Smirnov's parafermionic observable at and away from criticality

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Percolation

On the planar lattice $\mathbb{Z}^2$, declare each bond *open* (resp. *closed*) with probability p (resp. $1 - p$). It is not difficult to prove that there is a *critical value* p_c such that:

- If $p < p_c$, then a.s. there is no infinite connected component of open edges, while
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One can also prove that at criticality there is no infinite component, and robust arguments show that the phase transition is *sharp*: there is exponential decay of the two-point function below p_c:

$$P_p(0 \leftrightarrow x) \leqslant C\, e^{-\eta_p \|x\|}.$$

Percolation

Getting more quantitative results is more difficult. On the square lattice, the value of the critical point is known:

Theorem (Kesten, 1980)

$$p_c(\mathbb{Z}^2) = \frac{1}{2}$$

The main arguments of the proof are *self-duality* (specific) and a *sharp-threshold* result (robust).

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More details are provided by *critical exponents*, known to exist in only very few cases. For instance, the optimal η_p behaves like $(p_c - p)^\nu$.

The random-cluster model

The RC model is a dependent variant of bond percolation. On a finite graph, the probability of a configuration is equal to

$$P_{p,q,\Lambda}(\{\omega\}) = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}}{Z_{p,q,\Lambda}}$$

where:

- $o(\omega)$ is the number of open bonds,
- $c(\omega)$ is the number of closed bonds, and
- $k(\omega)$ is the number of connected components.

Notice that $q = 1$ gives exactly bond percolation.

The random-cluster model : phase transition

One can define the FK model in a finite box with *boundary conditions* (for now, either *open* or *closed*). Then, use monotonicity in the boundary conditions to define the thermodynamical limit(s) $P^0_{p,q}$ and $P^1_{p,q}$ in the whole plane.

There is monotonicity in p as well, so one can define the critical point as in the case of percolation:

$$p_c(q) := \sup \left\{ p : P^*_{p,q}(0 \leftrightarrow \infty) = 0 \right\} = \inf \left\{ p : P^*_{p,q}(0 \leftrightarrow \infty) > 0 \right\}$$

Theorem (B., Duminil-Copin)

On $\mathbb{Z}^2$, $\quad p_c(q) = \dfrac{\sqrt{q}}{1+\sqrt{q}} \quad$ *for all* $q \geqslant 1$.

Coupling with the Potts model

The random-cluster model is closely related to the q-state Potts model (and in particular to the Ising model for $q = 2$), defined by

$$\mu_{q,\beta}(\{\sigma\}) = \frac{\exp\left[\beta \sum_{x\sim y} \delta_{\sigma_x}^{\sigma_y}\right]}{Z_{q,\beta}}$$

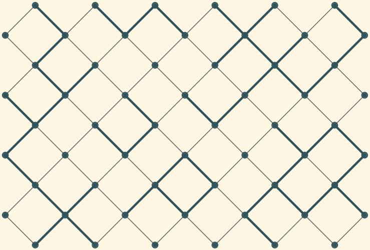
A classical coupling leads for $1 - p = e^{-\beta}$ to

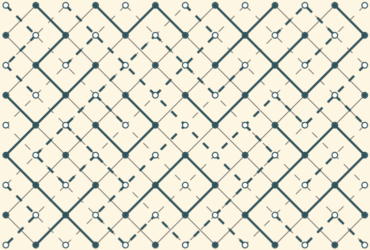
$$\mu_{q,e^{-p}}(\sigma_x = \sigma_y) - \frac{1}{q} = \frac{q-1}{q} P_{p,q}(x \leftrightarrow y)$$

Corollary (B., D.-C.)

$$\beta_c(q) = \log(1 + \sqrt{q}) \quad \textit{for all } q \geqslant 1.$$

Planar duality and percolation





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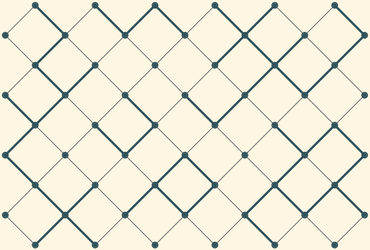

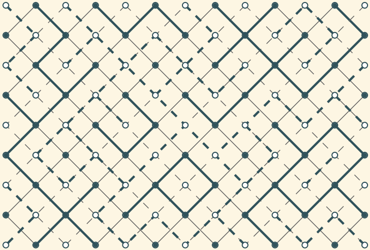

Easy to see: $(P_p)^* = P_{1-p}$. The self-dual point is $p_{sd} = \frac{1}{2}$:

