Random matrices and Aztec diamonds

Kurt Johansson

Florence, May, 2015.

Domino Tilings of the Aztec Diamond

Define an Aztec diamond, A_n, as the lattice squares contained in $\{(x, y) : |x| + |y| \leq n + 1\}$.

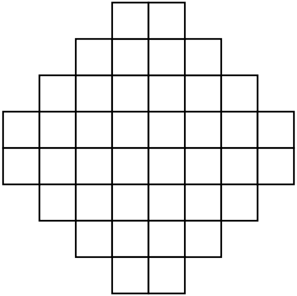

Figure: A_4

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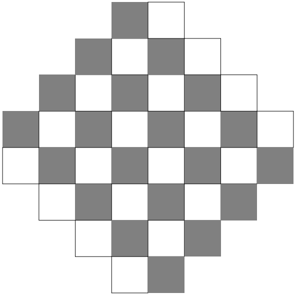

Figure: A_4 with a checkerboard coloring

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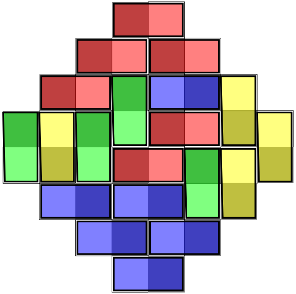


Figure: A_4 with a checkerboard coloring, tiled with dominos. Four types of dominoes N, E, S, W, here given different colors.

One-Periodic Weighting

One-periodic weighting of A_n: give weight 1 to horizontal dominos and weight a to vertical domino. For each tiling, take the product of the domino weights.

The partition function of domino tilings of A_n with the one-periodic weighting is $(1+a^2)^{n(n+1)/2}$. Computed by Elkies, Kuperberg, Larsen and Propp (1992).

To obtain a random tiling, pick each tiling T with probability proportional to the product of the domino weights of T.

For a one-periodic weighting, pick T with:

$$P(T) = \frac{a^{v(T)}}{(1+a^2)^{n(n+1)/2}}$$

where $v(T)$ is the number of vertical dominos for a tiling T.

Relatively large Aztec diamond with one-periodic weighting

Using the domino shuffle algorithm Propp, 2003

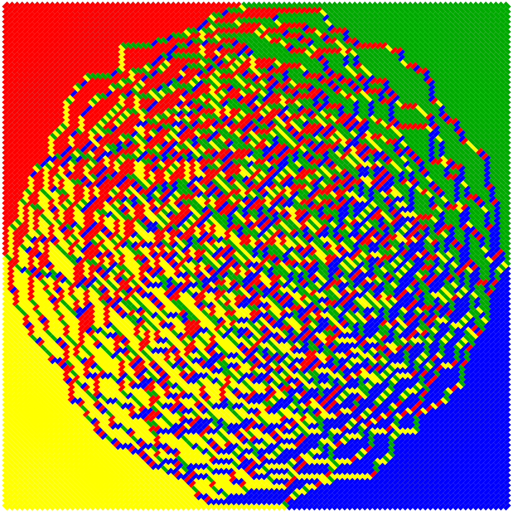
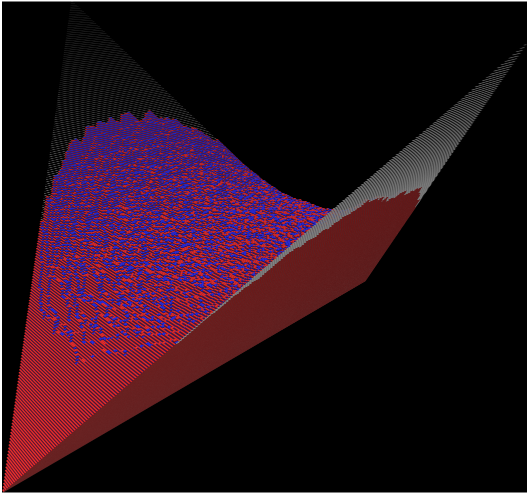


Figure: Random tiling $n = 100$, $a = 1$

Height function representation of a random tiling

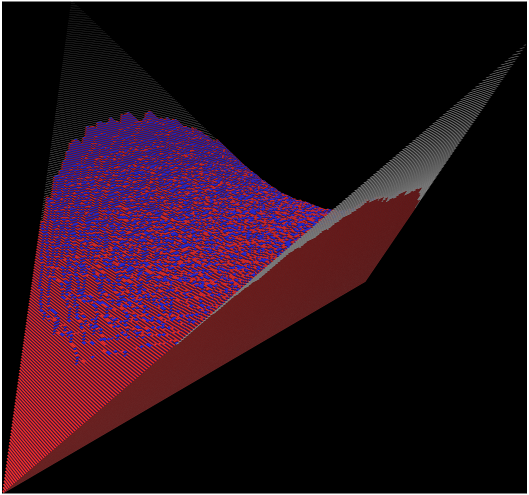
To each tiling of an Aztec diamond one can associate a height function.



