Combinatorics of exclusion processes with open boundaries

Sylvie Corteel (CNRS Paris 7)

GGI, Florence, May 19th 2015

Koornwinder moments and the two species ASEP

Sylvie Corteel (CNRS Paris 7)

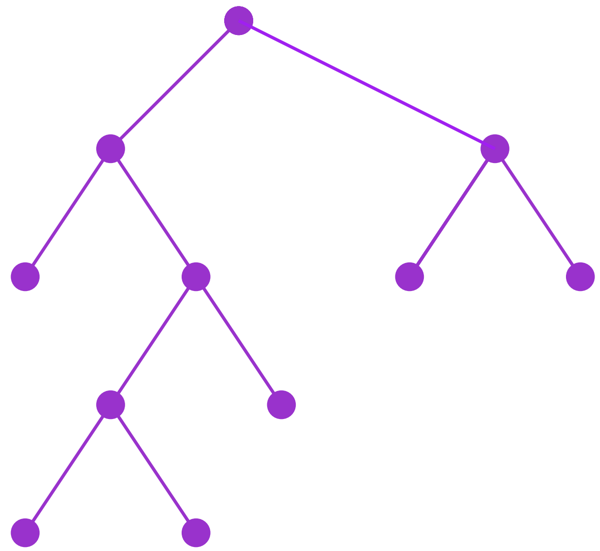
Lauren Williams (Berkeley)

Triangular staircase tableaux

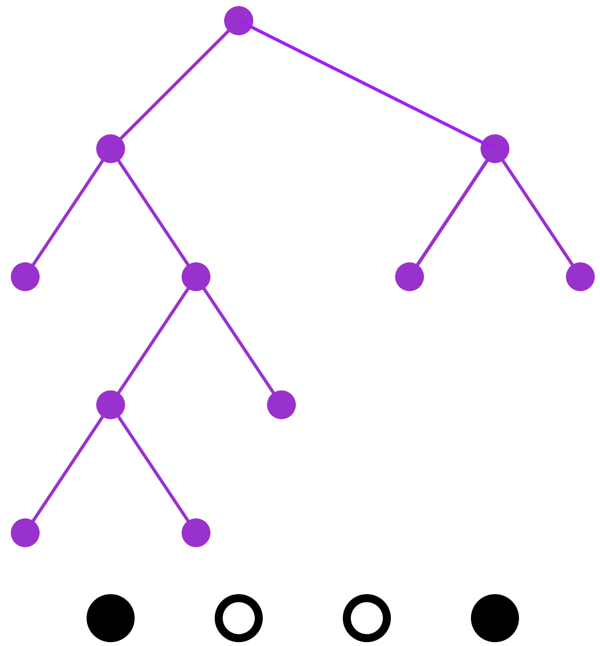
Sylvie Corteel, Olya Mandelshtam (Berkeley) and Lauren Williams (Berkeley)

⋮

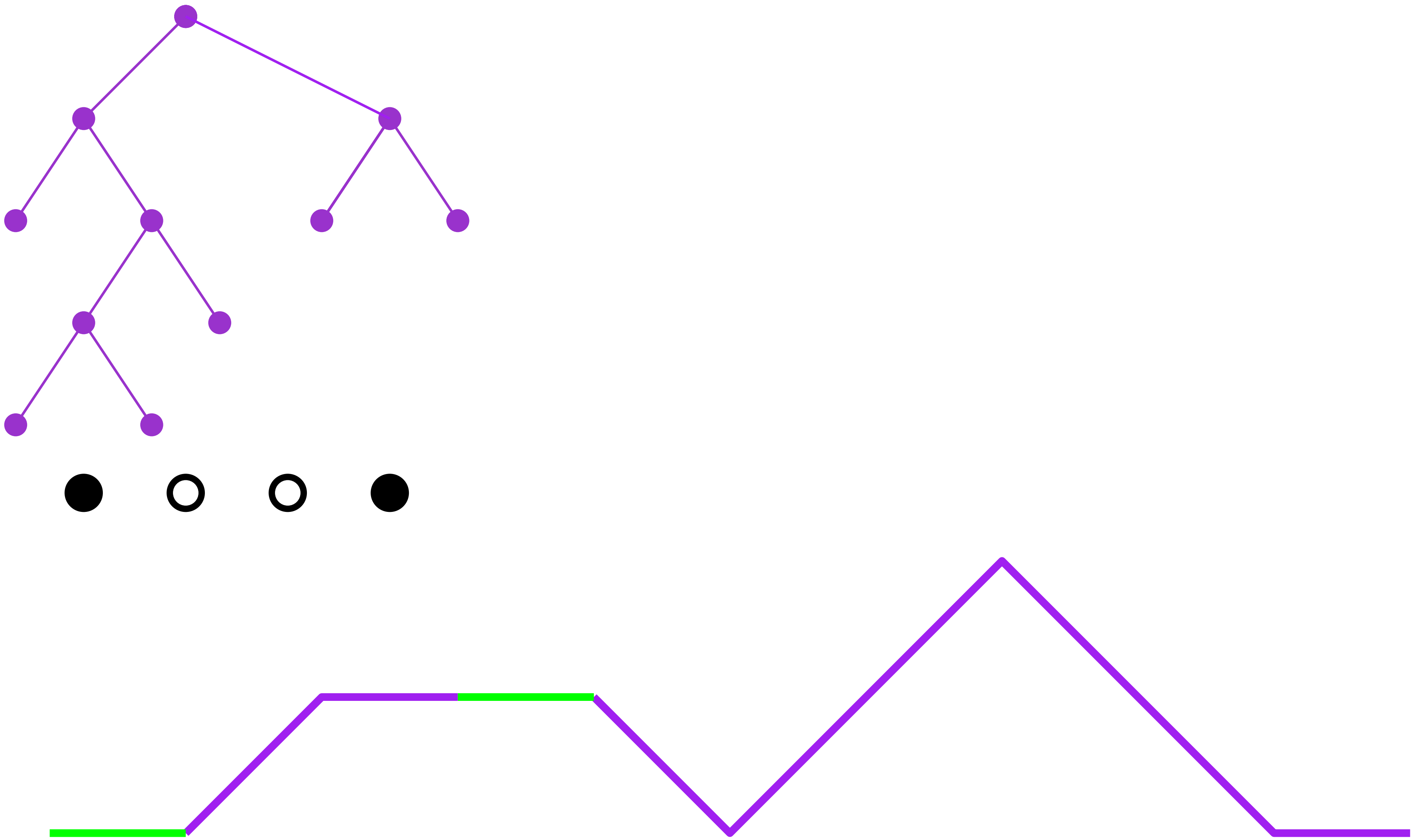
Binary trees, Paths and tableaux



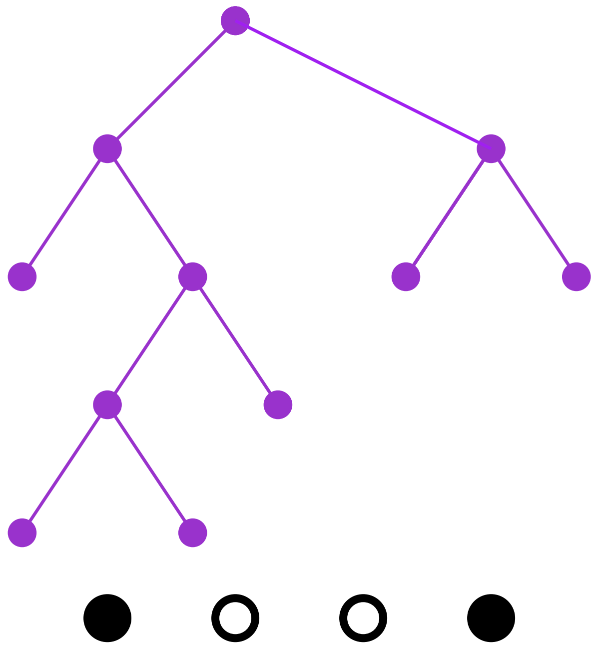
Binary trees, Paths and tableaux

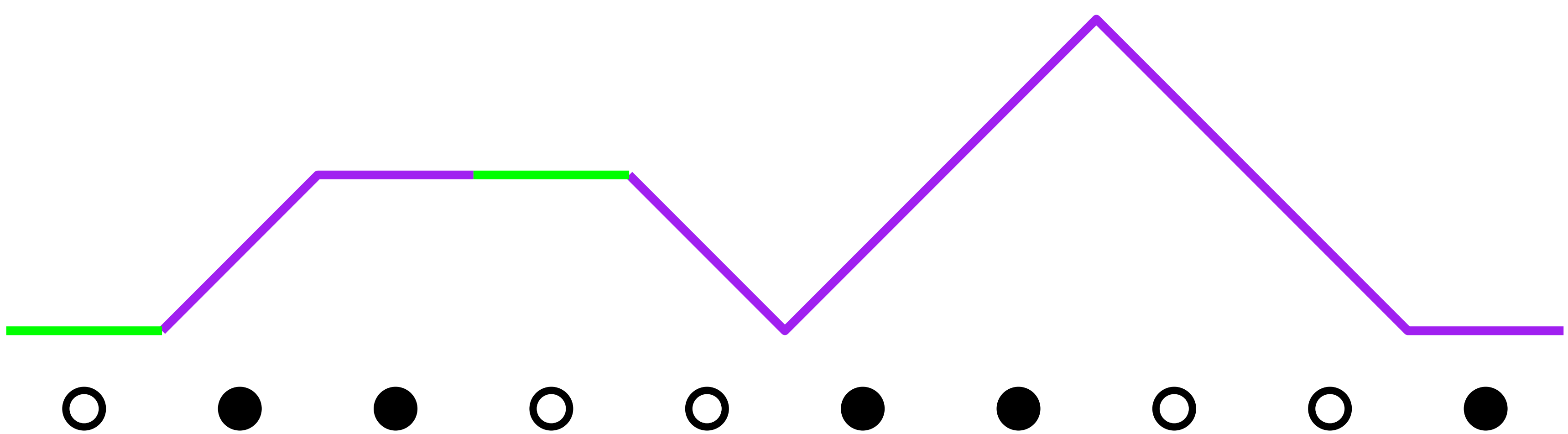


Binary trees, Paths and tableaux

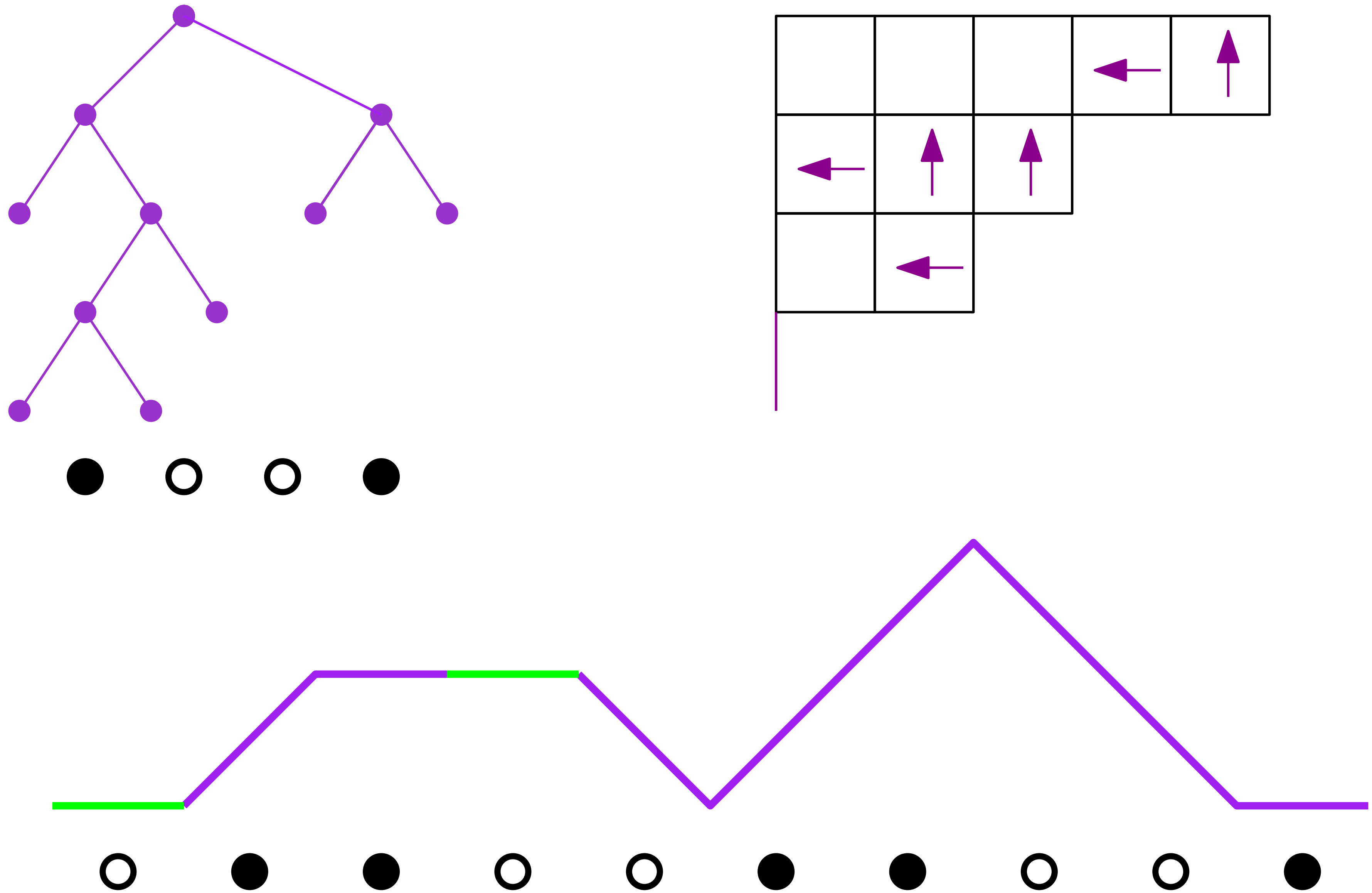


Binary trees, Paths and tableaux

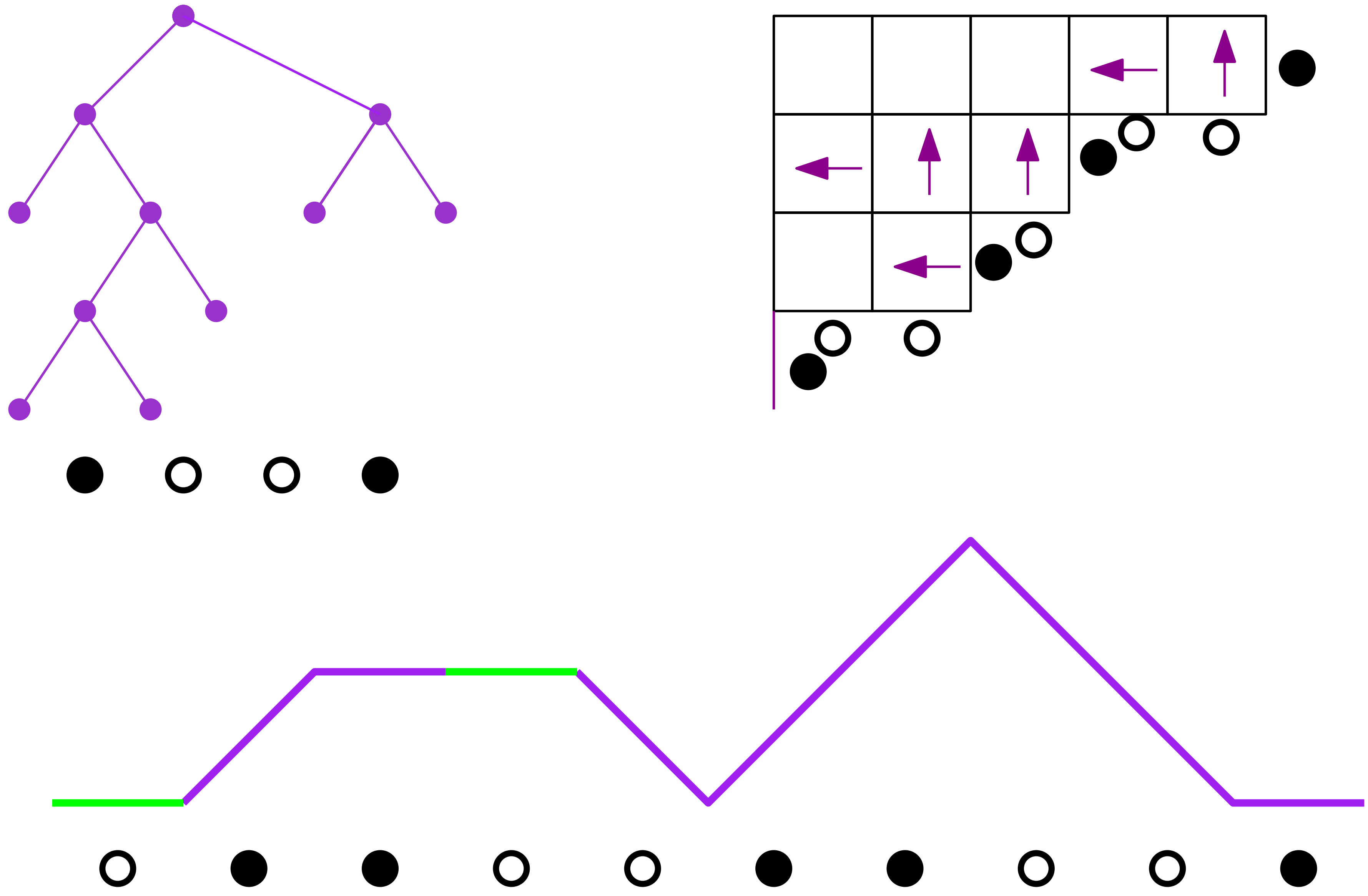




Binary trees, Paths and tableaux



Binary trees, Paths and tableaux



Binary trees, Paths and tableaux

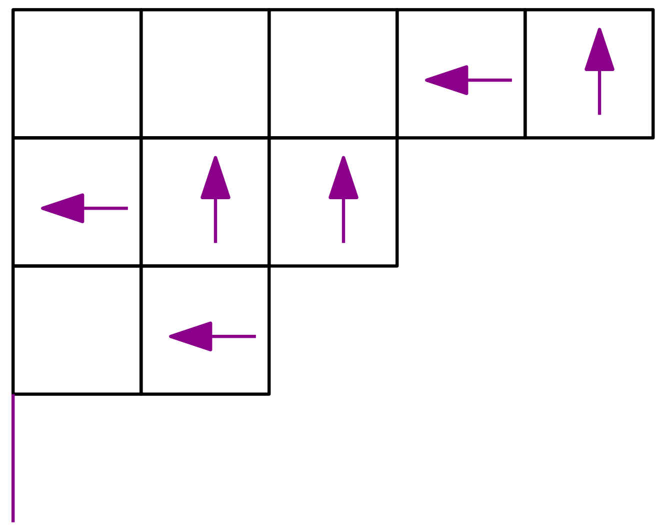
$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