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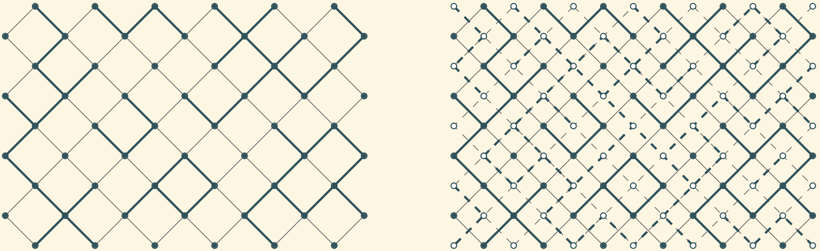

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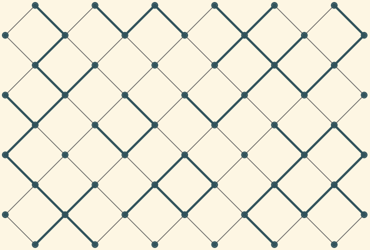
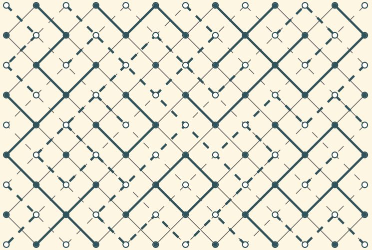
Theorem (Kesten)

For bond-percolation on $\mathbb{Z}^2$, one has $p_c = p_{sd}$.

Planar duality and the random-cluster model





The FK duality relation is $\dfrac{pp^*}{(1-p)(1-p^*)} = q$

Theorem (B., D.-C., restated)

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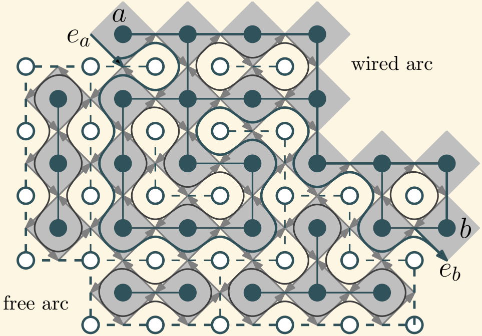
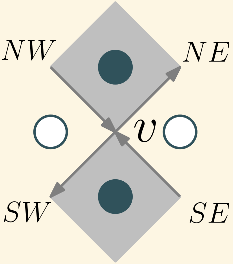
Loop representation of the FK model





$$\mathbb{P}_{G,a,b}(\{\omega\}) = \frac{x^{o(\omega)} q^{L(\omega)/2}}{\tilde{Z}(p,q,G)}, \quad \text{where} \quad x = \frac{p}{(1-p)\sqrt{q}} \quad (\text{so: } x_{sd} = 1)$$

Definition of the parafermionic observable

Smirnov defines the observable F_{e_a} for any edge $e \in E_\diamond$ by

$$F_{e_a}(e) := \mathbb{E}_{G,a,b}\left(\mathrm{e}^{-\mathrm{i}\sigma \mathsf{W}_\gamma(e_a,e)}\mathbb{1}_{e\in\gamma}\right),$$

where γ is the exploration path and σ is given by the relation

$$\cos(\sigma\pi/2) = \frac{\sqrt{q}}{2}.$$

On vertices: $F(v) := \sum\{F(e) : v \in e\}$.

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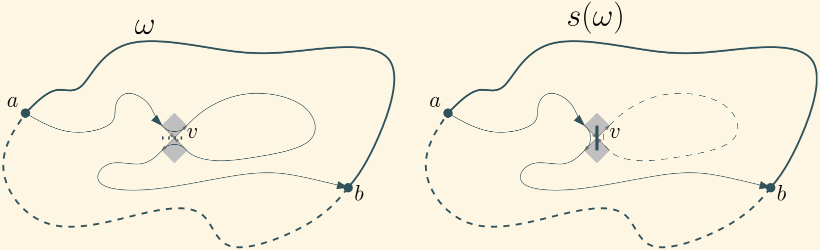
Notice that the observable is real and positive if $q > 4$, "real" (but not necessarily positive) if $q = 2$, and complicated in the other cases.

Glauber dynamics and the observable



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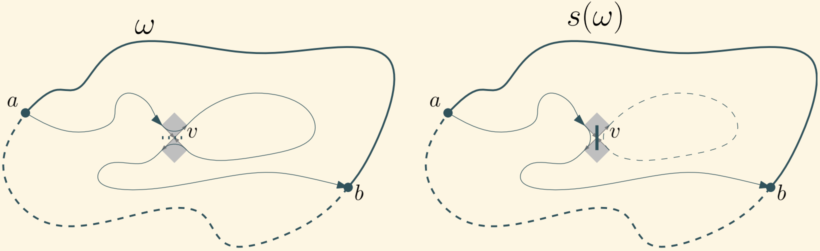

configuration	NW	SE	NE	SW
ω	NW_ω	$e^{i\sigma\pi} NW_\omega$	$e^{-i\sigma\pi/2} NW_\omega$	$e^{i\sigma\pi/2} NW_\omega$
$s(\omega)$	$x\sqrt{q} NW_\omega$	0	0	$e^{i\sigma\pi/2} x\sqrt{q} NW_\omega$

Local relations

Pairing configurations, we get the local relation

$$F(NW) + F(SE) = \Lambda(x)\,[F(SW) + F(NE)]$$

around each vertex, where

$$\Lambda(x) := \frac{e^{i\sigma\pi/2} + x}{e^{i\sigma\pi/2}x + 1}.$$

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Problem: too many variables ...

