q, v)$.

- Jacobsen and Scullard initially gave a contraction-deletion method, but later give a probabilistic, geometric interpretation.
- Consider two copies of the basis separated by an arbitrary distance. If connected, we say there is an infinite 2D cluster.
- Denote the *weight* of this event as $W(2D; B)$.
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- J-S reformulated the cluster representation as a loop model, adapted to the lattice geometry.
- Many details need sorting to build the TM. Different tricks typically needed for each lattice.
- For the kagome lattice with $q = 1$ they can get to bases of size 7 in this way, giving the result quoted above.
- Convergence is very fast. At least $\mathrm{O}(1/|B|^4)$ often even faster than $\mathrm{O}(1/|B|^6)$.
- Another exact method for cases that can be exactly solved.
- Fails to solve most cases that we've previously been unable to solve, but does provide lots of extra information (e.g. antiferromagnetic regime, Beraha number solution).
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- A gas of dilute non-intersecting loops.
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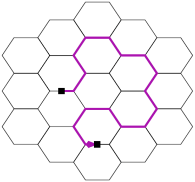
SELF-AVOIDING WALKS ON THE HEXAGONAL LATTICE



A self-avoiding walk on the honeycomb lattice, starting and finishing on a mid-edge.

These are known to 105 steps (Iwan Jensen 2006).

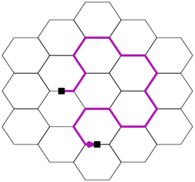
O.g.f : $C(x) = \sum c_n \cdot x^n$.

Conjecture: Nienhuis 1982

$$\mu = 1/x_c = \sqrt{2+\sqrt{2}}.$$

Proved by Smirnov and Duminil-Copin 2010

Self-avoiding walks on the hexagonal lattice



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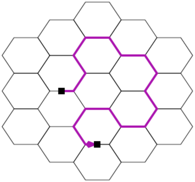

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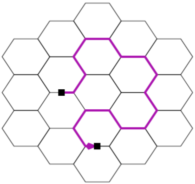

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HEXAGONAL LATTICE GEOMETRY

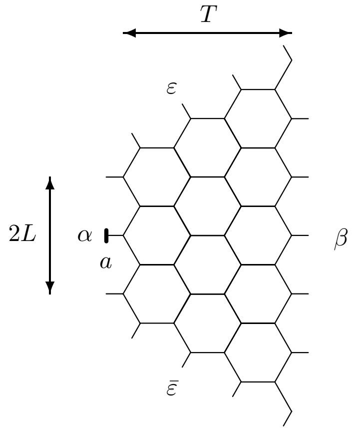


Figure: The figure shows the domain of width T and height $2L$. Walks start at point a and finish internally, or on the α, β or ε ($\bar{\varepsilon}$) wall. Corresponding g.f.'s $A(x)$, $B(x)$, $E(x)$.

Smirnov's hexagonal lattice observable.

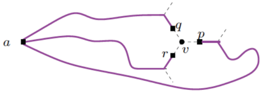
- The holomorphic observable is

$$F_z(x) = \sum_{\omega \subset \Omega:\, a \to x} e^{-i\sigma W_\omega(a,x)} z^{l(\omega)}.$$

- ω is a walk from boundary point a to x in Ω. $\sigma \in \mathbb{R}$ and $z \geq 0$.
- $l(\omega)$ is the $|\omega|$, and $W_\omega(a,b)$ is the rotation when ω is traversed.
- When $z = z_c = 1/\sqrt{2+\sqrt{2}}$ and $\sigma = 5/8$, F_{z_c} is discretely holomorphic, and satisfies

$$(p-v)F_{z_c}(p) + (q-v)F_{z_c}(q) + (r-v)F_{z_c}(r) = 0,$$

where p, q, r are the mid-edges of the three edges adjacent to v.



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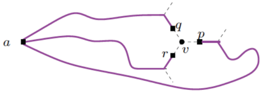

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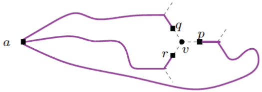

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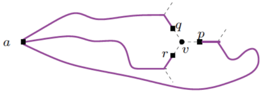
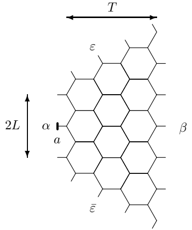


Consequence of observable.