Binary trees, Paths and tableaux



$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

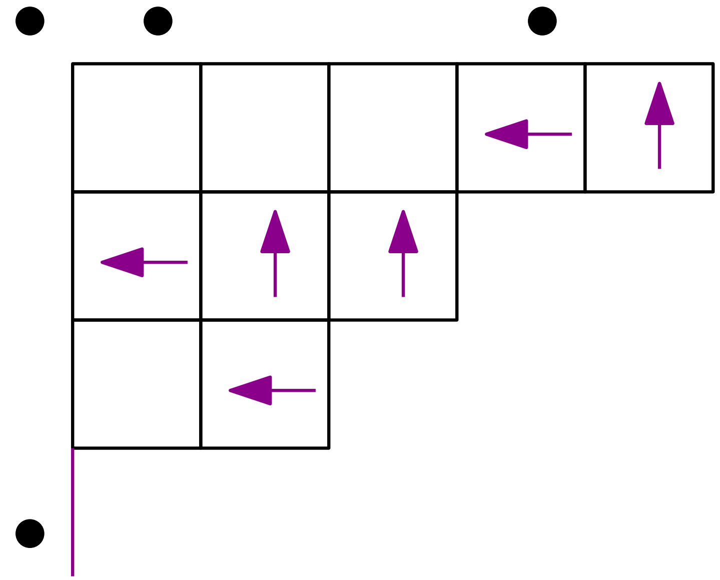
$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

$$B(\tau) = M(\tau) = C(\tau)$$

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Binary trees, Paths and tableaux



$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

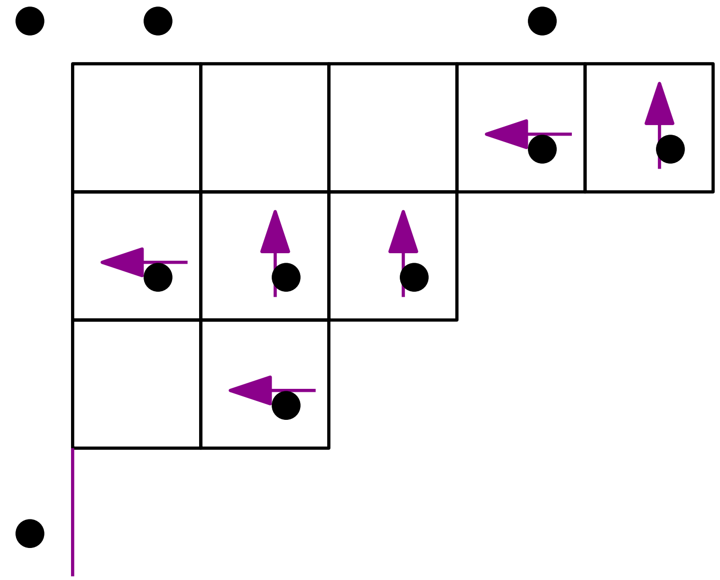
$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

$$B(\tau) = M(\tau) = C(\tau)$$

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Binary trees, Paths and tableaux



$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

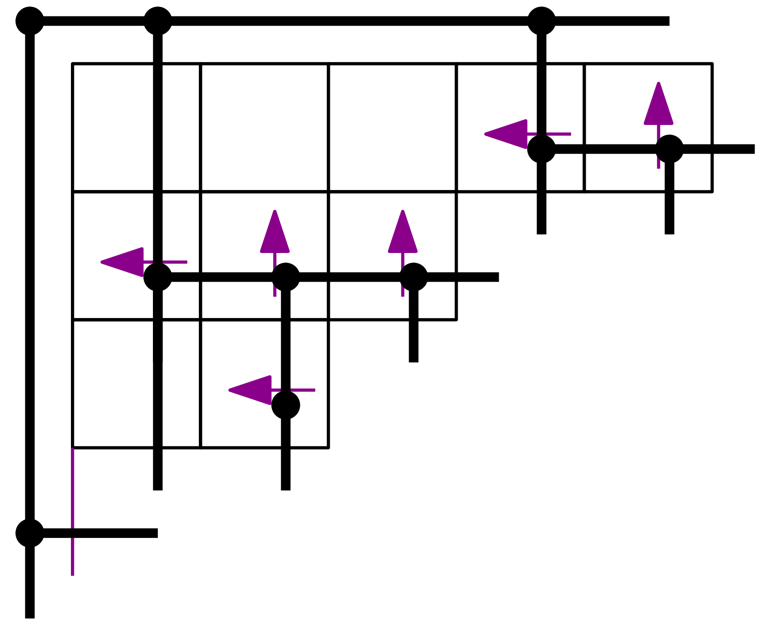
$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

$$B(\tau) = M(\tau) = C(\tau)$$

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Binary trees, Paths and tableaux



$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

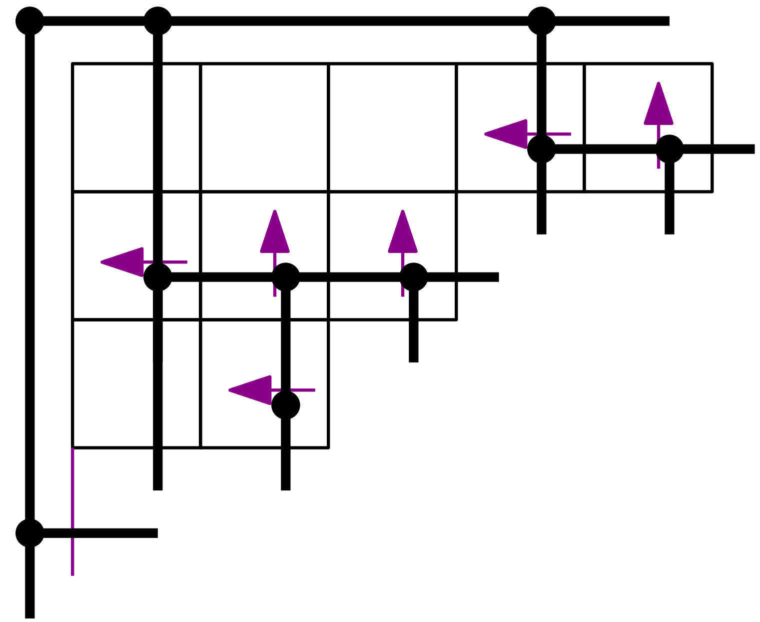
$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

$$B(\tau) = M(\tau) = C(\tau)$$

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Binary trees, Paths and tableaux



$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

$M(\tau)$ number of paths of shape τ

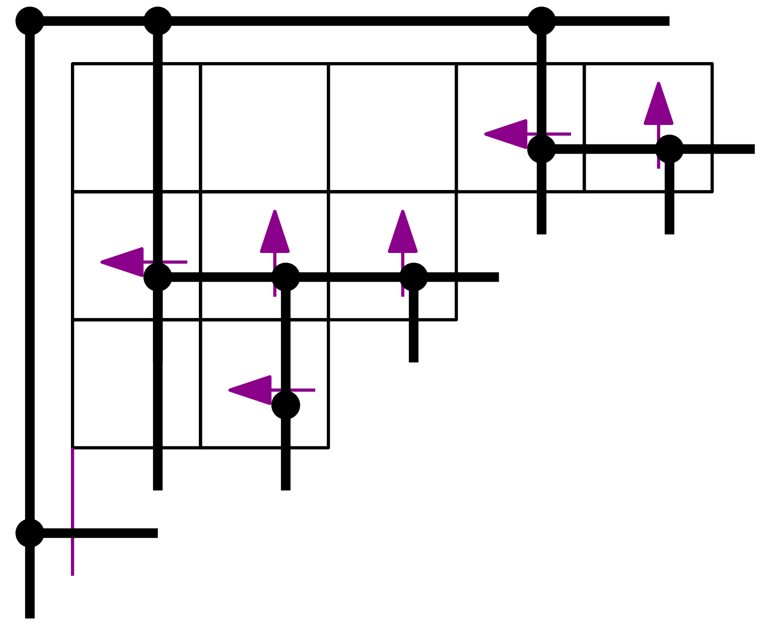
$C(\tau)$ number of tableaux of shape τ

$B(\tau)/C_{n+1}$ is the probability
to be in state τ of the TASEP with open
boundaries and N sites

$$B(\tau) = M(\tau) = C(\tau)$$

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Binary trees, Paths and tableaux



$\tau \in \{\circ, \bullet\}^N$

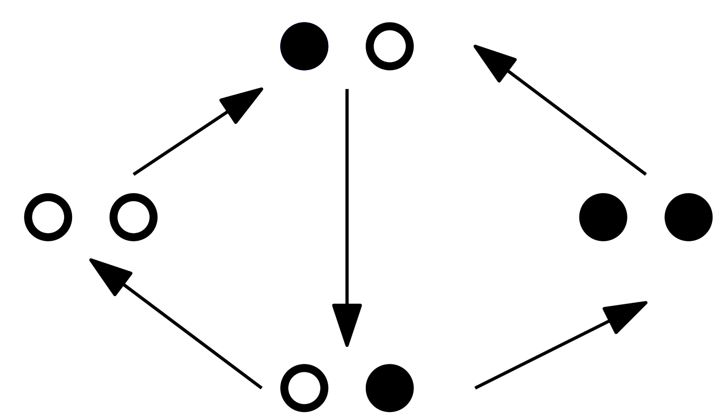
$B(\tau)$ number of trees of canopy τ

$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

$B(\tau)/C_{n+1}$ is the probability to be in state τ of the TASEP with open boundaries and N sites

$$B(\tau) = M(\tau) = C(\tau)$$



$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Matrix Ansatz [Derrida et al 93]

Matrices D and E, and vectors $\langle W|$ and $|V\rangle$

- $\langle W|E = \langle W|$
- $D|V\rangle = |V\rangle$
- $DE = D + E$

$$Z_N = \langle W|(D+E)^N|V\rangle.$$

Steady state $\tau \in \{\circ, \bullet\} = \{0,1\}^N$

$$P(\tau) = \frac{\langle W|\prod_{i=1}^N[\tau_i D + (1-\tau_i)E]|V\rangle}{Z_N}.$$

Matrix Ansatz [Derrida et al 93]

Matrices D and E, and vectors $\langle W|$ and $|V\rangle$

- $\langle W|E = \langle W|$
- $D|V\rangle = |V\rangle$
- $DE = D + E$

$$Z_N = \langle W|(D+E)^N|V\rangle.$$

Steady state $\tau \in \{\circ, \bullet\} = \{0,1\}^N$

$$P(\tau) = \frac{\langle W|\prod_{i=1}^N [\tau_i D + (1-\tau_i)E]|V\rangle}{Z_N}.$$

Solution: $\langle W| = (1, 0, \ldots)$, $|V\rangle = (1, 0, \ldots)^T$

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & & & & \vdots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & & & & \vdots \end{pmatrix}$$

Motzkin paths [Zeilberger, Duchi and Schaeffer, Brak and Essam]

Matrix Ansatz [Derrida et al 93]

Matrices D and E, and vectors $\langle W|$ and $|V\rangle$

- $\langle W|E = \langle W|$
- $D|V\rangle = |V\rangle$
- $DE = D + E$

$$Z_N = \langle W|(D+E)^N|V\rangle.$$

Steady state $\tau \in \{\circ, \bullet\} = \{0,1\}^N$

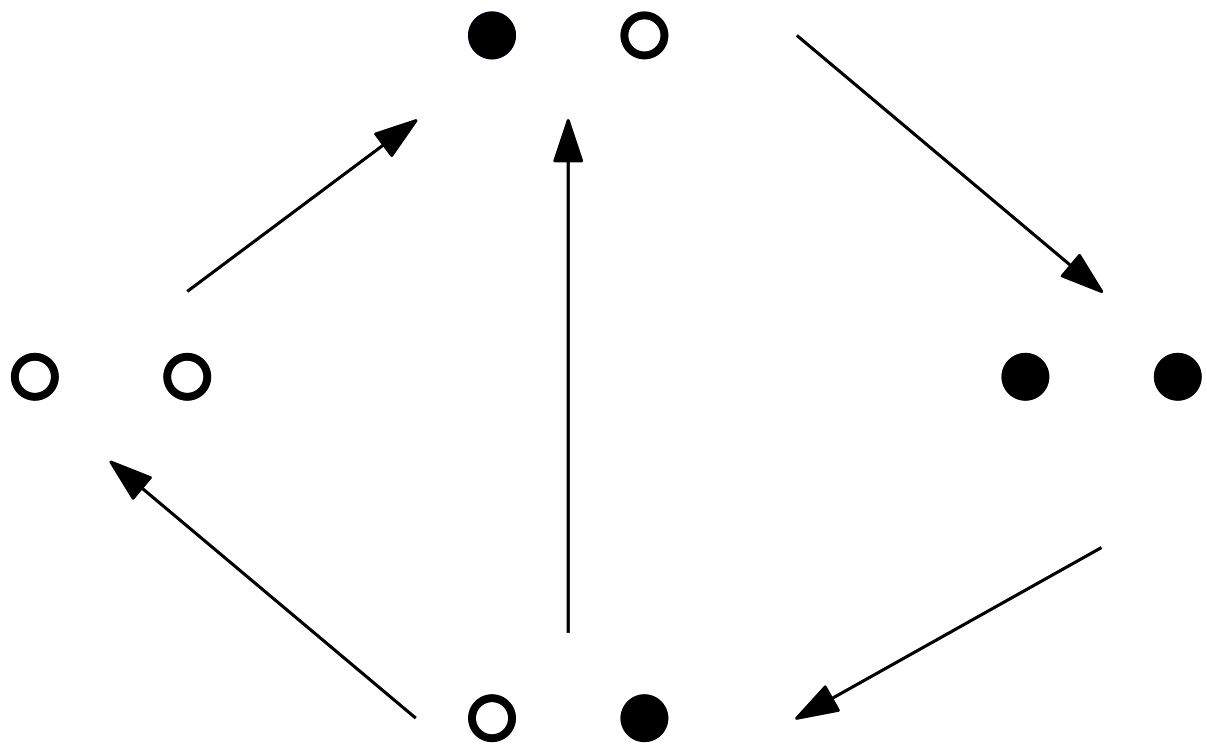
$$P(\tau) = \frac{\langle W| \prod_{i=1}^{N} [\tau_i D + (1-\tau_i)E] |V\rangle}{Z_N}.$$

