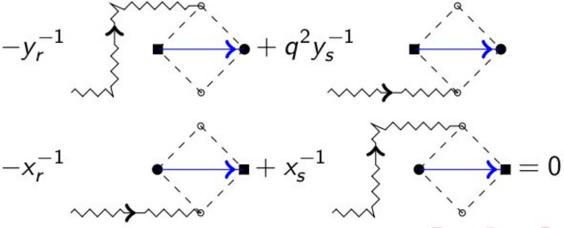
Discrete Holomorphicity in the Chiral Potts Model



Robert Weston

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GGI, 19/05/15

Plan

[*Ref:* Y. Ikhlef, RW, M. Wheeler and P. Zinn-Justin, J. Phys.A 46 (2013) 265205, arxiv:1302.4649; Y. Ikhlef and RW, (2015) arxiv:1502.04944]

The Talk in 1 Slide

- DH means a lattice analog of Cauchy-Riemann relations
- We use underlying quantum group to construct DH operators for stat-mech models
- DH follows from fact that lattice model weights are QG R-matrices
- DH relns in massless case are discrete version of $\partial_{\bar{z}}\Psi(z,\bar{z}) = 0$
- DH relns in massive case are of form $\partial_{\bar{z}}\Psi(z,\bar{z}) = \sum_i \chi_i(z,\bar{z})$

  where in CFT

$$\Psi(z)\Phi_i^{pert}(w,\bar{w}) = \cdots + \frac{\chi_i(w,\bar{w})}{z-w} + \cdots$$

- Can thus identify the CFT perturbing fields
- DH operators hopefully useful in rigorous proof of scaling to CFT

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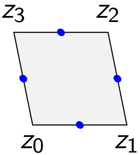
  Around elementary plaquette, we use:
  $f(z_{01})(z_1 - z_0) + f(z_{12})(z_2 - z_1) + f(z_{23})(z_3 - z_2) + f(z_{30})(z_0 - z_3) = 0$



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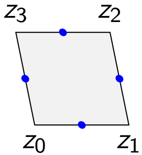


$$z_{ij} = (z_i + z_j)/2$$

- Can be written for this cycle as

$$\frac{f(z_{23}) - f(z_{01})}{z_2 - z_1} = \frac{f(z_{12}) - f(z_{30})}{z_1 - z_0}, \quad \text{a discrete C-R reln } \bar{\partial} f = 0$$

What is use of DH in SM/CFT?

- For review see [S. Smirnov, Proc. ICM 2006, 2010]
- DH of observables has been as a key tool in rigorous proof of existence and uniqueness of scaling limit to particular conformal field theories, e.g.,
  - planar Ising model [S. Smirnov, C. Hongler D. Chelkak . . . , 2001-] - convergence of interfaces to SLE(3)
  - site percolation on triangular lattice - Cardy's crossing formula and reln to SLE(6) [S. Smirnov: 2001]
- We find DH condition also useful in identifying the particular integrable CFT perturbation to which SM lattice model corresponds

DH and Integrability

- Observed by [Ikhlef, Cardy (09); de Gier, Lee, Rasmussen (09); Alam, Batchelor (12,14)] that candidate operators in various lattice models obey DH in the case when R-matrix obeys Yang-Baxter
- Our construction explains this by showing how DH operators arise naturally from Quantum Groups

Non-local quantum group currents in vertex models

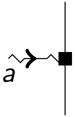
- Following Bernard and Felder [1991] we consider a set of elements $\{J_a, {\Theta_a}^b, {\widehat{\Theta}^a}_b\}$, $a, b = 1, 2, \ldots, n$, of a Hopf algebra U.