Picture by Benjamin Young

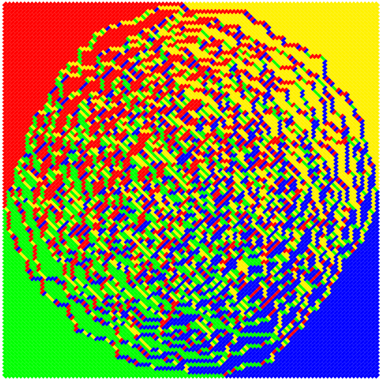
Height function representation of a random tiling

This is an idea that goes back to Thurston. One way to think about it is that as one goes around a domino the height goes up by 1 if the square to the left is white and down by one if it is black. In this way we get a certain class of random surface models.



Limit shape

Limit Shape: Jokusch, Propp and Shor (1995), Cohn, Elkies and Propp (1996), J. (2005), Romik (2011), Kenyon and Okounkov (2007).



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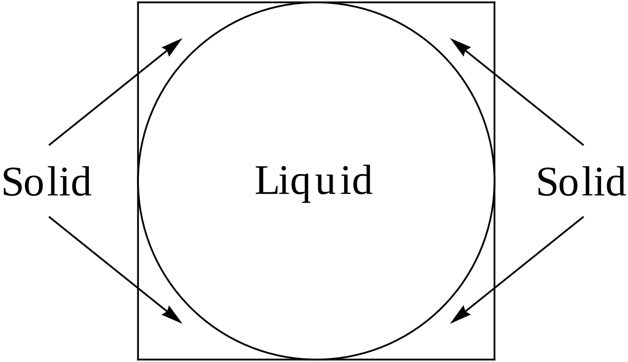

We have two types of phases in the limit called solid and liquid.

Particles

We can put particles on dominos. The particles are directly related to the height function.

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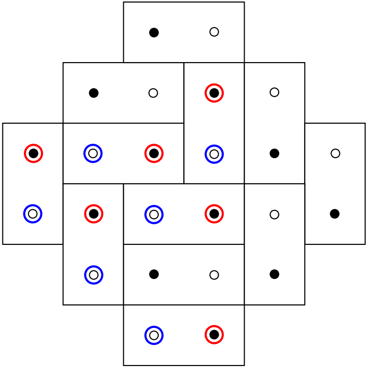
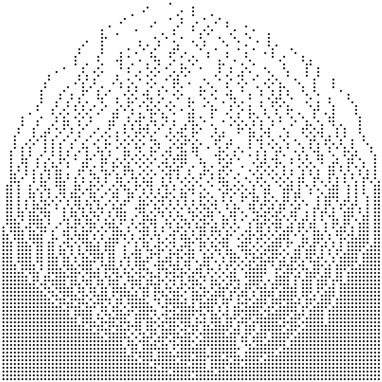
Interlacing particle system.



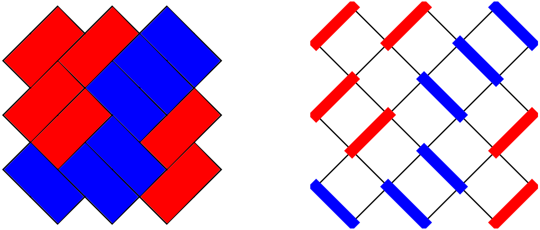
Particles

Interlacing particles defined by the Aztec diamond. These particles form a *determinantal point process*. Krawtchouk ensemble. Similar to eigenvalues of random matrices. Discrete analogue of GUE.



Dimers

Consider the graph theoretic dual of the Aztec diamond: each domino tiling is a dimer covering of the dual graph of the Aztec diamond.



A dimer covering is a subset of edges so that each vertex is incident to only one edge.
The weights of each domino are now edge weights.

Kasteleyn Matrix

Let $v : E \to \mathbb{R} > 0$ be the weights. The *Kasteleyn Orientation* is a signed weighting such that the product of the signed edge weights around each face is negative.

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Define, the *Kasteleyn Matrix*, K, by

$$K(b,w) = K_{bw} = \begin{cases} v(e) & \text{if } e \text{ is horizontal} \\ v(e)\mathrm{i} & \text{if } e \text{ is vertical} \\ 0 & \text{otherwise (i.e. no edge between } b \text{ and } w) \end{cases}$$

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Theorem (Montroll, Potts, Ward (1963), Kenyon (1997))

If $e_i = (b_i, w_i)$, *then*

$$\mathbb{P}(e_1, \ldots, e_m) = \left[\prod_{i=1}^{m} K(b_i, w_i)\right] \det\left(K^{-1}(w_i, b_j)\right)_{1 \le i,j \le m}$$

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This means that the dimers form a *determinantal point process*.

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L is the *correlation kernel*.

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For the one-periodic Aztec diamond it is possible to give a useful expression for K^{-1} in the form of a double contour integral Chhita, Johansson, Young '12, Helfgott '98.
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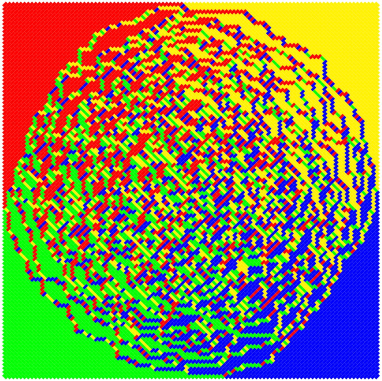
In this way dimer or random tiling models are sources of interesting determinantal point processes. In appropriate scaling limits we should get *universal limiting processes.*

Limiting processes. Fluctuations.