Solution: $\langle W| = (1, 0, \ldots)$, $|V\rangle = (1, 1, \ldots)^T$

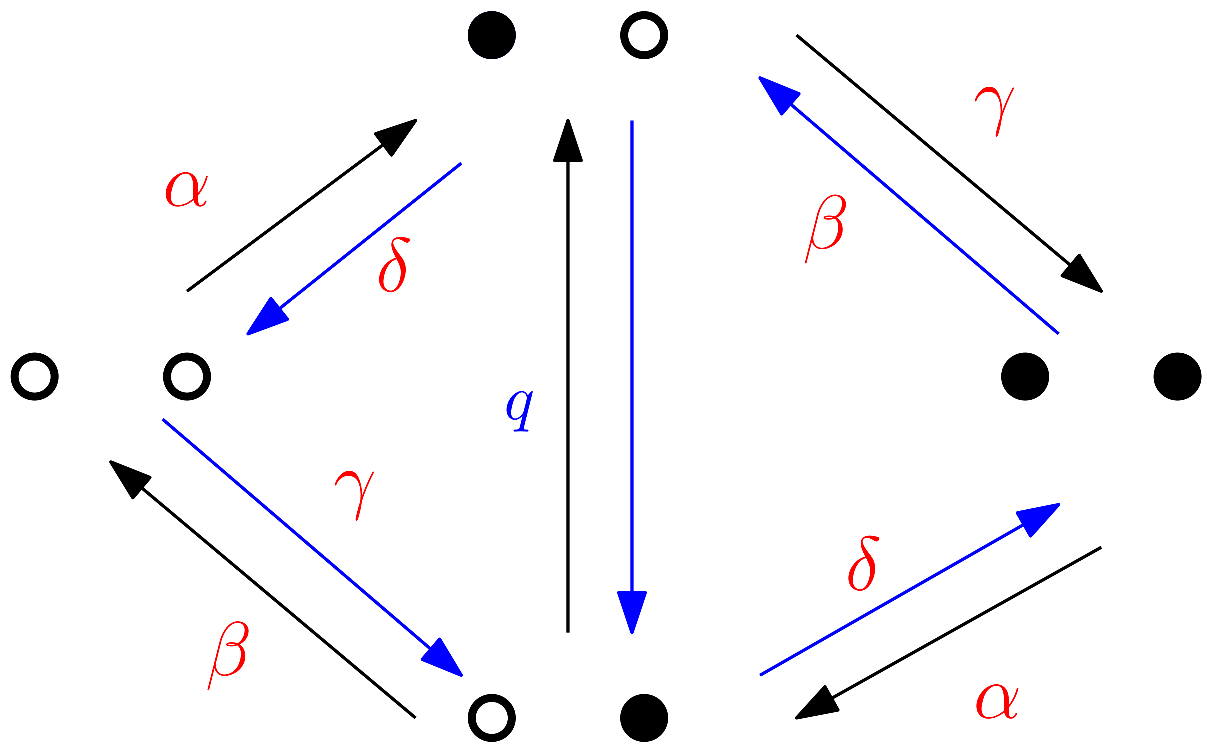
$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & & & & \vdots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ \vdots & & & & \vdots \end{pmatrix}$$

Lukasiewicz paths, Catalan tableaux

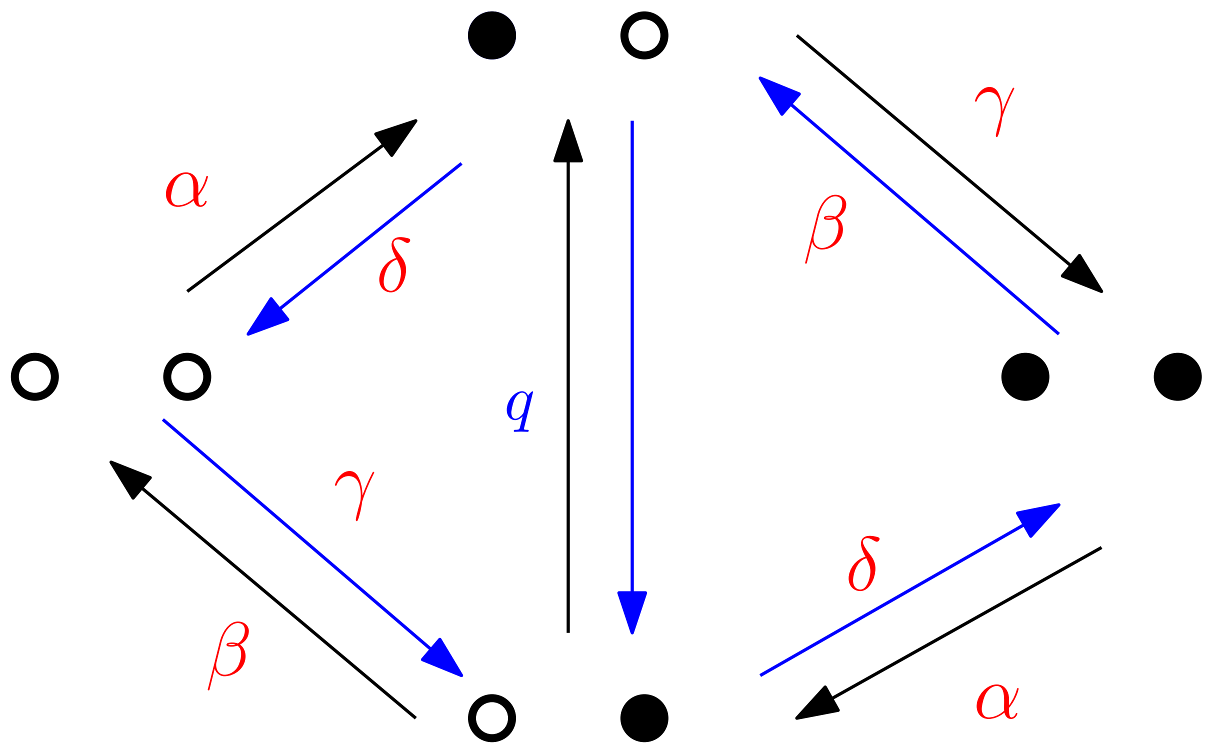
Asymmetric exclusion process with 5 parameters



Asymmetric exclusion process with 5 parameters



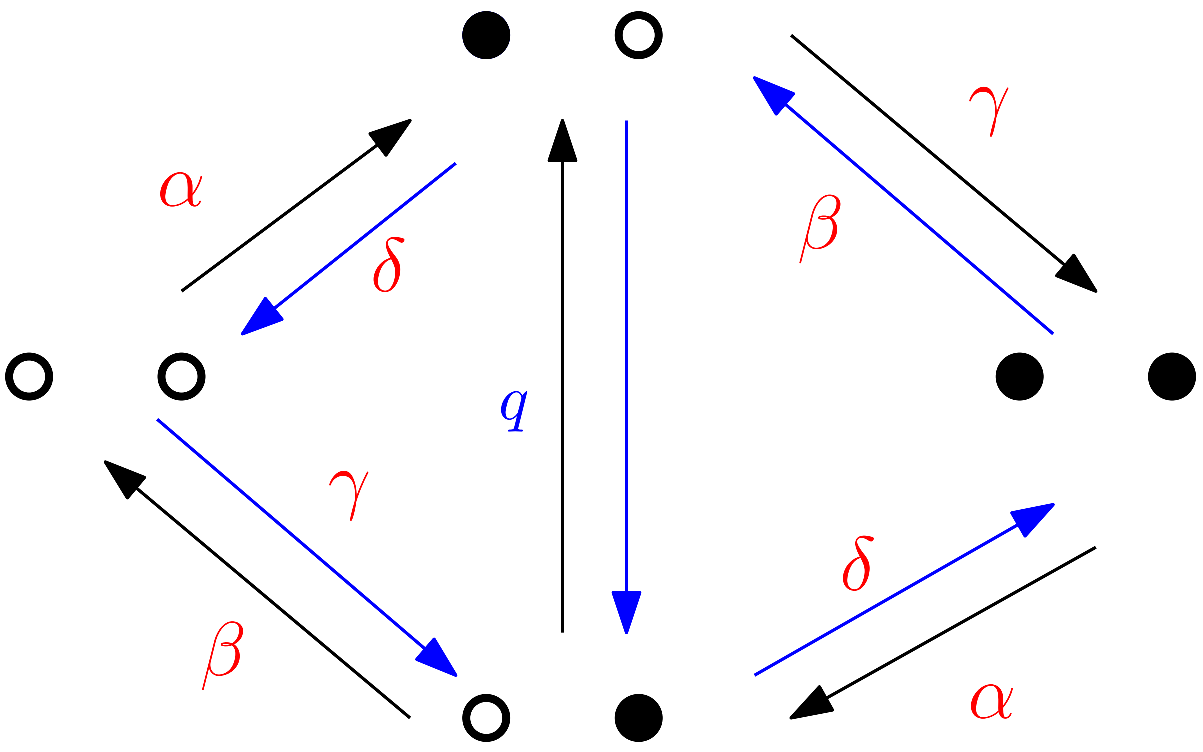
Asymmetric exclusion process with 5 parameters



Matrix Ansatz

- $\langle W|(\alpha E - \gamma D) = \langle W|$
- $(\beta D - \delta E)|V\rangle = |V\rangle$
- $DE = qED + D + E$

Asymmetric exclusion process with 5 parameters



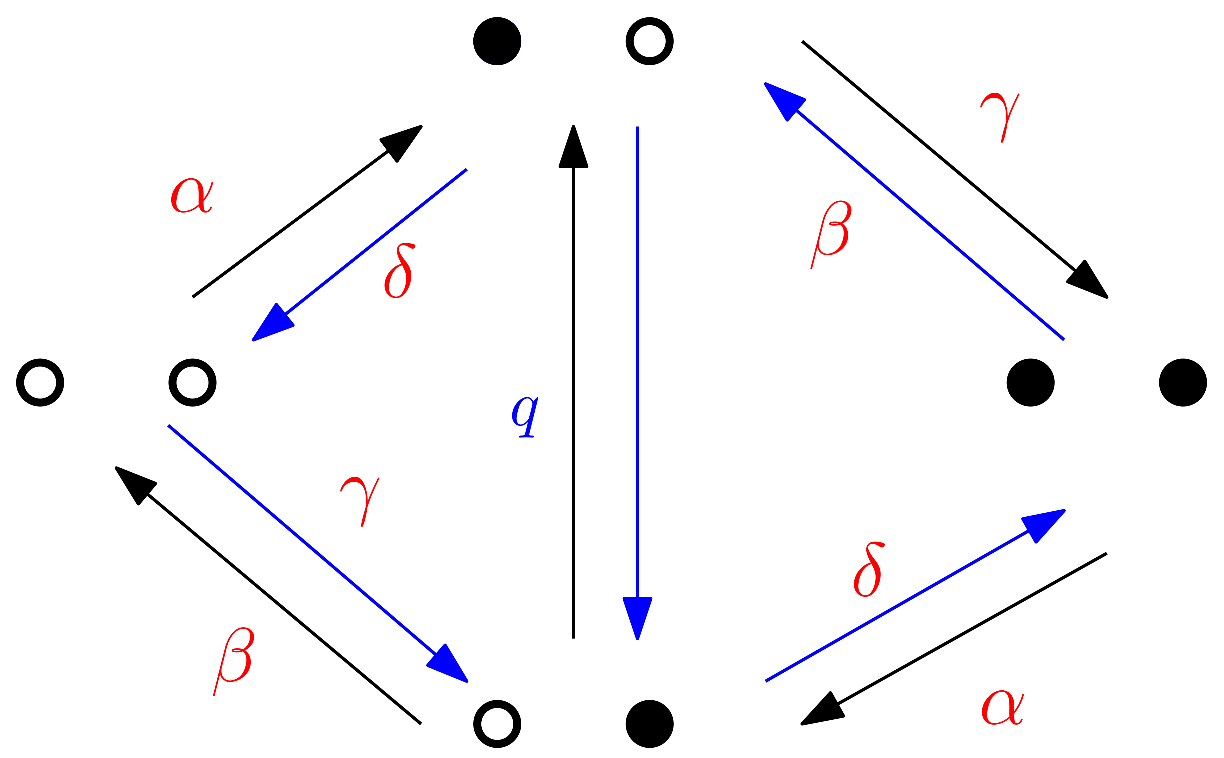
Matrix Ansatz

- $\langle W|(\alpha E - \gamma D) = \langle W|$
- $(\beta D - \delta E)|V\rangle = |V\rangle$
- $DE = qED + D + E$

$\gamma = \delta = 0$

- Trees $\Rightarrow$ tree like tableaux
- Paths $\Rightarrow$ moments of AlSalam-Chihara Polynomials
- Tableaux $\Rightarrow$ Permutation tableaux, Alternative tableaux

Asymmetric exclusion process with 5 parameters



Matrix Ansatz

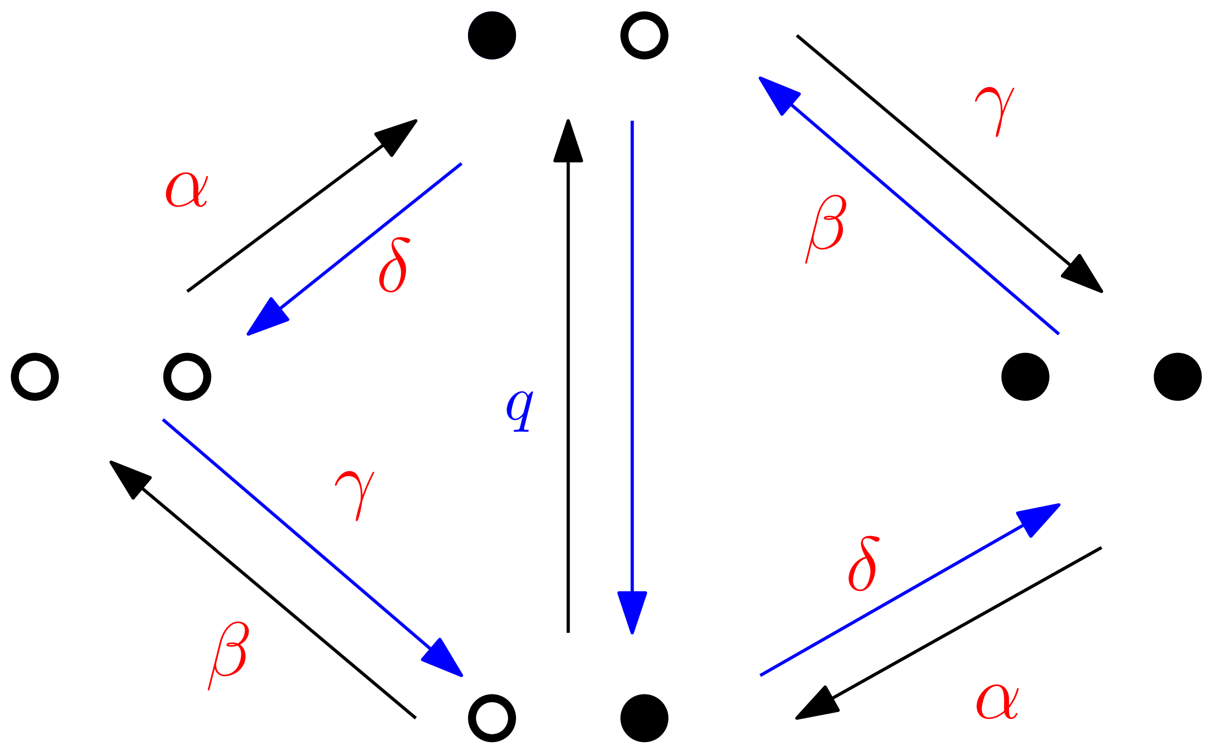
- $\langle W|(\alpha E - \gamma D) = \langle W|$
- $(\beta D - \delta E)|V\rangle = |V\rangle$
- $DE = qED + D + E$

$\gamma = \delta = 0$

- Trees ⇒ tree like tableaux
- Paths ⇒ moments of AlSalam-Chihara Polynomials
- Tableaux ⇒ Permutation tableaux, Alternative tableaux

[Aval, Boussicault, C. Josuat-Vergès, Nadeau, Viennot, Williams. . .]