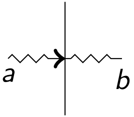
$$\text{Properties:} \quad {\Theta_a}^b {\widehat{\Theta}^c}_b = \delta_{a,c} \quad \text{and} \quad {\widehat{\Theta}^b}_a {\Theta_b}^c = \delta_{a,c}$$

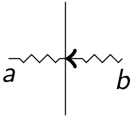
- Co-product Δ and antipode S are (with summation convention):

$$\Delta(J_a) = J_a \otimes 1 + {\Theta_a}^b \otimes J_b \qquad S(J_a) = -{\widehat{\Theta}^b}_a J_b$$
$$\Delta({\Theta_a}^b) = {\Theta_a}^c \otimes {\Theta_c}^b \qquad S({\Theta_a}^b) = {\widehat{\Theta}^b}_a$$
$$\Delta({\widehat{\Theta}^a}_b) = {\widehat{\Theta}^a}_c \otimes {\widehat{\Theta}^c}_b \qquad S({\widehat{\Theta}^a}_b) = {\Theta_b}^a.$$

- Acting on rep of U, we represent as

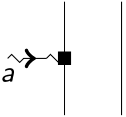
$J_a =$ 

, ${\Theta_a}^b =$ 

, ${\widehat{\Theta}^a}_b =$ 

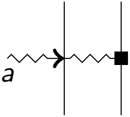
- Coproducts pictures are:

$\Delta(J_a) =$



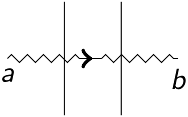
$J_a \otimes 1$

$+$



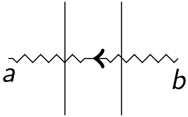
$\Theta_a{}^b \otimes J_b$

$\Delta(\Theta_a{}^b) =$



$\Theta_a{}^c \otimes \Theta_c{}^b$,

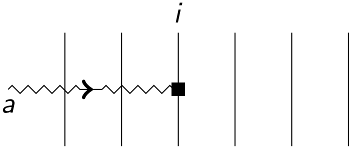
$\Delta(\widehat{\Theta}^a{}_b) =$

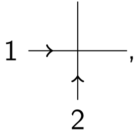


$\widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b$

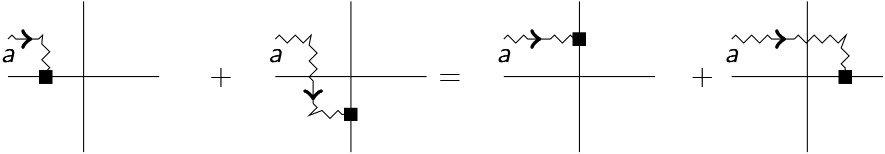
- with obvious extensions to $\Delta^{(N)}(x)$:

$\Delta^{(N)}(J_a) = \sum_i$

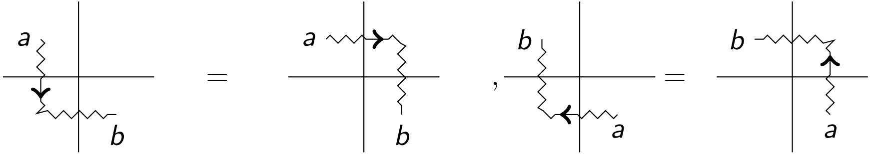


- With $\check{R} : V_1 \otimes V_2 \to V_2 \otimes V_1$ 

, $\check{R}\Delta(x) = \Delta(x)\check{R}$ is

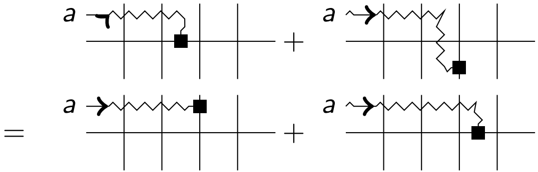


$$\check{R}(J_a \otimes 1) + \check{R}(\Theta_a{}^b \otimes J_b) = (J_a \otimes 1)\check{R} + (\Theta_a{}^b \otimes J_b)\check{R}$$



$$\check{R}(\Theta_a{}^c \otimes \Theta_c{}^b) = (\Theta_a{}^c \otimes \Theta_c{}^b)\check{R} \qquad \check{R}(\widehat{\Theta}^b{}_c \otimes \widehat{\Theta}^c{}_a) = (\widehat{\Theta}^b{}_c \otimes \widehat{\Theta}^c{}_a)\check{R}$$

- So we have non-local currents



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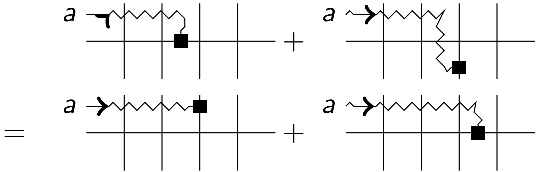