We are particularly interested in the behaviour near the boundaries between phases.



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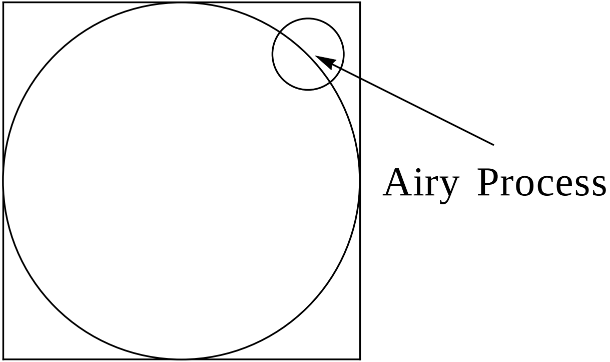
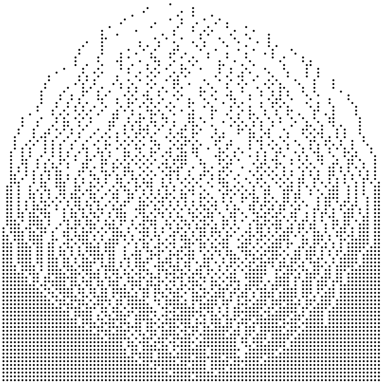
The Airy Process J. (2005). Fluctuation exponents $1/3$ and $2/3$ (*KPZ-universality*).



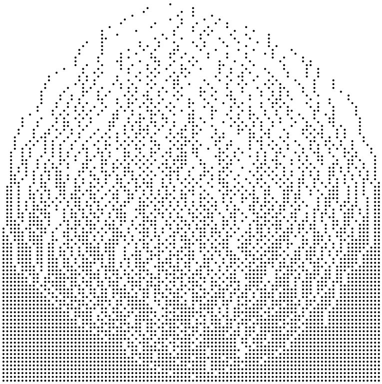
Limiting processes. Fluctuations.

Particles around the edge converge to the Airy kernel point process.



Limiting processes. Fluctuations.

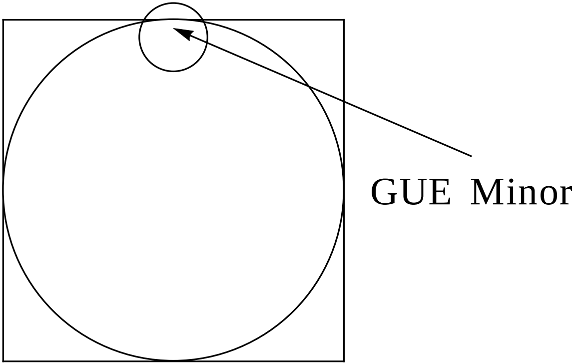
Tangency points



Limiting processes. Fluctuations.

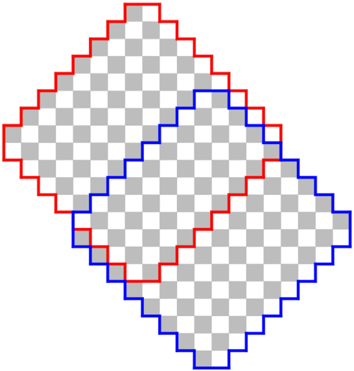
Tangency points

The GUE minor process



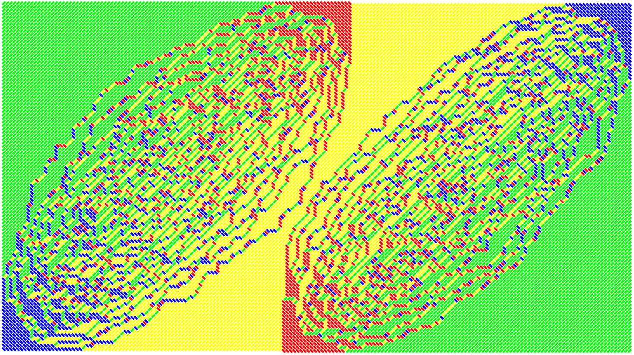
Other limiting processes. The double Aztec diamond.

The shape of a double Aztec diamond



Other limiting processes. The double Aztec diamond.

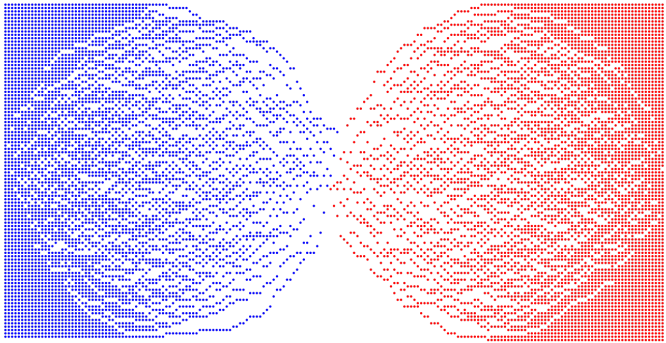
A simulation of a double Aztec diamond in a *tacnode situation*.