Asymmetric exclusion process with 5 parameters



General model

- Moments of Askey Wilson polynomials [Uchiyama, Sasamoto, Wadati 04]
- Staircase tableaux [C., Williams 10]

Askey Wilson polynomials

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

$$b_n = 1/2(a + 1/a - A_n - C_n) \qquad \lambda_n = A_{n-1}C_n/4$$

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n})(1-abcdq^{2n-1})}$$

symmetric in a, b, c, d $\qquad C_n = \frac{(1-abq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-q^n)}{a(1-abcdq^{2n-2})(1-abcdq^{2n-1})}$

Askey Wilson polynomials

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

$$b_n = 1/2(a + 1/a - A_n - C_n) \qquad \lambda_n = A_{n-1}C_n/4$$

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n})(1-abcdq^{2n-1})}$$

symmetric in a, b, c, d

$$C_n = \frac{(1-abq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-q^n)}{a(1-abcdq^{2n-2})(1-abcdq^{2n-1})}$$

orthogonal

$$\oint_C \frac{dz}{4\pi iz} w\left(\frac{z+z^{-1}}{2}\right) P_m\left(\frac{z+z^{-1}}{2}\right) P_n\left(\frac{z+z^{-1}}{2}\right) = h_n\delta_{mn},$$

$$w(x) = \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty},\ x = (z + z^{-1})/2$$

$$h_n = \frac{(1-q^{n-1}abcd)(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-q^{2n-1}abcd)(abcd;q)_n}$$

Askey Wilson polynomials

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

$$b_n = 1/2(a + 1/a - A_n - C_n) \qquad \lambda_n = A_{n-1}C_n/4$$

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n})(1-abcdq^{2n-1})}$$

symmetric in a, b, c, d $\qquad C_n = \frac{(1-abq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-q^n)}{a(1-abcdq^{2n-2})(1-abcdq^{2n-1})}$

orthogonal

$$\oint_C \frac{dz}{4\pi iz} w\left(\frac{z+z^{-1}}{2}\right) P_m\left(\frac{z+z^{-1}}{2}\right) P_n\left(\frac{z+z^{-1}}{2}\right) = h_n \delta_{mn},$$

$$w(x) = \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}, \; x = (z + z^{-1})/2$$

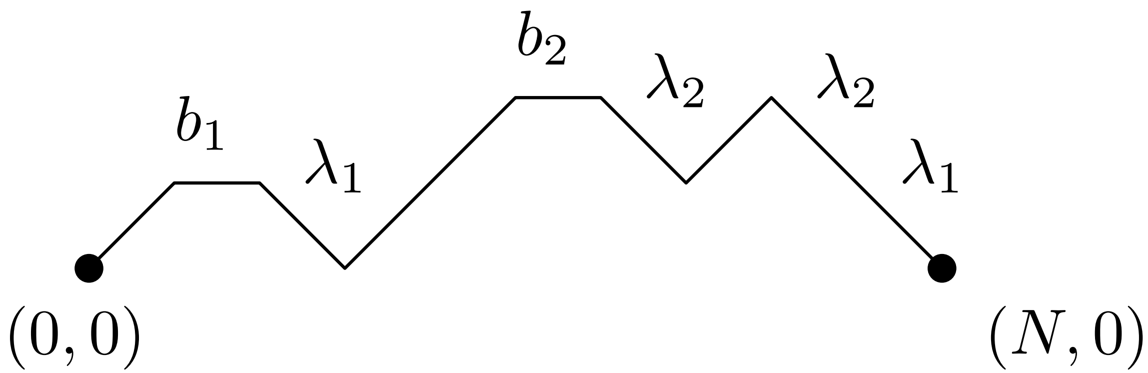
$$h_n = \frac{(1-q^{n-1}abcd)(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-q^{2n-1}abcd)(abcd;q)_n}$$

Moments $\qquad \mu_N^{AW} = \oint_C \frac{dz}{4\pi iz} w\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^N$

Combinatorics of moments

[Flajolet, Viennot 80s]

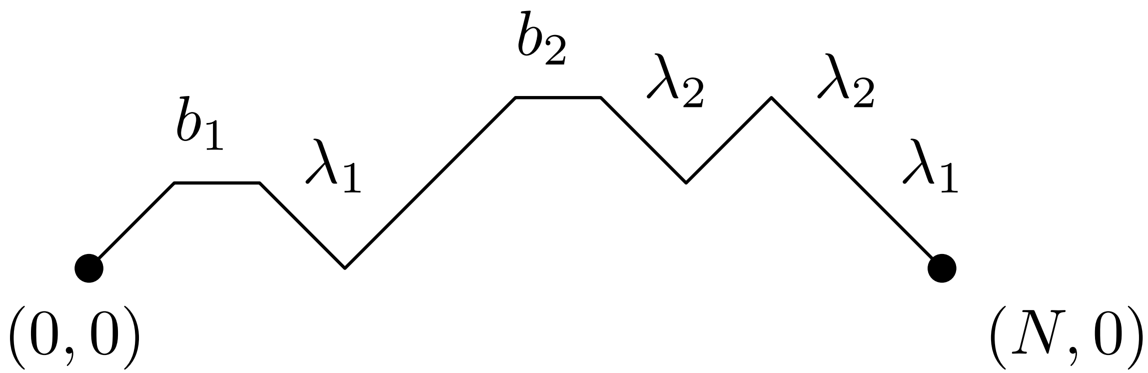
$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$



Combinatorics of moments

[Flajolet, Viennot 80s]

$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$



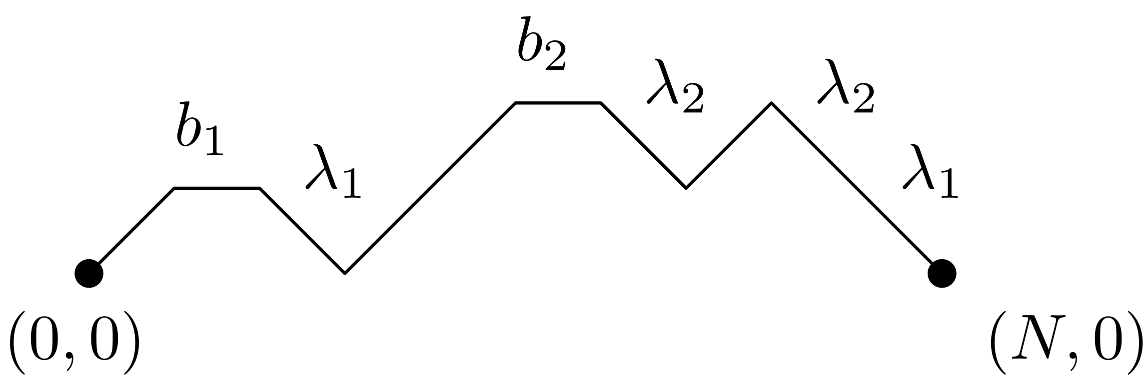
$\mu_N = \sum_P W(p)$

$W(P) = b_1 b_2 \lambda_1 \lambda_2^2$

Combinatorics of moments

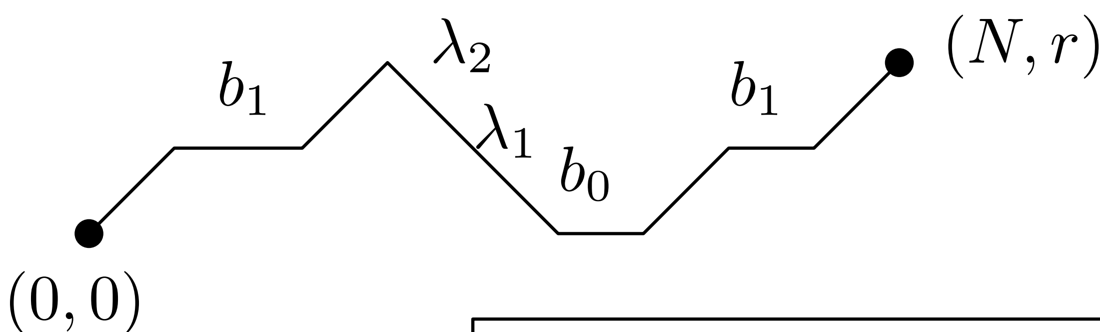
[Flajolet, Viennot 80s]

$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$



$\mu_N = \sum_P W(p)$

$W(P) = b_1 b_2 \lambda_1 \lambda_2^2$



$$\mu_{N,r} = \oint_C \frac{dz}{4\pi i z} w\left(\frac{z+z^{-1}}{2}\right) P_r\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^N$$

Solution of the 5 parameter model [USW 04]

$$\mathsf{d} = \begin{pmatrix} d_0^\natural & d_0^\sharp & 0 & \cdots \\ d_0^\flat & d_1^\natural & d_1^\sharp & \\ 0 & d_1^\flat & d_2^\natural & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$d_n^\flat = -\frac{q^n bd}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad e_n^\flat = \frac{1}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad d_n^\sharp = 1 \;\; e_n^\sharp = -q^n ac$$

$$\langle W| = (1, 0, \ldots),\; |V\rangle = (1, 0, \ldots)^T$$

Solution of the 5 parameter model [USW 04]

$$\mathsf{d} = \begin{pmatrix} d_0^\natural & d_0^\sharp & 0 & \cdots \\ d_0^\flat & d_1^\natural & d_1^\sharp & \\ 0 & d_1^\flat & d_2^\natural & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$d_n^\natural + e_n^\natural = b_n$$

$$d_n^\flat = -\frac{q^n bd}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad e_n^\flat = \frac{1}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad d_n^\sharp = 1 \;\; e_n^\sharp = -q^n ac$$

$$\langle W| = (1,0,\ldots),\ |V\rangle = (1,0,\ldots)^T$$

$$\boxed{\mu_N^{\mathrm{AW}} = \langle W|(\mathsf{d}+\mathsf{e})^N|V\rangle}$$

Solution of the 5 parameter model [USW 04]

$$\mathsf{d} = \begin{pmatrix} d_0^\natural & d_0^\sharp & 0 & \cdots \\ d_0^\flat & d_1^\natural & d_1^\sharp & \\ 0 & d_1^\flat & d_2^\natural & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$d_n^\natural + e_n^\natural = b_n$$

$$d_n^\flat = -\frac{q^n bd}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad e_n^\flat = \frac{1}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad d_n^\sharp = 1 \ \ e_n^\sharp = -q^n ac$$

$$\langle W| = (1,0,\ldots),\ |V\rangle = (1,0,\ldots)^T$$

$$\boxed{\mu_N^{\mathrm{AW}} = \langle W|(\mathsf{d}+\mathsf{e})^N|V\rangle}$$

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha} \ \ b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}$$

$$D = \frac{1+\mathsf{d}}{1-q},\ E = \frac{1+\mathsf{e}}{1-q}$$

$$\boxed{Z_N = \langle W|(D+E)^N|V\rangle}$$

Koorwinder polynomials

Multivariate version of the AW polynomials $P_\lambda(z_1, \ldots, z_m; a, b, c, d|q, t)$

at $q = t$

$$P_\lambda(\mathrm{z}; a, b, c, d|q, q) = \mathrm{const} \cdot \frac{\det(p_{m-j+\lambda_j}(z_i;a,b,c,d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i;a,b,c,d|q))_{i,j=1}^m}$$

Density

$$\prod_{1\le i<j\le m}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/z_iz_j)\prod_{1\le i\le m} w\left(\frac{z_i+z_i^{-1}}{2}\right)$$

Koorwinder polynomials

Multivariate version of the AW polynomials $P_\lambda(z_1,\dots,z_m;a,b,c,d|q,t)$

AW-polynomials

at $q=t$

$$P_\lambda(\mathrm{z};a,b,c,d|q,q) = \mathrm{const}\cdot\frac{\det(p_{m-j+\lambda_j}(z_i;a,b,c,d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i;a,b,c,d|q))_{i,j=1}^m}$$

Density

$$\prod_{1\le i<j\le m}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/z_iz_j)\prod_{1\le i\le m} w\left(\frac{z_i+z_i^{-1}}{2}\right)$$

AW-density

Possible definition of moments

$$M_\lambda = I_k(s_\lambda(x_1,\dots,x_m);a,b,c,d;q,q).$$

Koorwinder polynomials

Multivariate version of the AW polynomials $P_\lambda(z_1, \dots, z_m; a, b, c, d|q, t)$

AW-polynomials

at $q = t$

$$P_\lambda(\mathrm{z}; a, b, c, d|q, q) = \mathrm{const} \cdot \frac{\det(p_{m-j+\lambda_j}(z_i;a,b,c,d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i;a,b,c,d|q))_{i,j=1}^m}$$

Density

$$\prod_{1\le i<j\le m}(1 - z_i z_j)(1 - z_i/z_j)(1 - z_j/z_i)(1 - 1/z_i z_j) \prod_{1\le i\le m} w\left(\frac{z_i + z_i^{-1}}{2}\right)$$

AW-density

Possible definition of moments

$$M_\lambda = I_k(s_\lambda(x_1, \dots, x_m); a, b, c, d; q, q).$$

Schur functions

Integrate with respect to the Koorwinder density

Koorwinder polynomials

Multivariate version of the AW polynomials $P_\lambda(z_1,\dots,z_m;a,b,c,d|q,t)$

at $q=t$

AW-polynomials

$$P_\lambda(\mathrm{z};a,b,c,d|q,q) = \mathrm{const}\cdot\frac{\det(p_{m-j+\lambda_j}(z_i;a,b,c,d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i;a,b,c,d|q))_{i,j=1}^m}$$

Density

$$\prod_{1\le i<j\le m}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/z_iz_j)\prod_{1\le i\le m} w\left(\frac{z_i+z_i^{-1}}{2}\right)$$

AW-density

Possible definition of moments

$$M_\lambda = I_k(s_\lambda(x_1,\dots,x_m);a,b,c,d;q,q).$$

Lemma
Rains

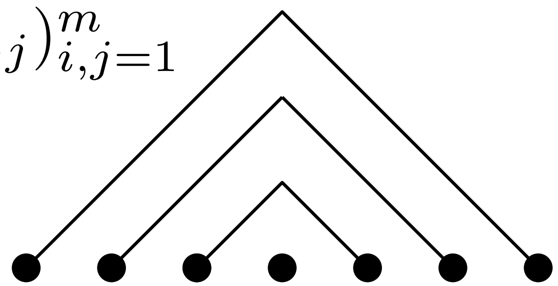
$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Koorwinder moments

$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Path interpretation

$\det(\mu_{2m-i-j})_{i,j=1}^m$



Lindström, Gessel, Viennot

$\prod_{i=1}^m \lambda_i^{m-i}$

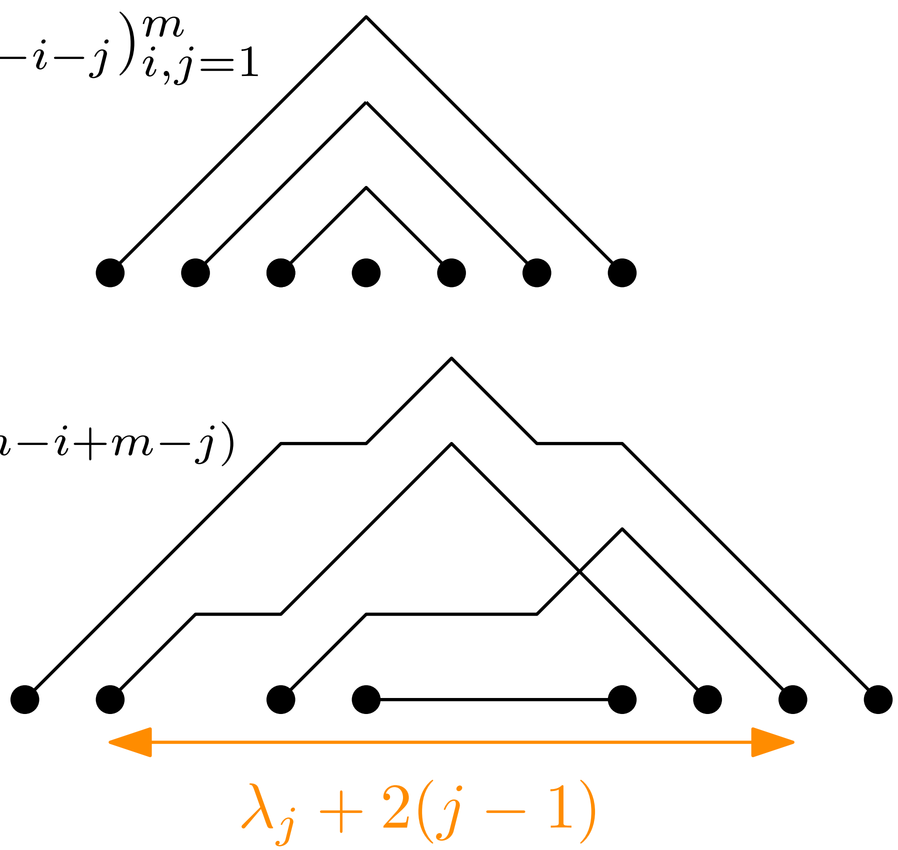
Koorwinder moments

$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Path interpretation

$\det(\mu_{2m-i-j})_{i,j=1}^m$

$\det(\mu_{\lambda_i+m-i+m-j})$



Lindström, Gessel, Viennot

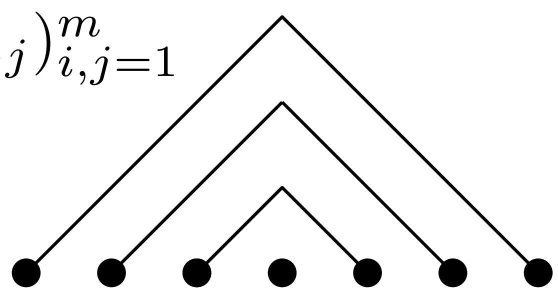
$\prod_{i=1}^m \lambda_i^{m-i}$

Koorwinder moments

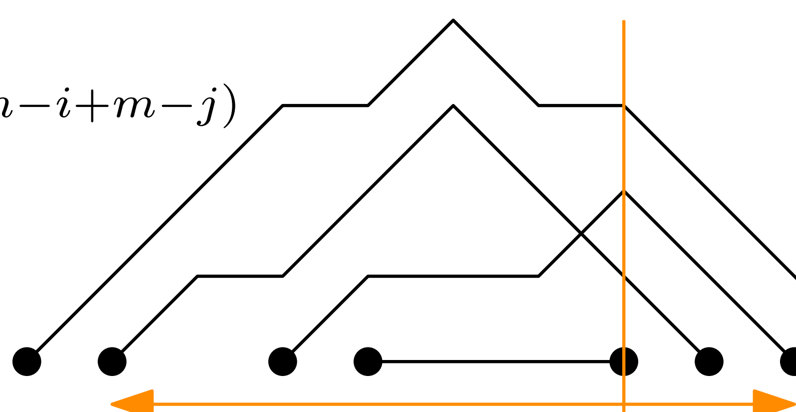
$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Path interpretation