At the critical point

In the Ising case, the observable is essentially real, in the sense that the argument is determined by the edge type; so half as many variables, and in principle the local relations are enough to know everything about the model.

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If in addition $x = 1$, we get a version of *discrete holomorphicity* (in this case, s-holomorphicity). More precisely: if $e = (xy)$ and $\ell(e)$ is the line of direction $\sqrt{e/e_a}$, then $F(x)$ and $F(y)$ have the same projection on $\ell(e)$ (namely, $F(e)$).

Besides, along the domain boundary, the winding is known exactly, so determining F turns into discrete Riemann-Hilbert BVP. These go very well to the continuous limit.

Scaling limit

Theorem (Smirnov)

Properly normalized by a factor $\delta^{-1/2}$, the observable in a domain Ω discretized at mesh δ converges to $\sqrt{\phi'}$, where

$$\phi : \Omega \xrightarrow{\sim} \mathbb{R} \times (0,1).$$

Corollary

The critical Ising model is conformally invariant in the scaling limit. The scaling limit of the curve γ is chordal SLE(16/3).

Massive harmonicity

If $x \neq 1$, inside the domain, we get massive harmonicity of F, in the sense:

$$\Delta F = \frac{1 - \cos 2\alpha}{\cos 2\alpha} F$$

where the real parameter α is given by the relation

$$\mathrm{e}^{\mathrm{i}\alpha} = \frac{\mathrm{e}^{\mathrm{i}\pi/4} + x}{\mathrm{e}^{\mathrm{i}\pi/4} x + 1} = \Lambda(x).$$

This implies a random ralk representation: $F(X) \simeq E^X[F(W_\tau) m^\tau]$ with $m = \cos 2\alpha < 1$. In particular, F is exponentially small away from the domain boundary.

Massive harmonicity

In terms of the Ising model two-point function in the bulk:

Theorem (Onsager; Messikh; B., D.-C.)

Let $\beta < \beta_c(q = 2)$ and $a = (a_1, a_2) \in \mathbb{R}^2$. Then,

$$\lim_{n\to\infty} -\frac{1}{n} \ln\left(\mathbb{E}_\beta[\sigma(0)\sigma(na)]\right) = a_1 \operatorname{arcsinh} sa_1 + a_2 \operatorname{arcsinh} sa_2,$$

where s solves the equation

$$\sqrt{1 + sa_1^2} + \sqrt{1 + sa_2^2} = \sinh 2\beta + \sinh^{-1} 2\beta$$

and $\mathbb{E}_\beta$ is the (unique) infinite-volume Ising measure at temperature β.

Computing the exponent

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To conclude, one has to prove that the observable and the two-point function have comparable asymptotics. This involves controlling the winding term in the observable, and can be done by introducing a boundary again.

What depends on q?

If $q < 4$, the winding factor in the definition of the observable has modulus 1, so the observable is bounded above by the corresponding two-point function.

If on the other hand $q > 4$, the prefactor can be very large or very small, and no *a priori* bound holds, but the observable should be an upper bound.

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In fact, one expects to have a first-order phase transition if $q > 4$, and uniform exponential decay of correlations all the way to p_c. But the mass in the previous equation does vanish at p_c, so in this case F and the two-point function really are of different order.

$q > 4$: estimating the observable

Recall our main equation:

$$F(NW) + F(SE) = \Lambda(x)\left[F(SW) + F(NE)\right].$$

Summing it over a finite domain A makes each inner bond occur twice, once in each role. This leads to a relation like

$$\sum_{e \in A} F(e) = \frac{1}{1 - \Lambda(x)} \sum_{e \in \partial A} c_e F(e)$$

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If S_n is the sum of F over a box of size n, this leads to $S_n \leqslant c S_{n+1}$ with $c \in (0,1)$. From this, exponential decays follows.

Consequence: observable-based proof that $p_c = p_{sd}$

From what we just did, the observable has exponential decay as soon as $x < 1$. By positivity of the winding term, this implies exponential decay for the connectivities.

From there: the dual model is super-critical, so $p^* \geqslant p_c$ whenever $p < p_{sd}$. Letting $p \uparrow p_{sd}$ we get the bound

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On the other hand, it is known that there is no infinite cluster at the self-dual point (Burton-Keane), so p_{sd} is not super-critical: in other words,

$$p_{sd} \leqslant p_c.$$

A few questions

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- Say anything for $q \in (0, 4) \setminus \{2\}$.