Adler, Johansson, van Moerbeke (2011)

Other limiting processes. The double Aztec diamond.

Particles in a double Aztec diamond. *Tacnode GUE-minor process*. Universal limiting process.



Adler, Chhita, Johansson, van Moerbeke (2013)

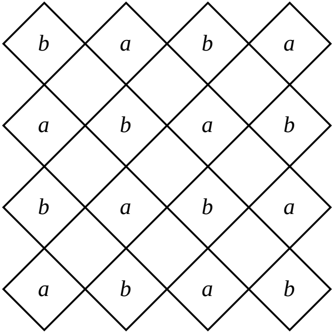
Two Periodic Weighting

Joint work with Sunil Chhita.

Two Periodic Weighting

We consider a weighting which is called a *two-periodic weighting* of the Aztec diamond.

For a two coloring of the faces, the edge weights around a particular colored face alternate between a and b. We shall set $b = 1$. E.g. for $n = 4$



Large two periodic weightings

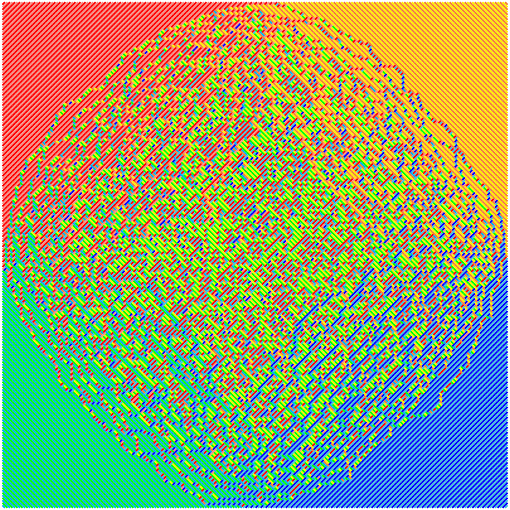


Figure: $n = 200, a = 0.5, b = 1$ with 8 colors

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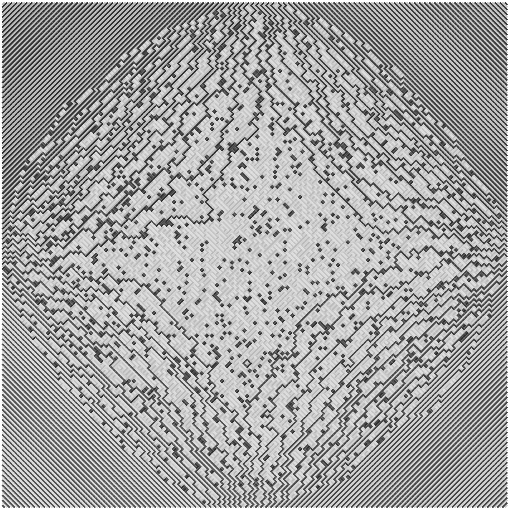

Figure: $n = 200, a = 0.5, b = 1$ with 8 grayscale colors

Limit Shape of Two-periodic Model

Using techniques from Kenyon-Okounkov (2007), one can find a formula for the limit shape of the boundaries. This is a degree 8 curve.

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$$
\begin{aligned}
&-64c^6\left(x^2-1\right)\left(y^2-1\right)+16c^4\left(x^4\left(16y^4-20y^2+3\right)\right.\\
&\left.+x^2\left(-20y^4+27y^2-6\right)+3\left(y^2-1\right)^2\right)\\
&+4c^2\left(x^6\left(8y^2+3\right)+x^4\left(-16y^4+13y^2-9\right)\right.\\
&\left.+x^2\left(8y^6+13y^4-30y^2+9\right)+3\left(y^2-1\right)^3\right)\\
&+\left(x^4-2x^2\left(y^2+1\right)+\left(y^2-1\right)^2\right)^2=0,
\end{aligned}
$$

where $c=a/(1+a^2)$ for a rescaled Aztec diamond with corners $(\pm 1,\pm 1)$.

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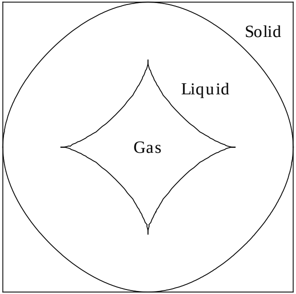

The limit shape has three regions where we get different types of phases, solid, liquid and gas.