$\det(\mu_{2m-i-j})_{i,j=1}^m$

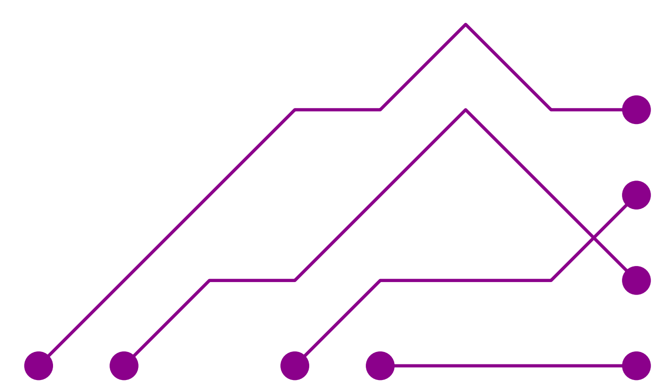


$\det(\mu_{\lambda_i+m-i+m-j})$



Lindström, Gessel, Viennot

$\prod_{i=1}^m \lambda_i^{m-i}$

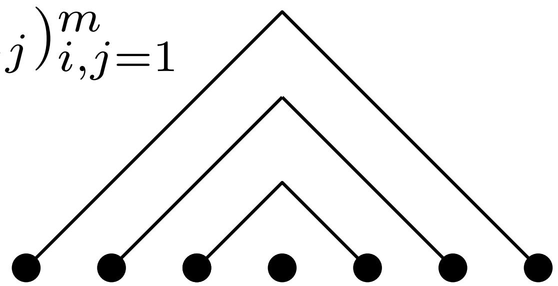


Koorwinder moments

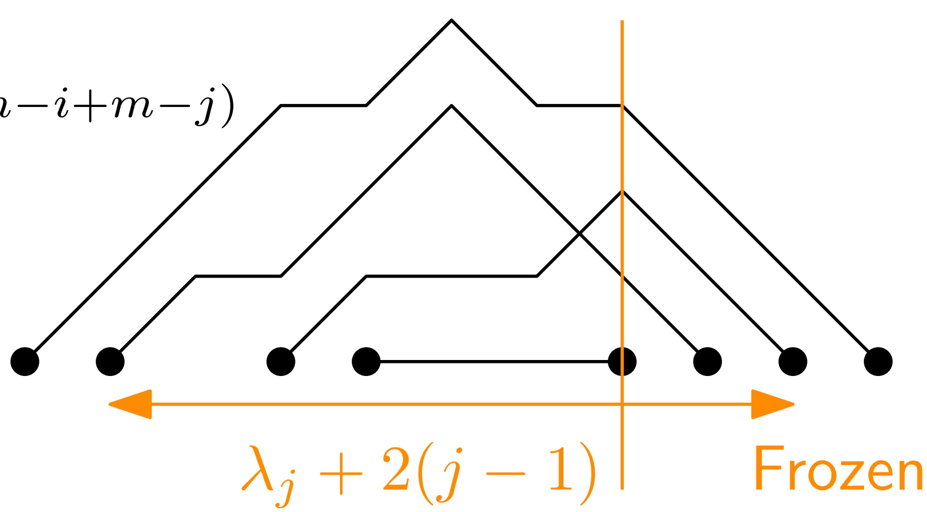
$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Path interpretation

$\det(\mu_{2m-i-j})_{i,j=1}^m$



$\det(\mu_{\lambda_i+m-i+m-j})$



Lindström, Gessel, Viennot

$\prod_{i=1}^m \lambda_i^{m-i}$

$$\boxed{M_\lambda = \det(\mu_{\lambda_i+n-i+m-j,j})}$$

More Koornwinder moments

$$\lambda_1 \geq \ldots \geq \lambda_m \geq 0$$

$$K_\lambda = \frac{\det(Z_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(Z_{2m-i-j})_{i,j=1}^m}$$

$$K_\lambda = \det(K_{(\lambda_i+j-i,0,0,\ldots,0)})_{i,j=1}^n$$

Conjecture [C., Rains, Williams 14]

The Koornwinder moment K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients (up to a normalizing factor).

More Koornwinder moments

$\lambda_1 \geq \ldots \geq \lambda_m \geq 0$

$$K_\lambda = \frac{\det(Z_{\lambda_i+m-i+m-j})_{i,j=1}^{m}}{\det(Z_{2m-i-j})_{i,j=1}^{m}}$$

$$K_\lambda = \det(K_{(\lambda_i+j-i,0,0,\ldots,0)})_{i,j=1}^{n}$$

Conjecture [C., Rains, Williams 14]

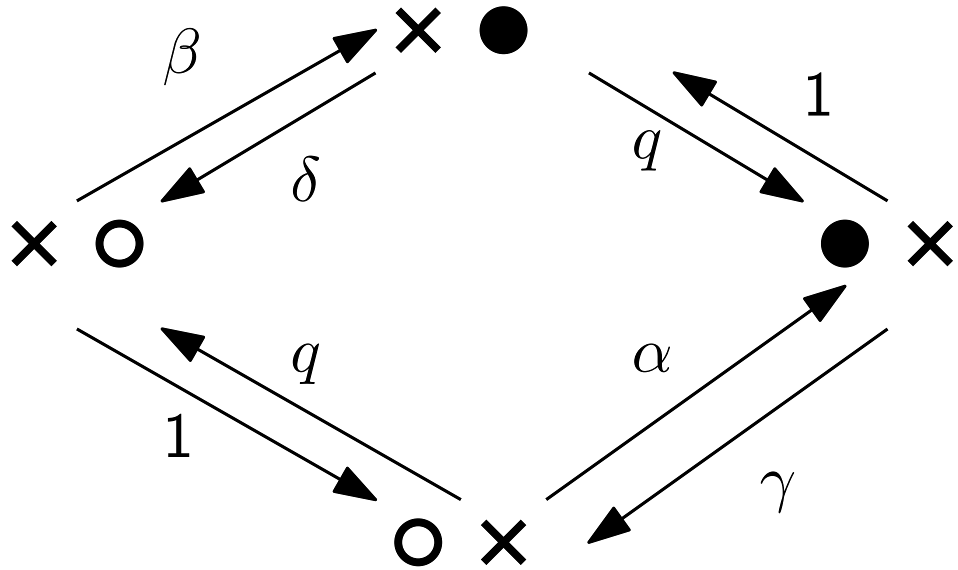
The Koornwinder moment K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients (up to a normalizing factor).

True for $\lambda = (N-r, \underbrace{0, \ldots, 0}_{r})$

Theorem [C., Williams 15; Cantini 15]

$K_{(N-r.0,\ldots,0)}$ Partition function of the two species ASEP

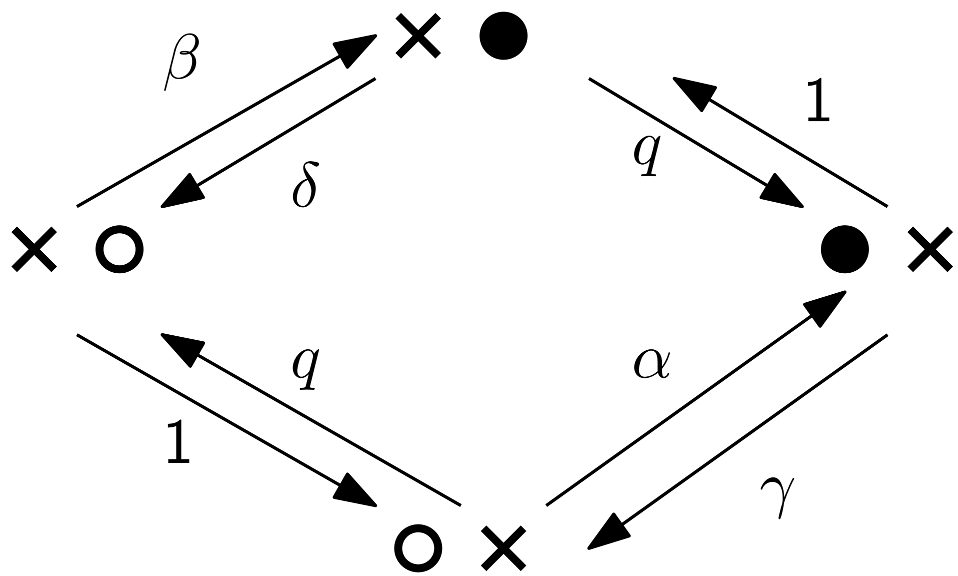
Two species ASEP



N sites

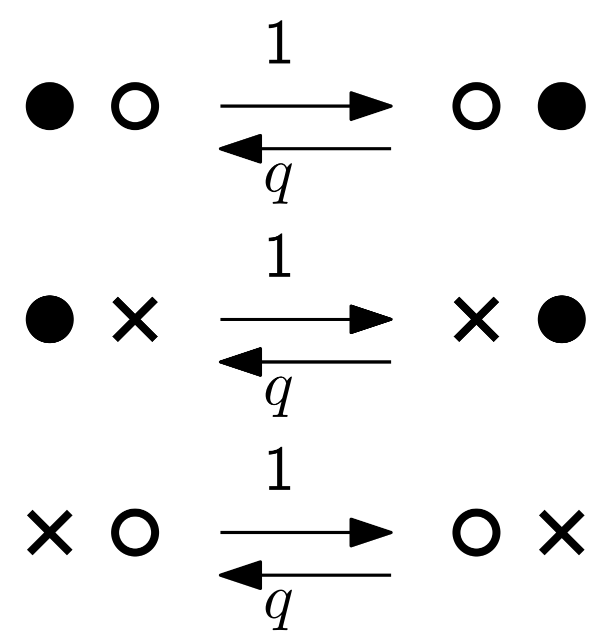
r particles equal to $\times$

Two species ASEP

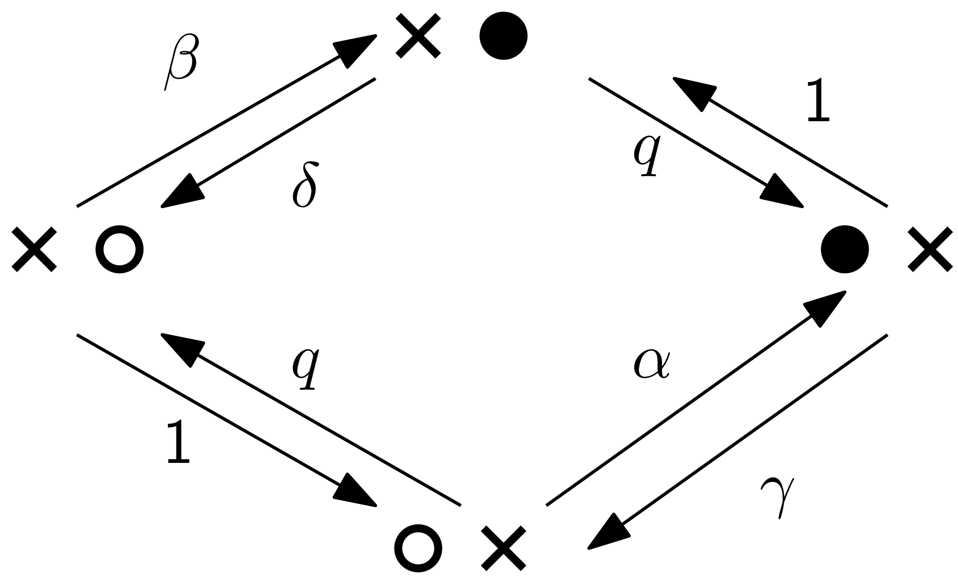


N sites

r particles equal to $\times$



Two species ASEP



N sites

r particles equal to $\times$

Matrix Ansatz [Uchiyama 08]

- $\langle W|(\alpha E - \gamma D) = \langle W|$
- $(\beta D - \delta E)|V\rangle = |V\rangle$
- $DE - qED = D + E$
- $DA = qAD + A$
- $AE = qEA + A.$

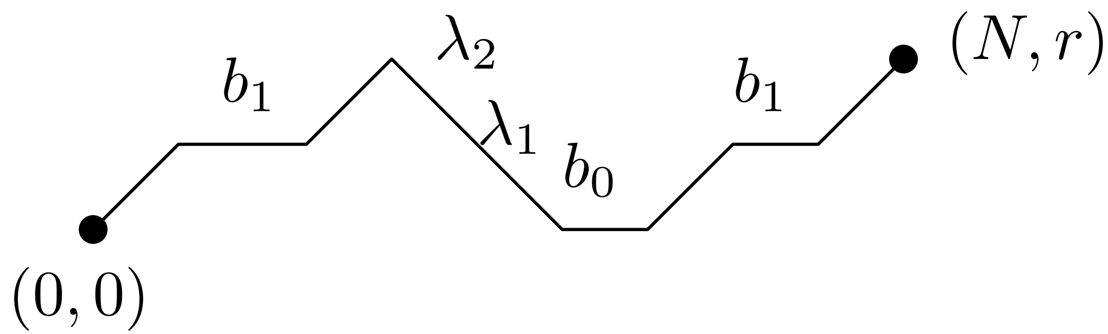
Partition function

$$Z_{N,r} = [y^r]\frac{\langle W|(D+E+yA)^N|V\rangle}{\langle W|A^r|V\rangle}.$$

Two species ASEP

$$Z_{N,r} = [y^r]\frac{\langle W|(D+E+yA)^N|V\rangle}{\langle W|A^r|V\rangle}.$$

$$K_{(N-r.0,\dots,0)} = \langle W|(D+E)^N|V^r\rangle \qquad |V^r\rangle = (0,\dots,0,1,0,\dots)^T$$



$$K_{(N-r,0\dots,0)} = \mu_{N,r} = \oint_C \frac{dz}{4\pi i z} w\left(\frac{z+z^{-1}}{2}\right) P_r\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^N$$

Theorem. $Z_{N,r} = \frac{\alpha^r(1-q)^r}{\alpha+q^i\gamma} \times K_{(n-r,0,\dots,0)}$

Sketch of proof

Lemma. The theorem is true if $\langle W|D^N|V^r\rangle \alpha^r(1-q)^r = [y^r]\frac{\langle W|(D+yA)^N|V\rangle}{\langle W|A^r|V\rangle}$.

Sketch of proof

Lemma. The theorem is true if $\langle W|D^N|V^r\rangle\alpha^r(1-q)^r = [y^r]\frac{\langle W|(D+yA)^N|V\rangle}{\langle W|A^r|V\rangle}$.

Proof. Matrix Ansatz

Sketch of proof

Lemma. The theorem is true if $\langle W|D^N|V^r\rangle \alpha^r(1-q)^r = [y^r]\frac{\langle W|(D+yA)^N|V\rangle}{\langle W|A^r|V\rangle}$.

Proof. Matrix Ansatz

$D = (1+\mathsf{d})/(1-q)$

Lemma. The theorem is true if $\langle W|\mathsf{d}^N|V^r\rangle = \begin{bmatrix} N \\ r \end{bmatrix}_q \frac{\langle W|A^r\mathsf{d}^{N-r}|V\rangle}{\langle W|A^r|V\rangle}$.

Proof. Matrix Ansatz

Sketch of proof

Lemma. The theorem is true if $\langle W|D^N|V^r\rangle \alpha^r (1-q)^r = [y^r]\frac{\langle W|(D+yA)^N|V\rangle}{\langle W|A^r|V\rangle}$.

Proof. Matrix Ansatz

$D = (1+\mathsf{d})/(1-q)$

Lemma. The theorem is true if $\langle W|\mathsf{d}^N|V^r\rangle = \begin{bmatrix} N \\ r \end{bmatrix}_q \frac{\langle W|A^r\mathsf{d}^{N-r}|V\rangle}{\langle W|A^r|V\rangle}$.

Proof. Matrix Ansatz

"Guess and check"

Proposition

$$\frac{\langle W|A^r\mathsf{d}^{N-r}|V\rangle}{\langle W|A^r|V\rangle} = \frac{\sum_{i=0}^{N-r}(-1)^i \begin{bmatrix} N-r \\ i \end{bmatrix}_q q^{\binom{i}{2}} (bdq^r)^i B_{N-r-i}(b,d,q) B_i(a,c,1/q)}{\prod_{i=0}^{N-r-1}(1-abcdq^{2r+i})}$$

$$B_m(b,d,q) = \left(\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q b^j d^{m-j} \right)$$

Enumeration formula

Theorem. [Stanton 15]

$$Z_{N,r} = \sum_{k=0}^{N} \sum_{j=0}^{k} F_{k,r} q^k \frac{q^{-j^2} a^{-2j}}{(q, q^{1-2j}/a^2; q)_j (q, a^2 q^{1+2j}; q)_{k-j}} (1 + aq^j + 1/(aq^j))^N / 2^N$$

$$F_{k,r} = (-a)^r \begin{bmatrix} k \\ r \end{bmatrix}_q \frac{(abq^r, acq^r, adq^r, q)_{k-r}}{(abcdq^{2r}, q)_{k-r}} \frac{(q;q)_r}{(abcd;q)_{2r}} (ab, ac, ad, bc, bd, cd; q)_r q^{\binom{r}{2}}$$

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha},\ b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}$$

Enumeration formula

Theorem. [Stanton 15]

$$Z_{N,r} = \sum_{k=0}^{N} \sum_{j=0}^{k} F_{k,r} q^k \frac{q^{-j^2} a^{-2j}}{(q, q^{1-2j}/a^2; q)_j (q, a^2 q^{1+2j}; q)_{k-j}} (1 + aq^j + 1/(aq^j))^N / 2^N$$

$$F_{k,r} = (-a)^r \begin{bmatrix} k \\ r \end{bmatrix}_q \frac{(abq^r, acq^r, adq^r, q)_{k-r}}{(abcdq^{2r}, q)_{k-r}} \frac{(q;q)_r}{(abcd;q)_{2r}} (ab, ac, ad, bc, bd, cd; q)_r q^{\binom{r}{2}}$$

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha},\ b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}$$

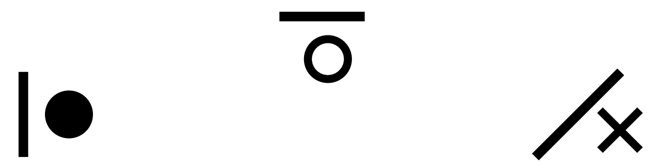
Remark. $Z_{N,r}$ is a polynomial with positive coefficients in $\alpha, \beta, \gamma, \delta$ and q with $4^{N-r}(n-r)!\binom{n}{r}^2$ terms

Can we extract the combinatorics of the two species ASEP?

Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

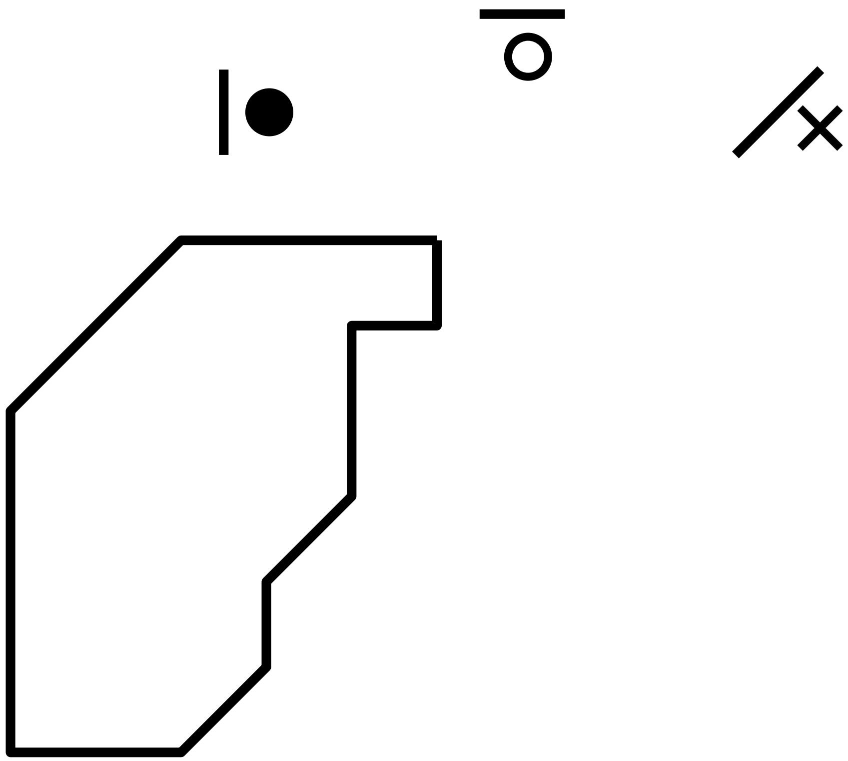
$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

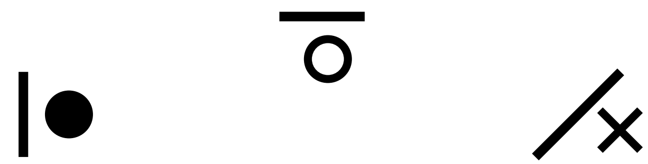
$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]

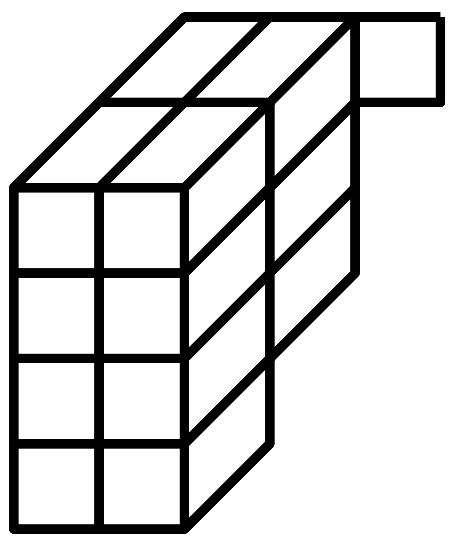


Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]

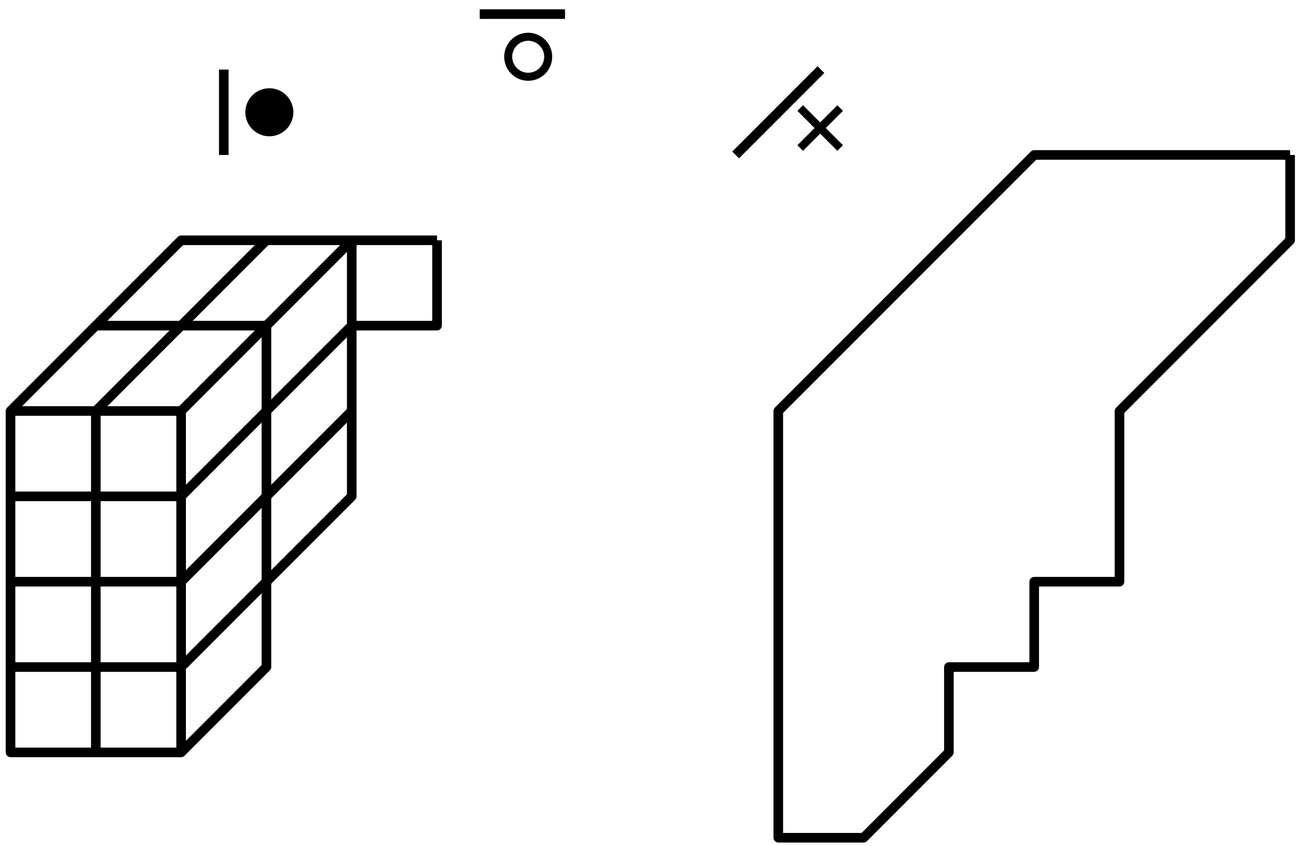




Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

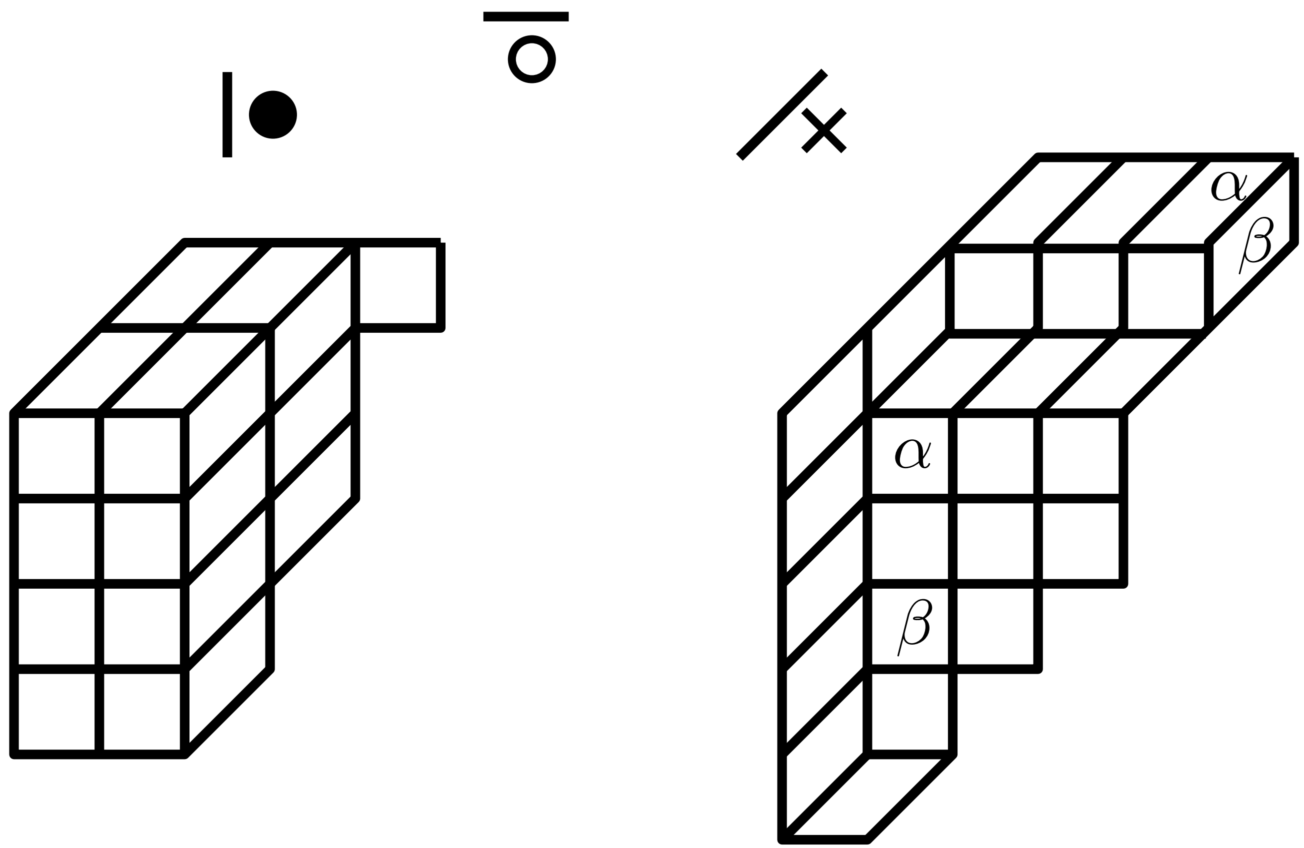
$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

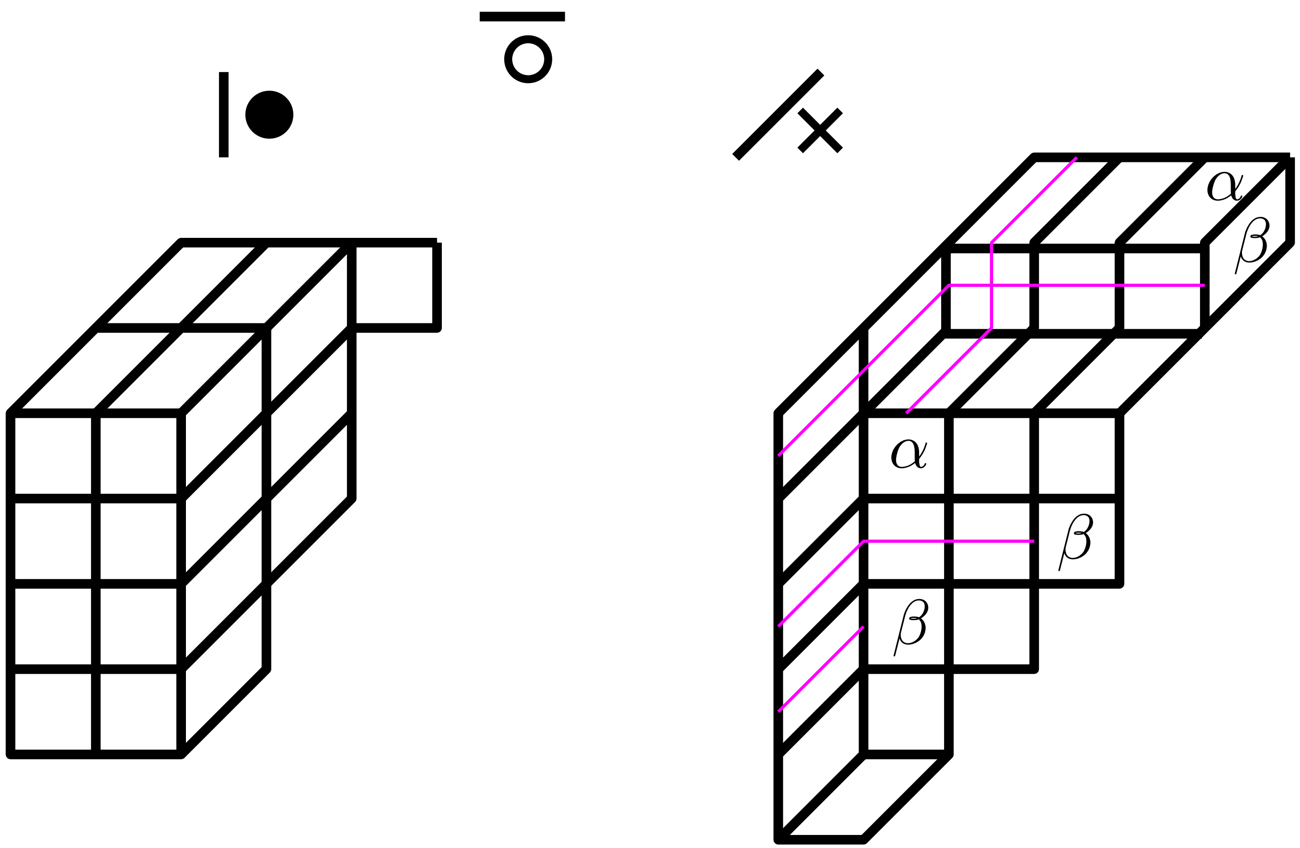
$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

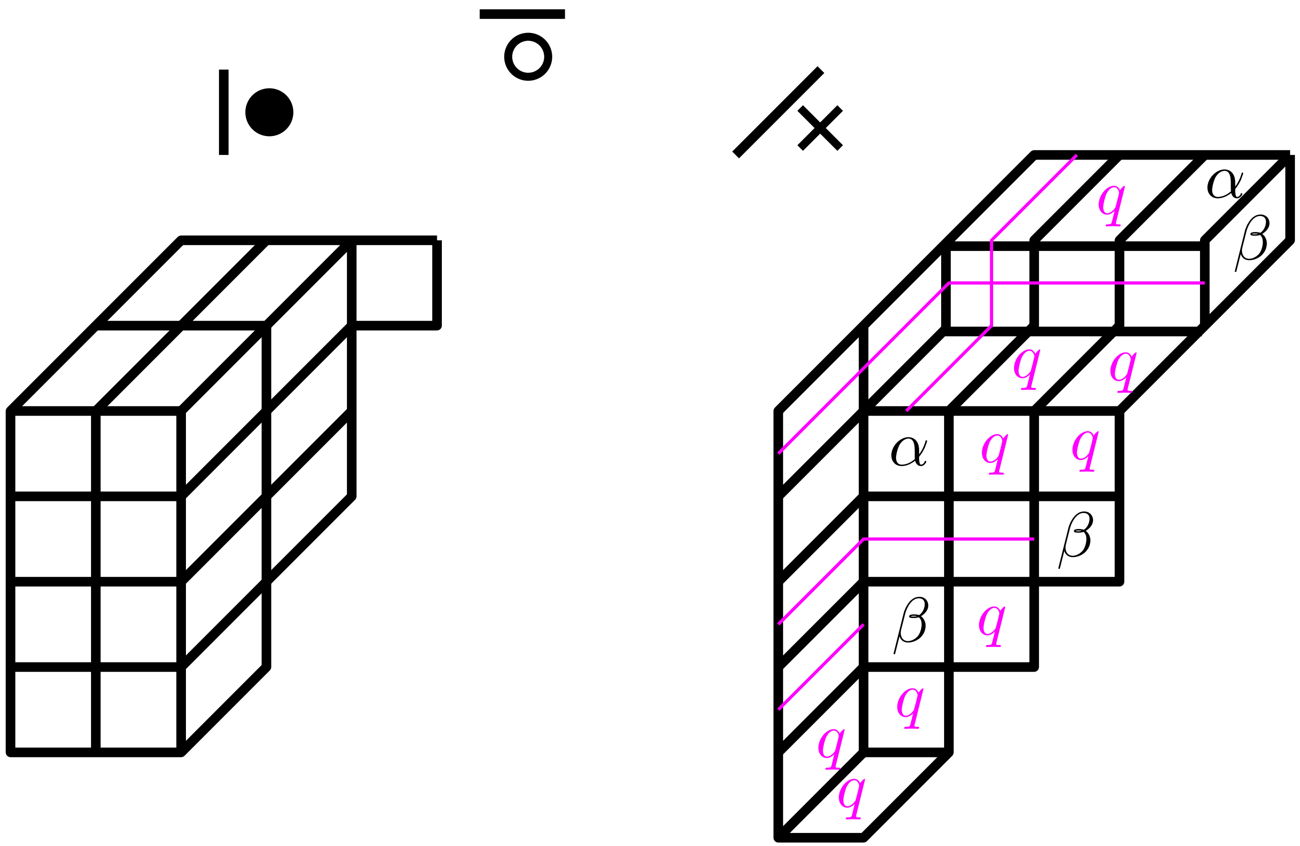
$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