- Gives

$$j_a(x - \frac{1}{2}, t) - j_a(x + \frac{1}{2}, t) + j_a(x, t - \frac{1}{2}) - j_a(x, t + \frac{1}{2}) = 0$$

when inserted into a correlation function

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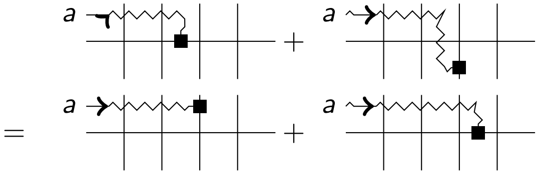

- Gives

$$j_a(x-\frac{1}{2},t)-j_a(x+\frac{1}{2},t)+j_a(x,t-\frac{1}{2})-j_a(x,t+\frac{1}{2})=0$$

when inserted into a correlation function

- Idea: Construct DH operators in terms of such currents:
  - Dense ($U_q(\widehat{sl}_2)$) and dilute loop models ($U_q(A_2^{(2)})$): [Ikhlef, RW, Wheeler, Zinn-Justin (13)]
  - Chiral Potts ($U_q(\widehat{sl}_2)$) : [Iklef, RW (15)]

The Integrable $Z(N)$ Chiral Potts Model

- Introduced by
  [Howes, Kadonoff, den Nijs (83); Au-Yang, Perk, McCoy, Tang, Yan, Sah (87); Baxter, Perk, Au-Yang (88)].
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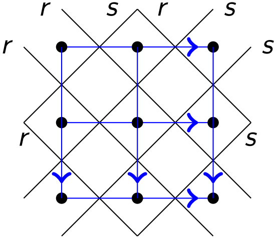
- Heights $a \in \{0, 1, \cdots, N-1\}$ on vertices:



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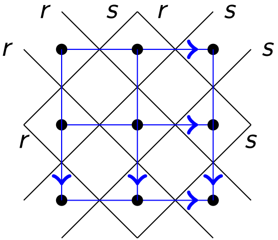
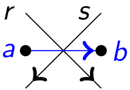
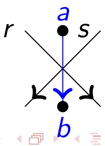


- Heights $a \in \{0, 1, \cdots, N-1\}$ on vertices:

- Boltzmann weights are

$W_{rs}(a-b) =$  ,

$\overline{W}_{rs}(a-b) =$  .

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$$x^N + y^N = k(1 + x^N y^N), \quad \mu^N = \frac{k'}{1 - kx^N} = \frac{1 - ky^N}{k'},$$

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- Obeys star-triangle

$$\sum_{d=0}^{N-1} \overline{W}_{rs}(a-d) W_{rt}(d-b) \overline{W}_{st}(d-c)$$
$$= \rho_{rst} \times W_{rs}(c-b) \overline{W}_{rt}(a-c) W_{st}(a-b)$$

but no difference property

The Integrable $Z(N)$ Chiral Potts Model ...

- Explicitly (with $\omega = \exp(2\pi i/N)$)

$$W_{rs}(a) = \left(\frac{\mu_r}{\mu_s}\right)^a \times \prod_{\ell=1}^{a} \frac{y_s - x_r\omega^\ell}{y_r - x_s\omega^\ell}$$

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- For general N, $k \neq 0$ the phase diagram *still* little understood

CP Representation Theory

- The CP models can be understood in terms of N dim. cyclic representations V_{rs} of $U_q(\widehat{\mathfrak{sl}}_2)$ at $q = -e^{i\pi/N}$, where $r, s \in \mathcal{C}_k$ [Bazhanov and Stroganov (90)]

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- $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ has generators $e_i, f_i, t_i^{\pm 1}, z_i$, $(i=0,1)$ with

$$\begin{aligned}\Delta(e_i) &= e_i\otimes 1 + t_i z_i\otimes 1, \quad \Delta(f_i)=f_i\otimes t_i^{-1}+z_i^{-1}\otimes f_i,\\ \Delta(t_i) &= t_i\otimes t_i, \quad \Delta(z_i)=z_i\otimes z_i,\end{aligned}$$

Useful to consider $\bar{e}_i := t_i f_i$ with $\Delta(\bar{e}_i)=\bar{e}_i\otimes 1 + t_i z_i^{-1}\otimes \bar{e}_i$.