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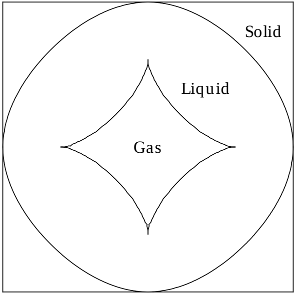
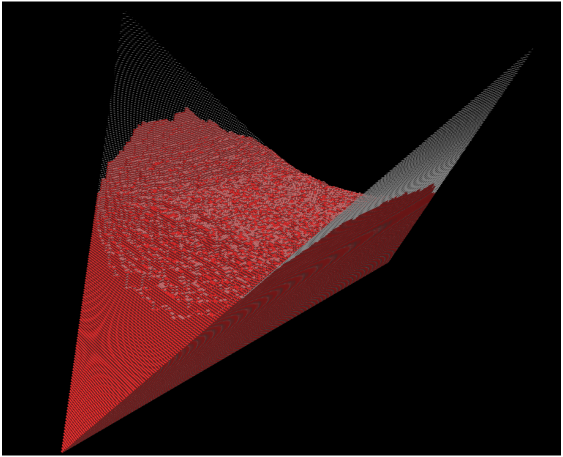


The limit shape has three regions where we get different types of phases, solid, liquid and gas.

Correlations between dominos decay polynomially (with distance) in the liquid region and exponentially (with distance) in the gas region.

Characterization of the three phases

In Kenyon, Okounkov and Sheffield (2006), the authors characterized the different limiting Gibbs measures that are possible for bipartite dimer models on the plane.



Picture by Benjamin Young

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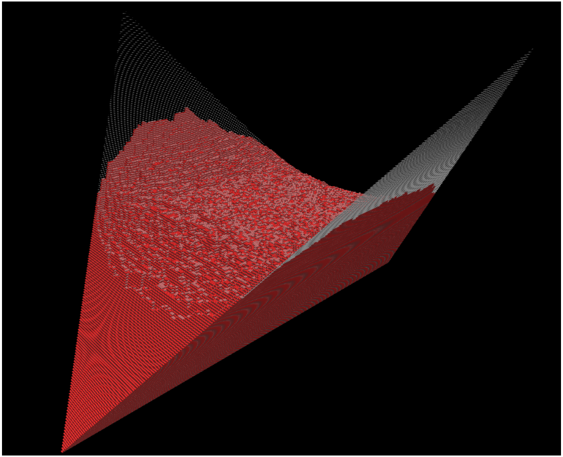
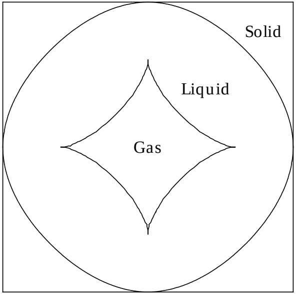


There are three classes of Gibbs measures defined via the limiting inverse Kasteleyn matrices $\mathbb{K}^{-1}_{\mathrm{solid}}$, $\mathbb{K}^{-1}_{\mathrm{liquid}}$ and $\mathbb{K}^{-1}_{\mathrm{gas}}$. Which of these expressions that applies in a certain region is determined by the slope of the limiting height function.

Liquid-gas boundary

The liquid-gas boundary is a new feature that we did not have in the one-periodic Aztec diamond.



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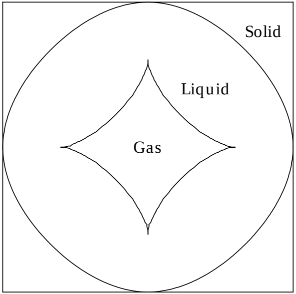

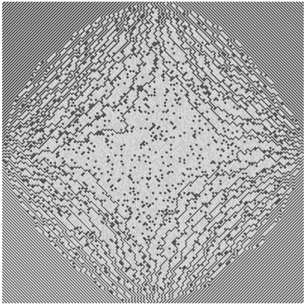

Can we find the correlation of the dominos at the liquid-gas boundary? Can we describe the boundary? Is it again given by an Airy process?

Formula for the inverse Kasteleyn matrix in the two-periodic case

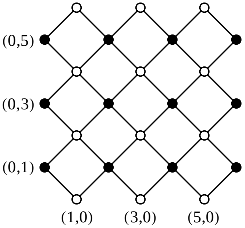

Figure: The coordinates

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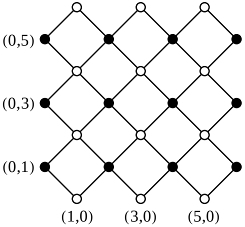

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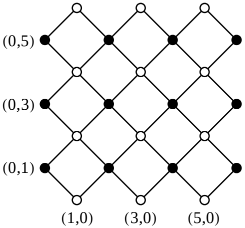

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$$G(w_1, w_2, b_1, b_2) = \sum_{\substack{(x_1,x_2)\in \mathtt{W} \\ (y_1,y_2)\in \mathtt{B}}} K^{-1}((x_1, x_2), (y_1, y_2)) w_1^{x_1} w_2^{x_2} b_1^{y_1} b_2^{y_2}.$$

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This gives a formula for the inverse Kasteleyn matrix

$$K^{-1}((x_1, x_2), (y_1, y_2)) = \frac{1}{(2\pi\mathrm{i})^4} \int_\gamma \dots \int_\gamma \frac{G(w_1, w_2, b_1, b_2)}{w_1^{x_1} w_2^{x_2} b_1^{y_1} b_2^{y_2}} \frac{dw_1}{w_1} \dots \frac{db_2}{b_2}$$

for a positively oriented contour γ around 0.