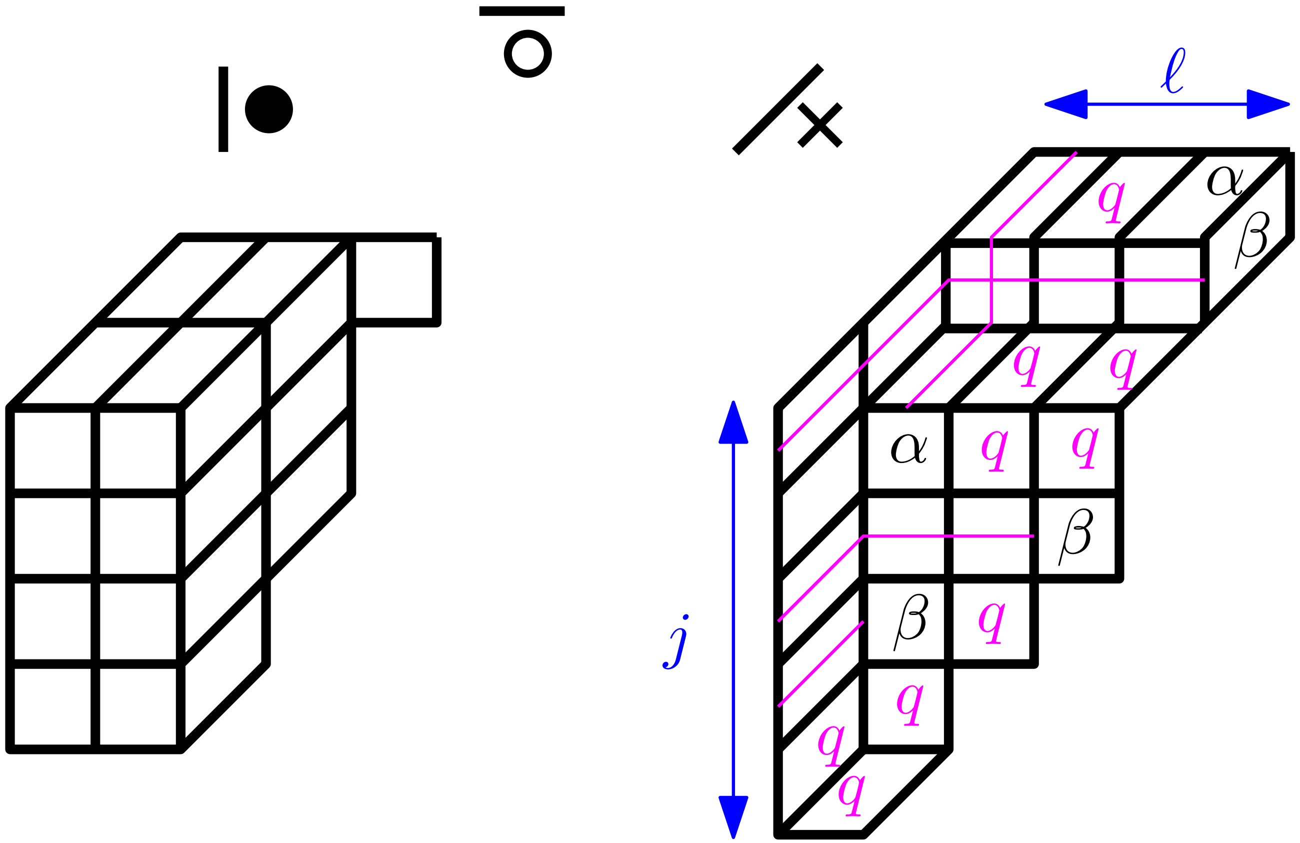
$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]

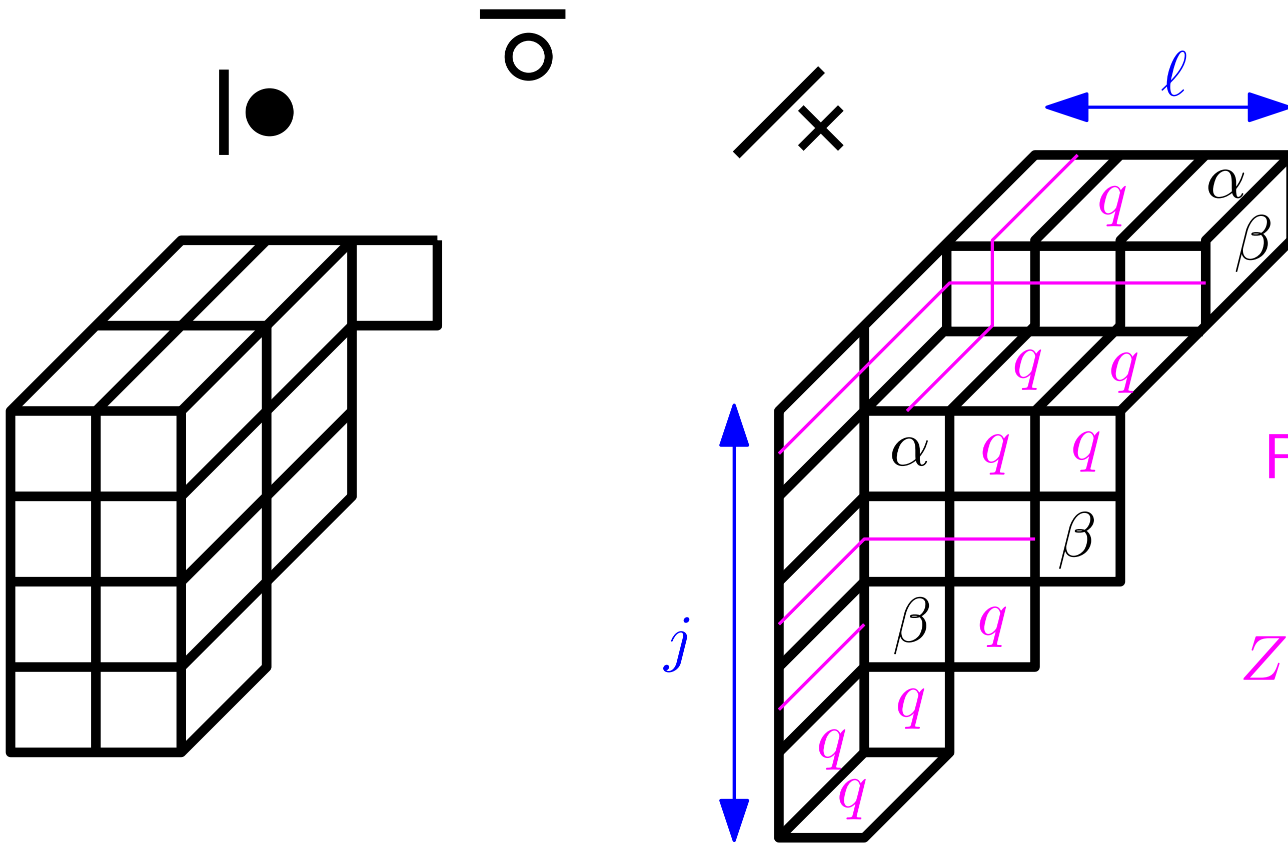


Weight= $\alpha^j \beta^\ell \prod \text{entries}$

Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Weight$= \alpha^j \beta^\ell \prod \text{entries}$

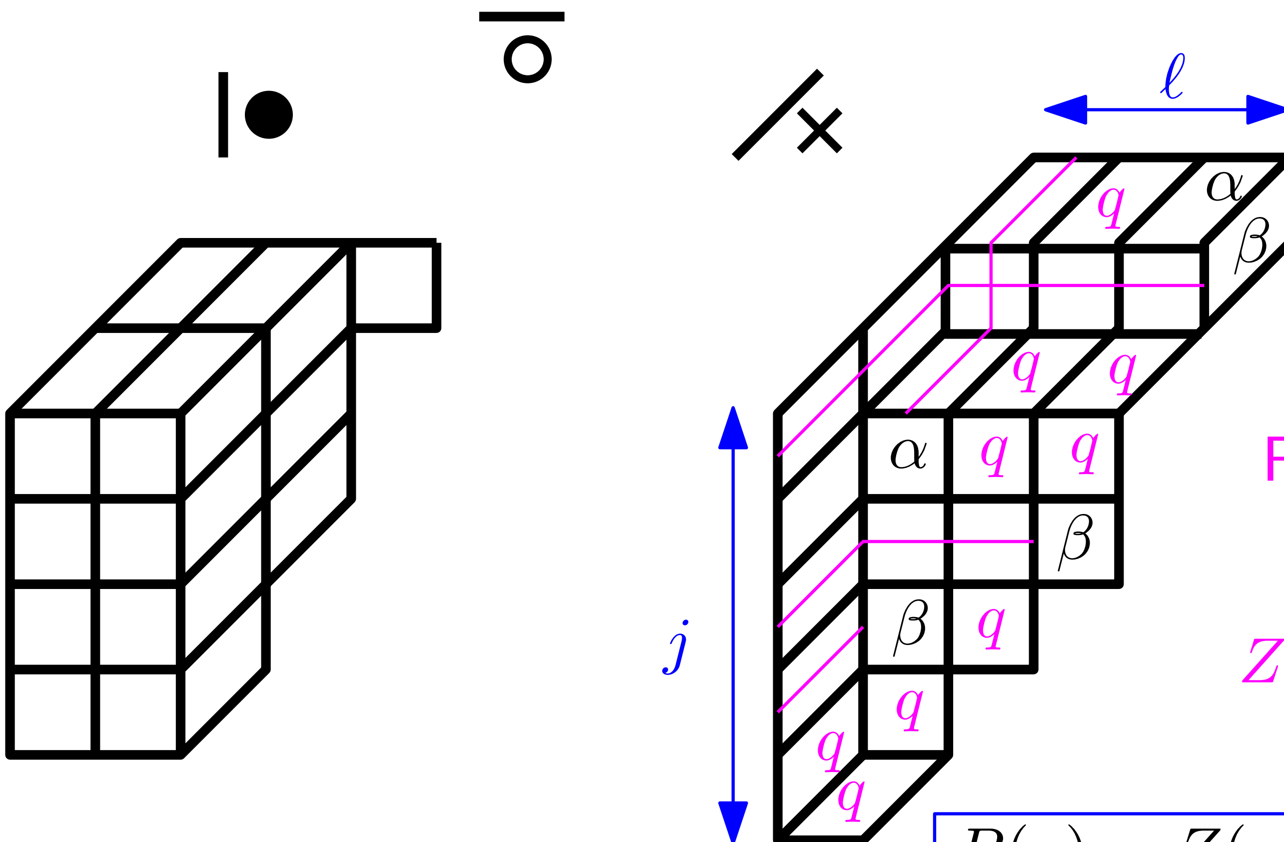
Fix a tiling t

$Z(\tau, t) = \sum_T \text{weight}(T)$

Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Weight$= \alpha^j \beta^\ell \prod \mathrm{entries}$

Fix a tiling t

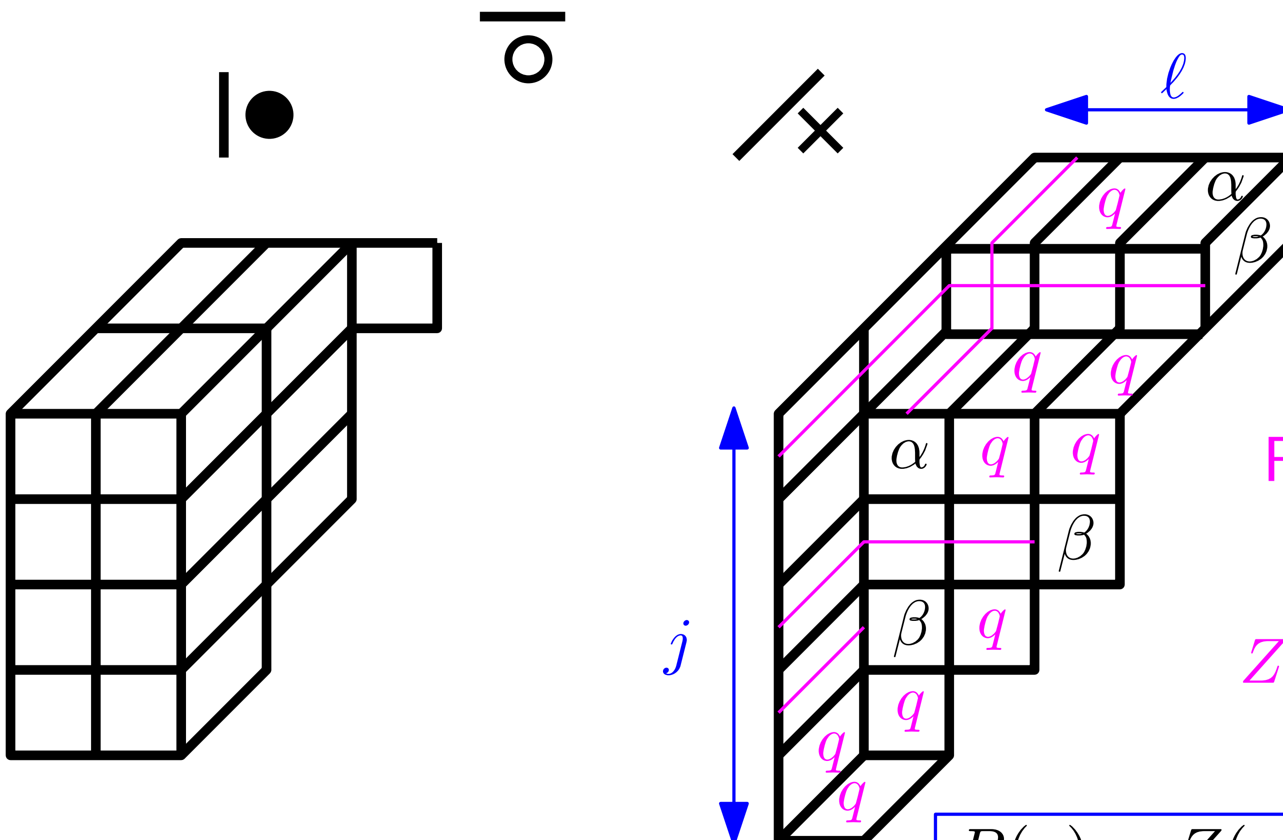
$Z(\tau, t) = \sum_T \mathrm{weight}(T)$

$P(\tau) = Z(\tau, t) / \sum_\tau Z(\tau, t)$

Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Weight$= \alpha^j \beta^\ell \prod \text{entries}$

Fix a tiling t

$Z(\tau, t) = \sum_T \text{weight}(T)$

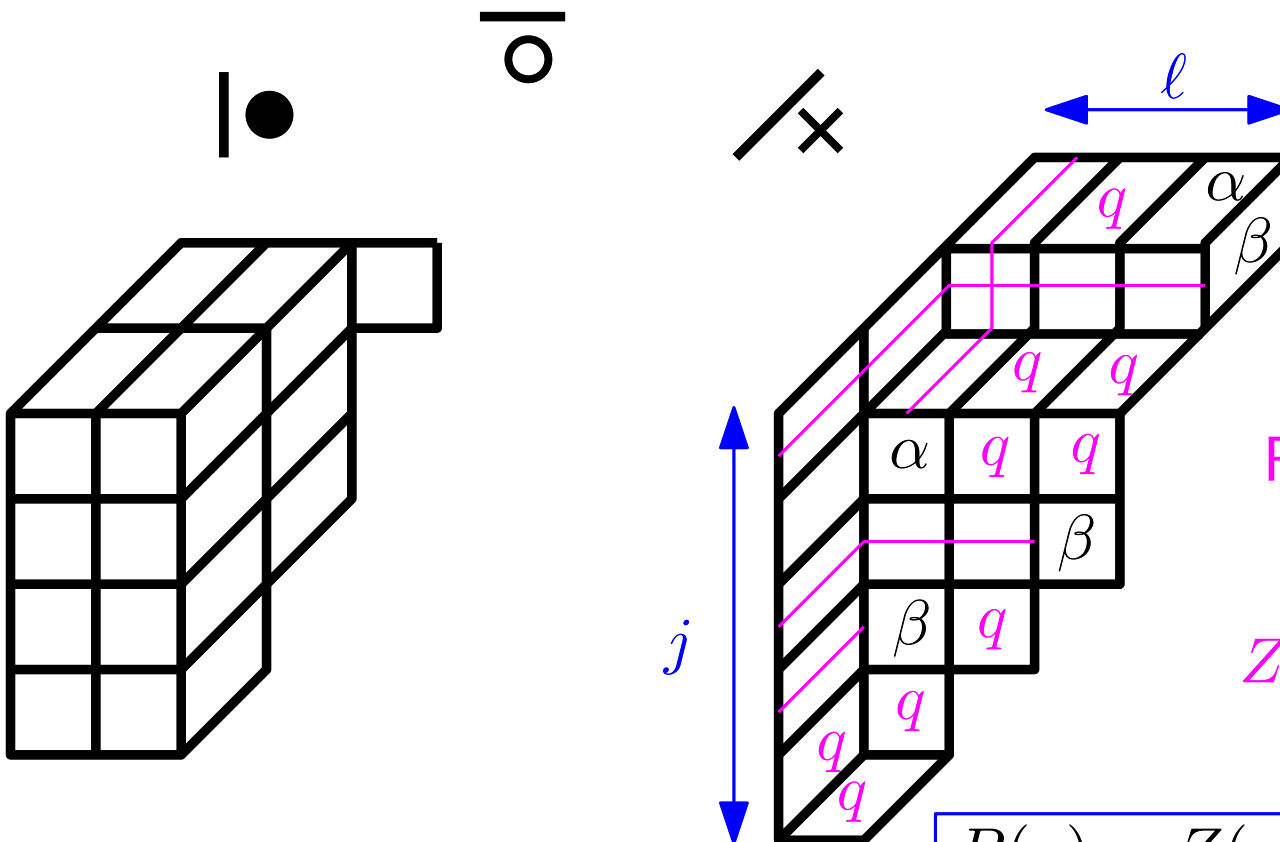
$P(\tau) = Z(\tau, t) / \sum_\tau Z(\tau, t)$