CP Representation Theory...

- Write action on V_{rs} in terms of $N \times N$ matrices X, Y:

$$X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & w & 0 & \cdots & 0 & 0 \\ 0 & 0 & w^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & w^{N-1} \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

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- e.g. $\pi_{rs}(e_1) = \dfrac{\kappa_1}{q - q^{-1}}(x_r\mu_r\mu_s Z - y_s)X$.
  where $r = (x_r, y_r, \mu_r) \in \mathcal{C}_k$.

CP Representation Theory. . .

- Suppose R-matrix $\check{R}(rr', ss') : V_{rr'} \otimes V_{ss'} \to V_{ss'} \otimes V_{rr'}$ is of form:

$$\check{R}(rr', ss') = S_{r's}(T_{r's'} \otimes T_{rs})S_{rs'}$$

$$S_{rs'} : V_{rr'} \otimes V_{ss'} \to V_{s'r'} \otimes V_{sr}, \quad T_{rs} : V_{sr} \to V_{rs}$$

$$S_{rs}(v_{\varepsilon_1} \otimes v_{\varepsilon_2}) = W_{rs}(\varepsilon_1 - \varepsilon_2)(v_{\varepsilon_2} \otimes v_{\varepsilon_1}), \quad T_{rs} v_\varepsilon = \sum_{a=0}^{N-1} \overline{W}_{rs}(a) v_{\varepsilon - a}.$$

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- Then $\check{R}(rr', ss')[\pi_{rr'} \otimes \pi_{ss'}(\Delta(x))] = [\pi_{ss'} \otimes \pi_{rr'}(\Delta(x))]\check{R}(rr', ss')$ is ensured by stronger 'sufficiency conditions':

$$\begin{aligned}
S_{rs'}[\pi_{rr'} \otimes \pi_{ss'}(\Delta(x))] &= [\pi_{s'r'} \otimes \pi_{sr}(\Delta(x))]S_{rs'} \\
(1 \otimes T_{rs})[\pi_{s'r'} \otimes \pi_{sr}(\Delta(x))] &= [\pi_{s'r'} \otimes \pi_{rs}(\Delta(x))](1 \otimes T_{rs}) \\
(T_{r's'} \otimes 1)[\pi_{s'r'} \otimes \pi_{rs}(\Delta(x))] &= [\pi_{r's'} \otimes \pi_{rs}(\Delta(x))](T_{r's'} \otimes 1) \\
S_{r's}[\pi_{r's'} \otimes \pi_{rs}(\Delta(x))] &= [\pi_{ss'} \otimes \pi_{rr'}(\Delta(x))]S_{r's}
\end{aligned}$$

CP Representation Theory...

- Defining $\check{R}(rr',ss')^{ab}_{cd}$ by $\check{R}(rr',ss')(v_a\otimes v_b)=\sum_{c,d}\check{R}(rr',ss')^{ab}_{cd}(v_d\otimes v_c)$, we have

$$\check{R}(rr',ss')^{ab}_{cd}=W_{r's}(d-c)\overline{W}_{r's'}(a-d)\overline{W}_{rs}(b-c)W_{rs'}(a-b).$$

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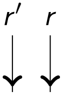
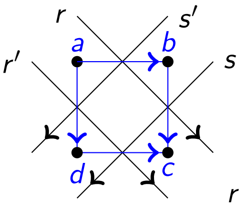
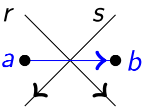
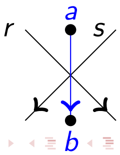
$$\check{R}(rr',ss')^{ab}_{cd} = W_{r's}(d-c)\overline{W}_{r's'}(a-d)\overline{W}_{rs}(b-c)W_{rs'}(a-b).$$

- Associating $V_{rr'}$ with  , we can represent $\check{R}(rr',ss')^{ab}_{cd}$ by



where the CP weights are represented by

$W_{rs}(a-b) =$  , $\overline{W}_{rs}(a-b) =$ 

Non-local QG currents

- Consider $\bar{e}_0 := t_0 f_0$ with $\Delta(\bar{e}_0) = \bar{e}_0 \otimes 1 + t_0 z_0^{-1} \otimes \bar{e}_0$ and

$$\pi_{rr'}(\bar{e}_0) \;=\; \alpha X \left[x_{r'}^{-1} - y_r^{-1} \pi_{rr'}(t_0 z_0^{-1}) \right], \; \pi_{rr'}(t_0 z_0^{-1}) = \frac{y_r y_{r'}}{q^2 x_r x_{r'} \mu_r \mu_{r'}} Z^{-1}$$