Simplified Formula

Theorem (Chhita-J.)

For an Aztec diamond of size n with the two-periodic weighting

$$K^{-1}((x_1,x_2),(y_1,y_2)) = \mathbb{K}^{-1}_{gas}((x_1,x_2),(y_1,y_2)) - \sum_{i=1}^{4} B_i((x_1,x_2),(y_1,y_2)),$$

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$$B_1(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \int_{|\omega_1|=r} \frac{d\omega_1}{\omega_1} \int_{|\omega_2|=1/r} d\omega_2 \frac{Y_{\epsilon_1,\epsilon_2}(\omega_1,\omega_2)}{\omega_2-\omega_1} \frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)}.$$

Simplified Formula

Theorem (Chhita-J.)

For an Aztec diamond of size n with the two-periodic weighting

$$K^{-1}((x_1,x_2),(y_1,y_2)) = \mathbb{K}^{-1}_{gas}((x_1,x_2),(y_1,y_2)) - \sum_{i=1}^{4} B_i((x_1,x_2),(y_1,y_2)),$$

where $\mathbb{K}^{-1}_{gas}((x_1,x_2),(y_1,y_2))$ *is the inverse Kasteleyn matrix on the plane in the gas region, and* $B_1,\dots,B_4$ *are related by a symmetry with* B_1 *having the form*

$$B_1(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \int_{|\omega_1|=r} \frac{d\omega_1}{\omega_1} \int_{|\omega_2|=1/r} d\omega_2 \frac{Y_{\epsilon_1,\epsilon_2}(\omega_1,\omega_2)}{\omega_2-\omega_1} \frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)}.$$

Here $Y_{\epsilon_1,\epsilon_2}(\omega_1,\omega_2)$ *is a complicated non-asymptotic factor,*

$$H_{x_1,x_2}(\omega) = \frac{\omega^{n/2} G(\omega)^{n/2-x_1/2}}{G(1/\omega)^{n/2-x_2/2}}, \quad G(\omega) = \frac{1}{\sqrt{2c}}(\omega - \sqrt{\omega^2+2c}),$$

and $c = a/(1+a^2)$ *with* $0 < c < 1/2$.

Leading asymptotics

If we want to do a saddle point analysis of the double contour integral formula we are led to study

$$h_{\xi_1,\xi_2}(\omega) = \frac{1}{n/2} \log H_{x_1,x_2}(\omega) = \log \omega - \xi_1 \log G(\omega) + \xi_2 \log G(1/\omega)$$

where we have introduced rescaled coordinates with the origin at the center of the Aztec diamond,

$$x_1 = n + [n\xi_1], \;\; x_2 = n + [n\xi_2],$$

$-1 < \xi_1, \xi_2 < 1.$

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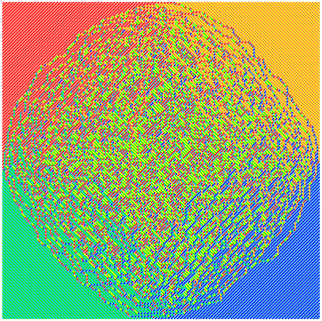
$$x_1 = n + [n\xi_1], \;\; x_2 = n + [n\xi_2],$$

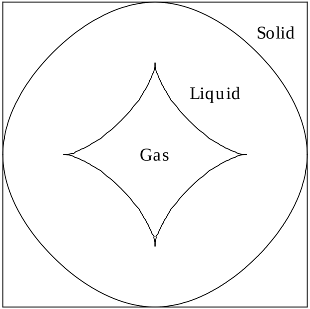
$-1 < \xi_1, \xi_2 < 1$.
To see the boundaries of the liquid region we look for second order critical points

$$h'_{\xi_1,\xi_2}(\omega_c) = h''_{\xi_1,\xi_2}(\omega_c) = 0.$$

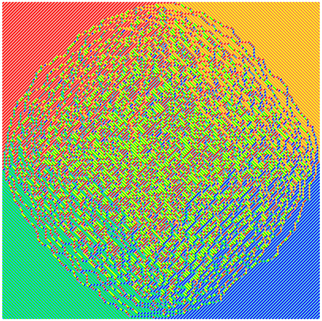
Eliminating ω_c leads to the degree 8 curve above.

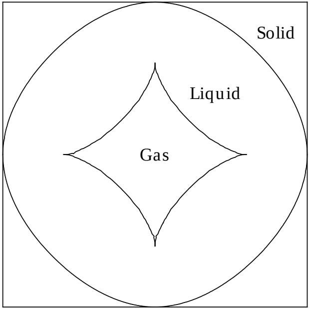
Asymptotics in each regime





Asymptotics in each regime





Theorem (Chhita-J.)