For t and t' tilings, $Z(\tau, t) = Z(\tau, t')$

Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



Weight$= \alpha^j \beta^\ell \prod \text{entries}$

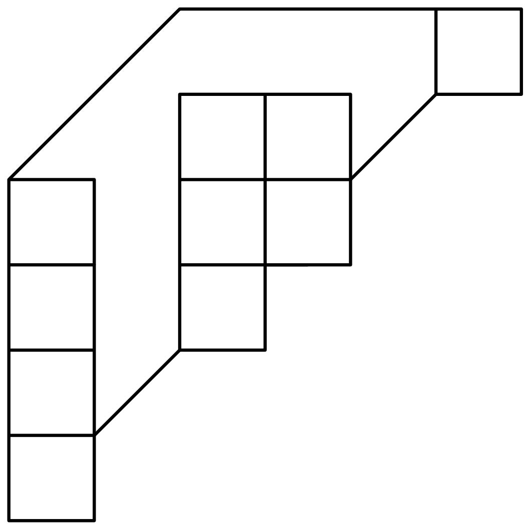
Fix a tiling t

$Z(\tau, t) = \sum_T \text{weight}(T)$

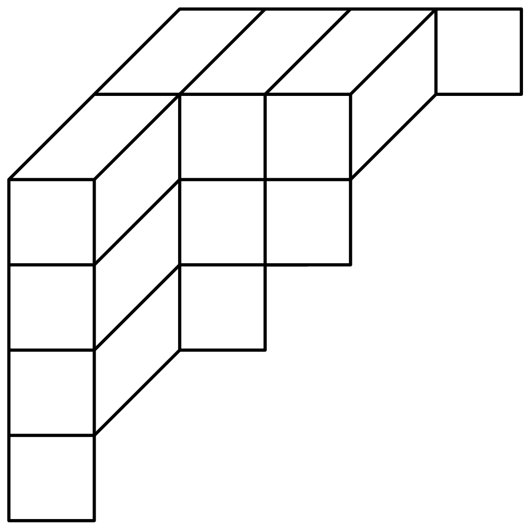
$$P(\tau) = Z(\tau,t) / \sum_\tau Z(\tau,t)$$

$\binom{N}{r} \frac{(N+1)!}{(r+1)!}$ tableaux

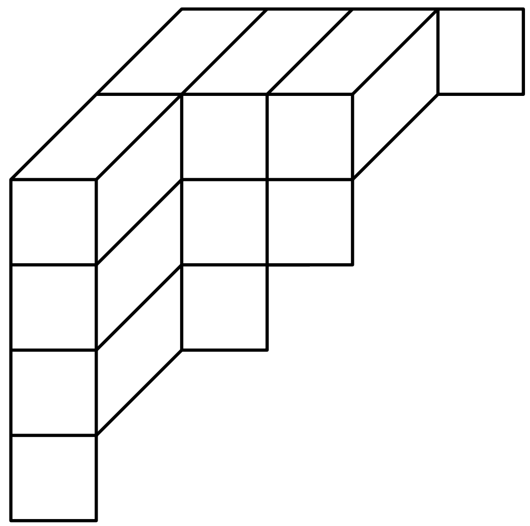
Triangular staircase tableaux [C., Mandelshtam, Williams 15]

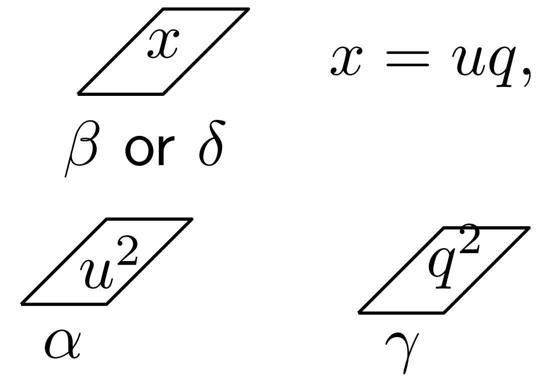


Triangular staircase tableaux [C., Mandelshtam, Williams 15]



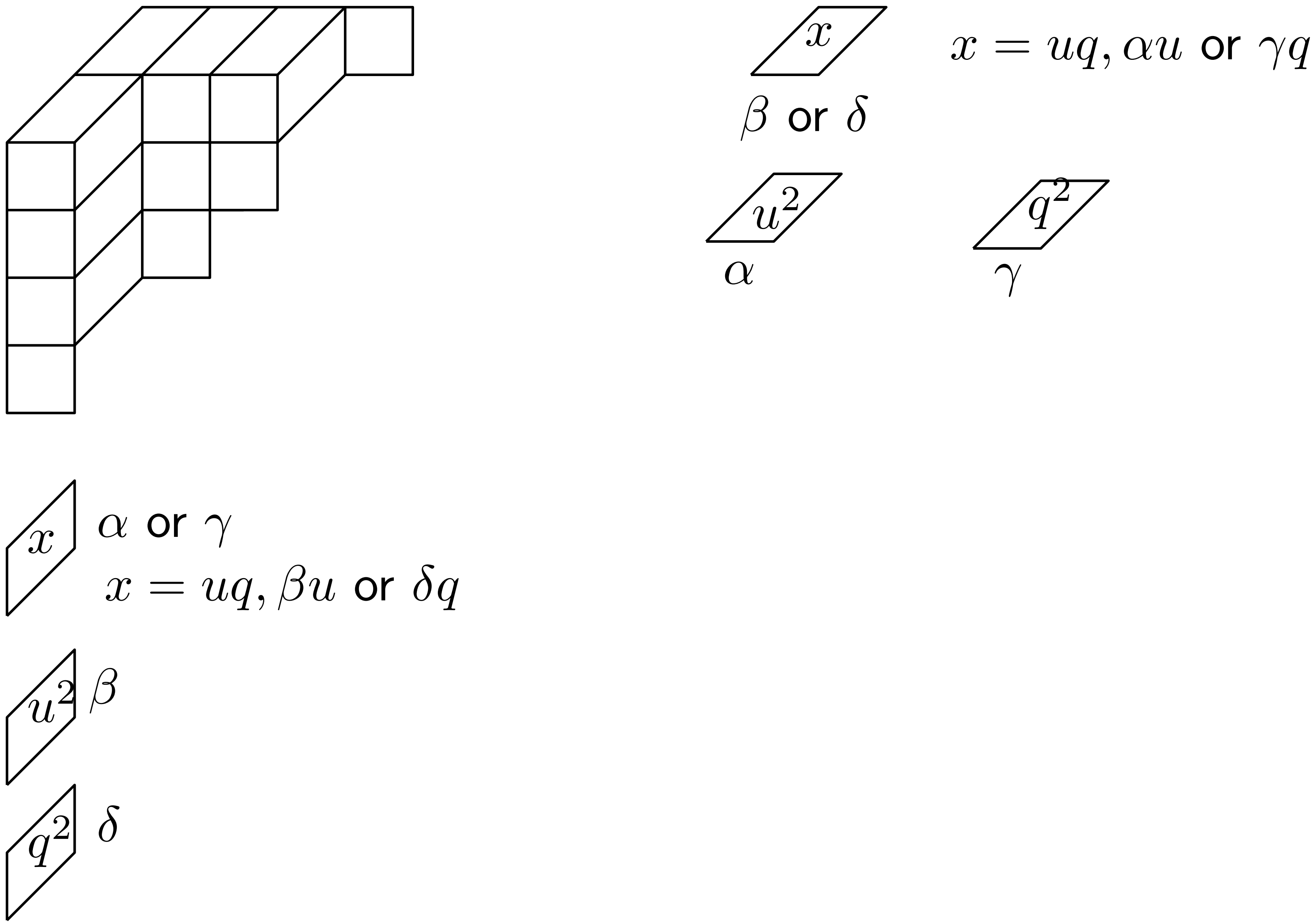
Triangular staircase tableaux [C., Mandelshtam, Williams 15]



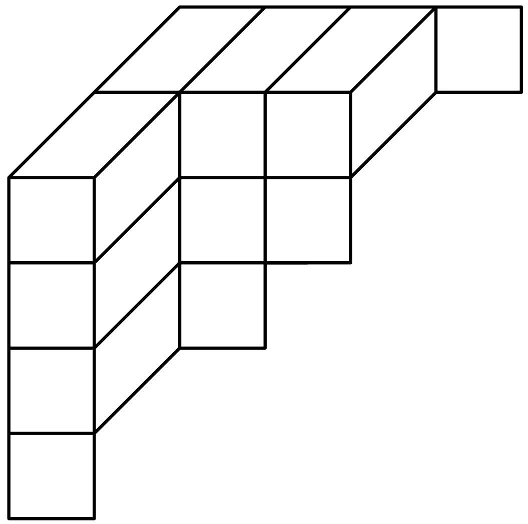


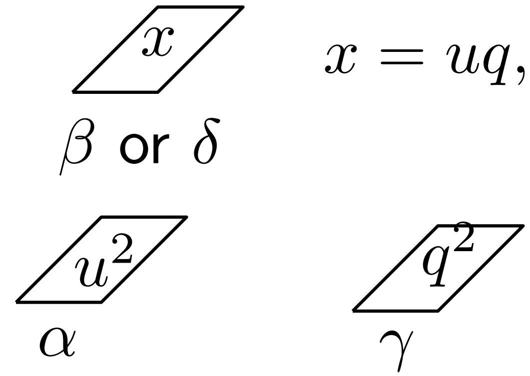
$x = uq, \alpha u$ or γq

Triangular staircase tableaux [C., Mandelshtam, Williams 15]

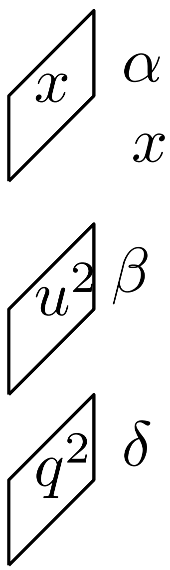


Triangular staircase tableaux [C., Mandelshtam, Williams 15]





$x = uq, \alpha u$ or γq



$x = uq, \beta u$ or δq

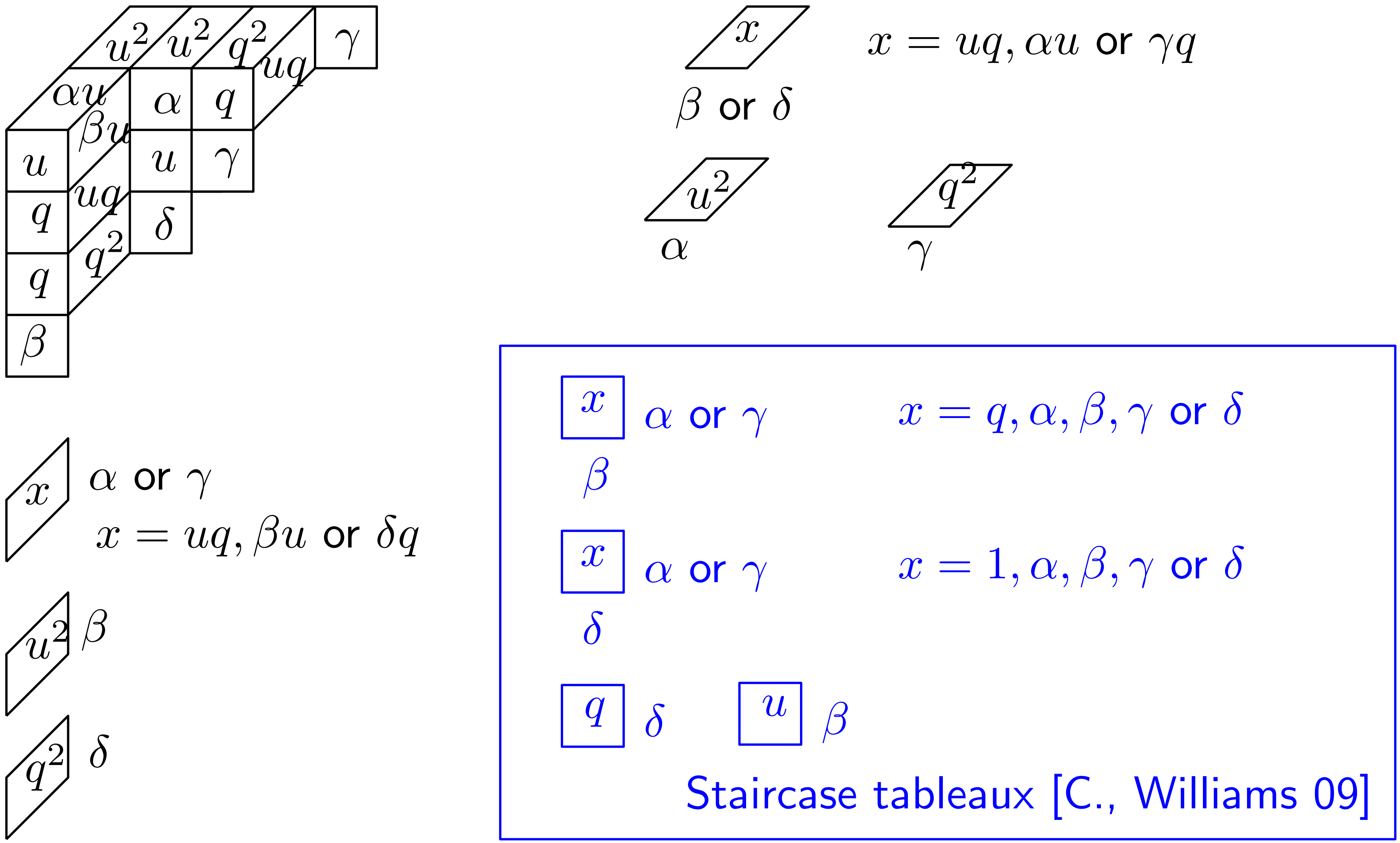
$\boxed{x}$ α or γ, below: β — $x = q, \alpha, \beta, \gamma$ or δ

$\boxed{x}$ α or γ, below: δ — $x = 1, \alpha, \beta, \gamma$ or δ

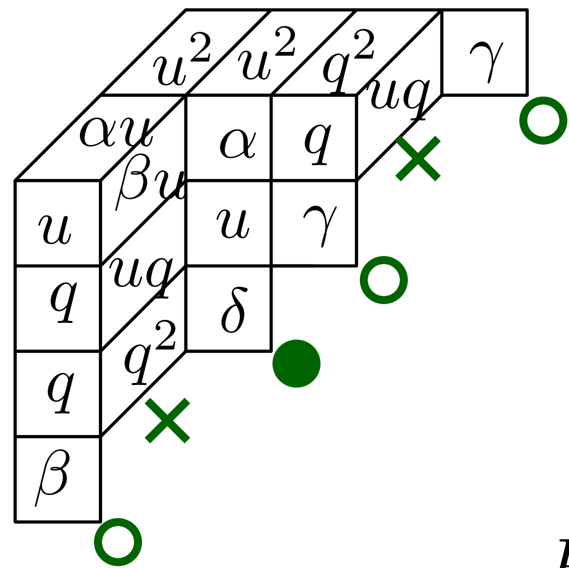
$\boxed{q}$ δ $\boxed{u}$ β

Staircase tableaux [C., Williams 09]

Triangular staircase tableaux [C., Mandelshtam, Williams 15]



Type



$$Z_\tau = \sum_{T \text{ type } \tau} W(T)$$

$$Z_n(\alpha, \beta, \gamma, \delta, q, u) = \sum_\tau Z_\tau$$

$$P(\tau) = Z_\tau / Z_n$$

$$Z_n(\alpha, \beta, \gamma, \delta, 1, 1) = \binom{n}{r} \prod_{i=r}^{n-1} ((\alpha+\gamma)(\beta+\delta)i + \alpha + \beta + \gamma + \delta)$$

$4^{n-r}(n-r)!\binom{n}{r}^2$ tableaux

Bijective proof?

More to do?

- Links with Affine Hecke algebras?

- How to prove the general conjecture?

Conj. K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with non-negative coefficients

More to do?

- Links with Affine Hecke algebras?
- How to prove the general conjecture?

Conj. K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with non-negative coefficients

Thanks!

Thanks!

Thanks!