Recall $(p-v)F_{z_c}(p) + (q-v)F_{z_c}(q) + (r-v)F_{z_c}(r) = 0.$
Now sum this over all vertices in the domain.

- Walks start at a and finish internally, or on the α, β or ε ($\bar{\varepsilon}$) wall.
- Gen. fns. $G_{T,L}(x)$, $A_{T,L}(x)$, $B_{T,L}(x)$ and $E_{T,L}(x)$ respectively.
- From DH condition, $G_{T,L}(x_c) = 0$. As $L \to \infty$, $E_{T,L}(x_c) \to 0$.
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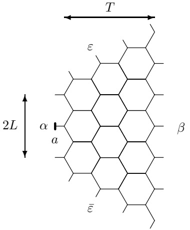

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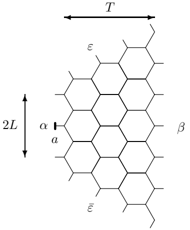

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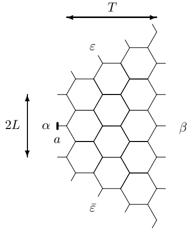

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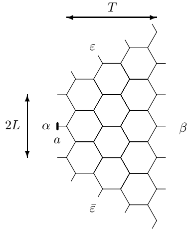

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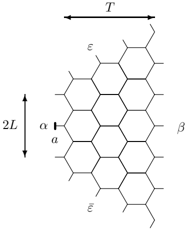

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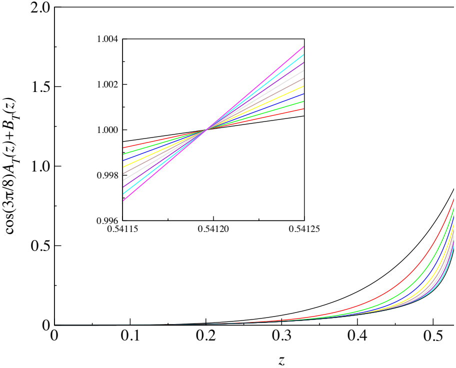

Figure: Bad picture with nice inset of $\cos\left(\frac{3\pi}{8}\right) A_T(x) + B_T(x)$ for honeycomb lattice walks in a strip of width $1, \cdots, 10$.

- There is no corresponding equation for SAW on other lattices.
- For the square lattice, Cardy and Ikhlef found a similar observable. The model describes osculating SAW with asymmetric weights.
- In the scaling limit, all SAW models should be identical, so "something similar" should be true for SAWs on other lattices.
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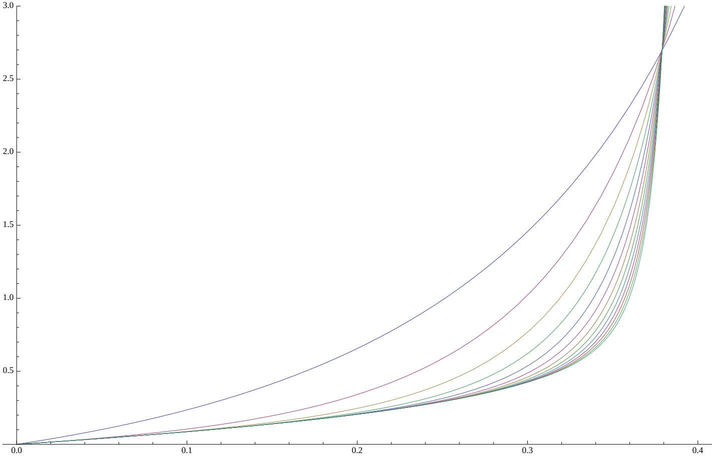

Figure: Square lattice $\cos\left(\frac{3\pi}{8}\right) A_T(x) + B(x)$ for walks in a strip of width $1, \cdots, 15$.