For an Aztec diamond of size n with the two-periodic weighting, set $x=(n+[n\xi]+x_1, n+[n\xi]+x_2)$, $y=(n+[n\xi]+y_1, n+[n\xi]+y_2)$ for $-1<\xi<0$ and let $c=a/(1+a^2)$ with $0<c<1/2$. Then,

$$K^{-1}(x,y)=\begin{cases} \mathbb{K}^{-1}_{\text{solid}}((x_1,x_2),(y_1,y_2))+O(e^{-dn}) & \text{if } -1<\xi<-1/2\sqrt{1+2c} \\ \mathbb{K}^{-1}_{\text{solid}}((x_1,x_2),(y_1,y_2))+O(n^{-1/3}) & \text{if } \xi=-1/2\sqrt{1+2c} \\ \mathbb{K}^{-1}_{\text{liquid}}((x_1,x_2),(y_1,y_2))+O(n^{-1/2}) & \text{if } -1/2\sqrt{1+2c}<\xi<-1/2\sqrt{1-2c} \\ \mathbb{K}^{-1}_{\text{gas}}((x_1,x_2),(y_1,y_2))+O(n^{-1/3}) & \text{if } \xi=-1/2\sqrt{1-2c} \\ \mathbb{K}^{-1}_{\text{gas}}((x_1,x_2),(y_1,y_2))+O(e^{-dn}) & \text{if } -1/2\sqrt{1-2c}<\xi<0 \end{cases}$$

Asymptotics in each regime

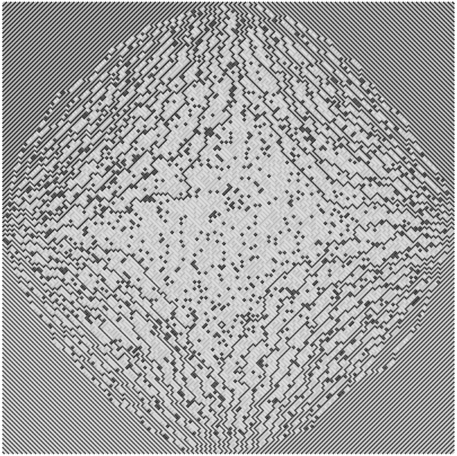
Theorem (Chhita-J.)

For an Aztec diamond of size n with the two-periodic weighting, set $x = (n + [n\xi] + x_1, n + [n\xi] + x_2)$, $y = (n + [n\xi] + y_1, n + [n\xi] + y_2)$ for $-1 < \xi < 0$ and let $c = a/(1+a^2)$ with $0 < c < 1/2$. Then,

$$K^{-1}(x,y) = \begin{cases} \mathbb{K}^{-1}_{\text{solid}}((x_1,x_2),(y_1,y_2)) + O(e^{-dn}) & \text{if } -1 < \xi < -1/2\sqrt{1+2c} \\ \mathbb{K}^{-1}_{\text{solid}}((x_1,x_2),(y_1,y_2)) + O(n^{-1/3}) & \text{if } \xi = -1/2\sqrt{1+2c} \\ \mathbb{K}^{-1}_{\text{liquid}}((x_1,x_2),(y_1,y_2)) + O(n^{-1/2}) & \text{if } -1/2\sqrt{1+2c} < \xi < -1/2\sqrt{1-2c} \\ \mathbb{K}^{-1}_{\text{gas}}((x_1,x_2),(y_1,y_2)) + O(n^{-1/3}) & \text{if } \xi = -1/2\sqrt{1-2c} \\ \mathbb{K}^{-1}_{\text{gas}}((x_1,x_2),(y_1,y_2)) + O(e^{-dn}) & \text{if } -1/2\sqrt{1-2c} < \xi < 0 \end{cases}$$

At the solid-liquid boundary and liquid-gas boundary, we can do a finer asymptotic analysis of the correlations between the dominos.

Liquid-gas correlations



Liquid-gas correlations

For $\xi = -1/2\sqrt{1-2c}$, suppose we have dimers $((x_1,x_2),(x_1-1,x_2+1))$ and $((y_1,y_2),(y_1-1,y_2+1))$ both having weight a, with

$$\left\{\begin{array}{l} x_1 = [n+\xi n+\alpha_x n^{1/3}+\beta_x n^{2/3}]+u_1 \\ x_2 = [n+\xi n+\alpha_x n^{1/3}-\beta_x n^{2/3}]+u_2 \end{array}\right\} \qquad \left\{\begin{array}{l} y_1 = [n+\xi n+\alpha_y n^{1/3}+\beta_y n^{2/3}]+v_1 \\ y_2 = [n+\xi n+\alpha_y n^{1/3}-\beta_y n^{2/3}]+v_2 \end{array}\right\}$$

Theorem (Chhita-J.)