Conjecture (best estimates of x_c):

$$1 = c_A(T)A_T(x_c) + c_B(T)B_T(x_c),$$

Successive widths $(T, T+1)$ give $c_A(T)$ and $c_B(T)$.
(Square lattice $T \leq 17$, triangular lattice $T \leq 11$).
Extrapolate:

$$\lim_{T\to\infty} \frac{c_A(T)}{c_B(T)} = \cos\left(\frac{3\pi}{8}\right)$$

to 6 sig. digits. Hence

$$\cos\left(\frac{3\pi}{8}\right) A_T(x_c) + B_T(x_c) = const. + correction$$

In fact $1.0249663(1 + const/T^{9/4} + O(T^{-13/4}))$, similarly for the triang. lattice.

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$$1 = c_A(T)A_T(x_c) + c_B(T)B_T(x_c),$$

Successive widths $(T, T+1)$ give $c_A(T)$ and $c_B(T)$.
(Square lattice $T \leq 17$, triangular lattice $T \leq 11$).
Extrapolate:

$$\lim_{T\to\infty} \frac{c_A(T)}{c_B(T)} = \cos\left(\frac{3\pi}{8}\right)$$

to 6 sig. digits. Hence

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To estimate x_c we solve

$$\cos\left(\frac{3\pi}{8}\right) A_T(x) + B_T(x) = \cos\left(\frac{3\pi}{8}\right) A_{T+1}(x) + B_{T+1}(x)$$

Successive values of T give

$$x_c(T) = x_c(1 + O(1/T^{13/4})).$$

Extrapolate $x_c(T)$ and find
$x_c(sq) = 0.37905227774(4)$
(c.f. old conjecture of G. that x_c is a root of $581x^4 + 7x^2 - 13 = 0$,
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Jacobsen's method for $O(n)$ model

- Semi-infinite cylinder of circumference L.
- Earlier work usually done on finite rectangles.
- Set up TM for SAWs, with weights z^n, (n monomers).
- Compute leading eigenvalue of the TM in two different sectors:
- (i) with an (open) strand from one end of the cylinder to the other. (A SAW with the ends at opposite ends of the cylinder).
- (ii) with no propagating loop strands. Basically SAPs.
- A loop on the cylinder has weight $n = 0$. Loops around the cylinder get weight n'.
- Setting $n' = n$ would give the ground state sector. However with $n' = -\sqrt{2-n}$, the sector exponents are equal.

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- The f.e/site is $f = -(1/L)\log(\Lambda_{max})$.
- f_0 is the ground state f.e., and f_i are the f.e's in other sectors. From CI, $$f_i - f_0 = (2\pi x_i)/L^2 + o(L^{-2}),$$ where x_i is a critical exponent.
- The exponent for paths in both sectors are known from CG arguments. The sector (2) exponent varies with n', which is chosen so that the exponents are equal.
- Therefore one obtains, right at the infinite-size critical point $$f_2 - f_1 = o(L^{-2}).$$
- Define a finite-size critical point $z_c(L)$ by finding the monomer fugacity s.t. $$f_2(L) = f_1(L)$$ then the finite-size corrections to $z_c(L)$ will be very small.

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- These corrections turn out to be exactly zero for solvable models (like that of Nienhuis on the hexagonal lattice), whereas for square and triangular SAWs they turn out to go like $1/L^4$ with subdominant $1/L^6$, $1/L^8$ etc terms.
- So we systematically extrapolate to eliminate terms $O(1/L^6)$, $O(1/L^8)$, $O(1/L^{10}), \ldots$. In this way the current result for the square lattice is $x_c = 0.3790522777533(2)$.
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CONCLUSION

- Four methods, all exact for some situations, not for others. Why?
- Non-D-finiteness is an answer in some cases.
- Maybe natural boundaries is another answer?
- Does an algebraic critical point imply integrability?
- For Y-B integrability one needs a model with one or two continuous parameters (“rapidities.") (One if you have a difference or quotient of the two rapidities.)
- With an alg. critical point, there is either a Y-B equation within the model, or one needs an extended model, or perhaps there is no Y-B equation.
- In any event, we now have a powerful suite of tools to obtain increasingly precise numerical estimates of critical parameters, and equally significantly, to give insight into the solvability of the underlying problem.

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