If $(\alpha_x,\beta_x)=(\alpha_y,\beta_y)$, *then the covariance between these two dimers is*

$$-a^2\mathbb{K}_{\text{gas}}^{-1}((u_1,u_2),(v_1-1,v_2+1))\mathbb{K}_{\text{gas}}^{-1}((v_1,v_2),(u_1-1,u_2+1))+O(n^{-1/3}) \tag{1}$$

Liquid-gas correlations

For $\xi = -1/2\sqrt{1-2c}$, suppose we have dimers $((x_1, x_2), (x_1 - 1, x_2 + 1))$ and $((y_1, y_2), (y_1 - 1, y_2 + 1))$ both having weight a, with

$$\left\{ \begin{array}{l} x_1 = [n + \xi n + \alpha_x n^{1/3} + \beta_x n^{2/3}] + u_1 \\ x_2 = [n + \xi n + \alpha_x n^{1/3} - \beta_x n^{2/3}] + u_2 \end{array} \right\} \qquad \left\{ \begin{array}{l} y_1 = [n + \xi n + \alpha_y n^{1/3} + \beta_y n^{2/3}] + v_1 \\ y_2 = [n + \xi n + \alpha_y n^{1/3} - \beta_y n^{2/3}] + v_2 \end{array} \right\}$$

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If $(\alpha_x, \beta_x) \neq (\alpha_y, \beta_y)$, *then the covariance between these two dimers is*

$$C n^{-2/3} \mathbb{A}((\alpha_x, \beta_x), (\alpha_y, \beta_y)) \mathbb{A}((\alpha_y, \beta_y), (\alpha_x, \beta_x)) \tag{2}$$

Liquid-gas correlations

For $\xi = -1/2\sqrt{1-2c}$, suppose we have dimers $((x_1, x_2), (x_1 - 1, x_2 + 1))$ and $((y_1, y_2), (y_1 - 1, y_2 + 1))$ both having weight a, with

$$\left\{ \begin{array}{l} x_1 = [n + \xi n + \alpha_x n^{1/3} + \beta_x n^{2/3}] + u_1 \\ x_2 = [n + \xi n + \alpha_x n^{1/3} - \beta_x n^{2/3}] + u_2 \end{array} \right\} \qquad \left\{ \begin{array}{l} y_1 = [n + \xi n + \alpha_y n^{1/3} + \beta_y n^{2/3}] + v_1 \\ y_2 = [n + \xi n + \alpha_y n^{1/3} - \beta_y n^{2/3}] + v_2 \end{array} \right\}$$

Theorem (Chhita-J.)

If $(\alpha_x, \beta_x) = (\alpha_y, \beta_y)$, *then the covariance between these two dimers is*

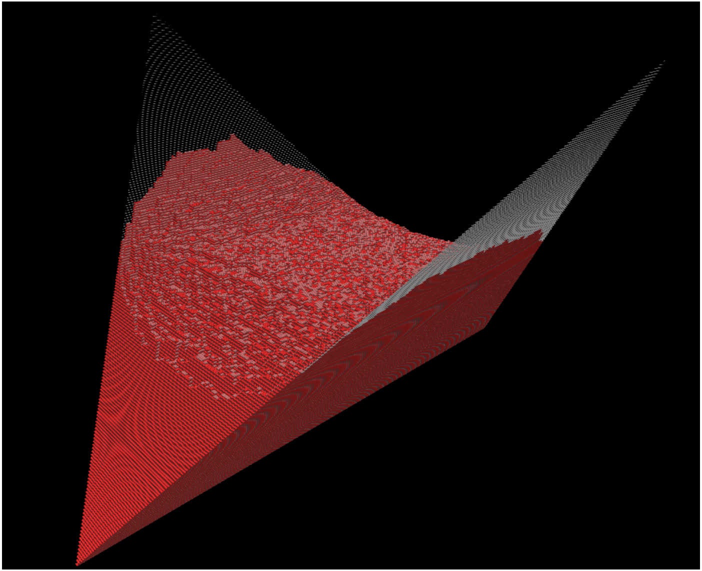
$$-a^2 \mathbb{K}^{-1}_{\mathrm{gas}}((u_1, u_2), (v_1 - 1, v_2 + 1)) \mathbb{K}^{-1}_{\mathrm{gas}}((v_1, v_2), (u_1 - 1, u_2 + 1)) + O(n^{-1/3}) \tag{1}$$

If $(\alpha_x, \beta_x) \neq (\alpha_y, \beta_y)$, *then the covariance between these two dimers is*

$$C n^{-2/3} \mathbb{A}((\alpha_x, \beta_x), (\alpha_y, \beta_y)) \mathbb{A}((\alpha_y, \beta_y), (\alpha_x, \beta_x)) \tag{2}$$

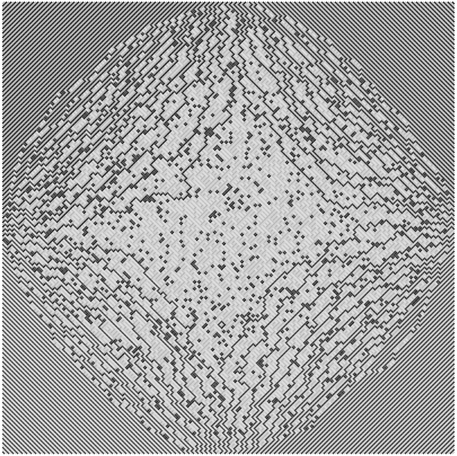
$\mathbb{A}((\alpha_x, \beta_x), (\alpha_y, \beta_y))$ is related to the extended Airy kernel. Note that if we had just a gaseous phase the correlation between the two dimers with this distance would be much smaler, like $exp(-dn^{2/3})$.

Discussion



Picture by Benjamin Young

Discussion



Thank you