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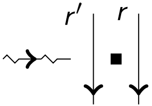
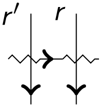
$X \sim$ 

, $\pi_{(rr')}(t_0 z_0^{-1}) \sim$ 

spin σ disorder μ

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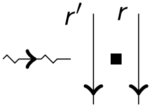
$X \sim$ 

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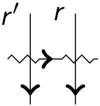
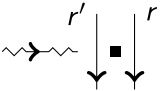
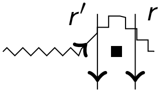
, $\pi_{(rr')}(t_0 z_0^{-1}) \sim$ 

disorder μ

- Current $\bar{e}_0(x,t)$ splits into two 'half-currents' thus:

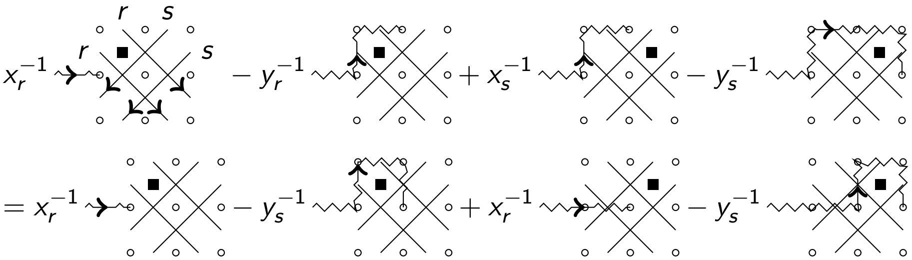
$\bar{e}_0(x,t) = x_{r'}^{-1} \cdots$ 

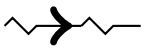
$-y_r^{-1} \cdots$ 

- Consider the sufficiency condition

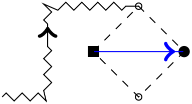
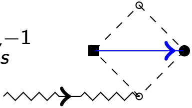
$$S_{rs'}[\pi_{rr'} \otimes \pi_{s,s'}(\Delta(x))] \quad = \quad [\pi_{s',r'} \otimes \pi_{s,r}(\Delta(x))]S_{rs'}$$

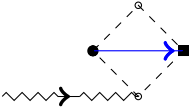
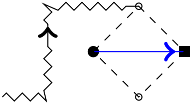
- For $x = \bar{e}_0$ (with $r' = r$, $s' = s$), this becomes:



- Note:
  1. The  line lives on the dual CP lattice
  2. There is cancellation

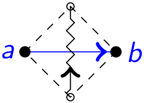
- Four terms cancel. Expressed in terms of CP weights:

$-y_r^{-1}$  $+\, q^2 y_s^{-1}$ 

$-x_r^{-1}$  $+\, x_s^{-1}$  $= 0$

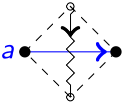
- The effect of the 'disorder' operator expressed purely in terms of CP Boltzmann weights is:

$$\frac{f_s}{f_r} W_{rs}(a-b-1) \quad = $$

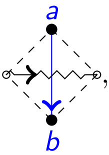


,

$$\frac{f_r}{f_s} W_{rs}(a-b+1) = $$



$$\frac{1}{f_r f_s} \overline{W}_{rs}(a-b-1) \quad = $$



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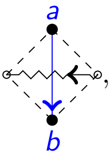
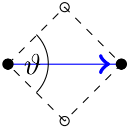
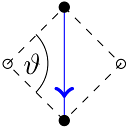


,

$$f_r = \frac{y_r}{-q x_r \mu_r} \qquad .$$

Embedding into the complex plane

- Define u, ϕ by $x = e^{i(u+\phi)/N}$ and $y = e^{i(u-\phi+\pi)/N}$ and embed with angle $\vartheta = u_s - u_r$:

$W_{rs} =$  , $\overline{W}_{rs} =$ ,

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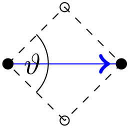
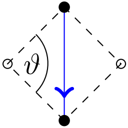
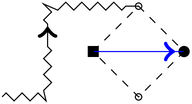
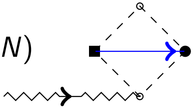
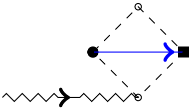
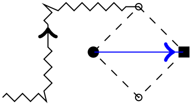
$W_{rs} =$  , $\overline{W}_{rs} =$  ,

- Above relation becomes

$-\exp(i(\vartheta + \phi_r - \pi)/N)$  $+ \exp(i(\phi_s + \pi)/N)$ 

$-\exp(i(\vartheta - \phi_r)/N)$  $+ \exp(-i\phi_s/N)$  $= 0$

- Now define $\mathcal{O}(z)$ to be the half current

$$\mathcal{O}((z_1+z_2)/2)) = \exp(-is\,\mathrm{Arg}(z_1-z_2))\,T(\mu(z_2)\sigma(z_1))$$

where

- $\sigma(z_1)$ is $X = \blacksquare$ at CP site z_1
- $\mu(z_2)$ is disorder operator ending at dual CP site z_2
- T is time ordering (largest $Im(z_i)$ to right)
- $\mathrm{Arg}(z)$ is principal argument of z
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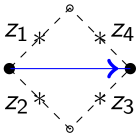

$$e^{i\phi_r/N}\delta z_1 \mathcal{O}(z_1) + e^{i\phi_s/N}\delta z_2 \mathcal{O}(z_2)$$
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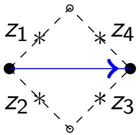

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- Get similar condition for vertical $\overline{W}$ plaquette.

CFT interpretation

- Want to interprete the 'twisted' DH cond

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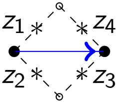
around 

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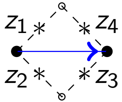
around 

1. Critical Fateev-Zamolodchikov case

- We have $\phi_r=\phi_s=k=0$ and $\mathcal{O}(z)$ is known $Z(N)$ F-Z lattice model parafermion with DH condition [Rajabpour & Cardy 07]

$$\delta z_1\mathcal{O}_1+\delta z_2\mathcal{O}_2+\delta z_3\mathcal{O}_3+\delta z_4\mathcal{O}_4=0$$

which is discrete version of $\bar{\partial}\mathcal{O}=0$

Described by CFT: $c=2(N-1)/(N+2)$, $\mathcal{O}=$ fund. spin $s=1-1/N$ parafermion.

CFT Interpretation . . .

2. General $N > 2$ Case

- Cardy (93), Watts (98) predict integrable CP identifiable as

$$S = S_{\mathrm{FZ}} + \int d^2r\,[\delta_+\Phi_+(z,\bar{z}) + \delta_-\Phi_-(z,\bar{z}) + \tau\varepsilon(z,\bar{z})]$$

  - spin 0 energy operator ε has conf. dim. $(h_\varepsilon, h_\varepsilon)$ with $h_\varepsilon = 2/(N+2)$
  - spin ± 1 $\Phi_\pm$ have conf. dim $(h_\varepsilon + 1, h_\varepsilon)$ and $(h_\varepsilon, h_\varepsilon + 1)$

CFT Interpretation . . .

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- CFT argument then implies

$$\bar{\partial}\mathcal{O}(z,\bar{z}) = \pi\Big(\delta_+ \; \chi_+(z,\bar{z}) + \delta_- \; \chi_-(z,\bar{z}) + \tau \; \chi_0(z,\bar{z})\Big)$$

where

$$\mathcal{O}(z)\Phi_\pm(w,\bar{w}) \quad = \quad + \cdots \frac{\chi_\pm(w,\bar{w})}{z-w} + \cdots \; ; \; \text{spin}(\chi_\pm) = s + 1 \mp 1$$

$$\mathcal{O}(z)\varepsilon(w,\bar{w}) \quad = \quad + \cdots \frac{\chi_0(w,\bar{w})}{z-w} + \cdots \; ; \; \text{spin}(\chi_0) = s - 1$$

CFT Interpretation ...

- By expanding around FZ point our DH condition

$$e^{i\phi_r/N}\delta z_1\mathcal{O}(z_1)+e^{i\phi_s/N}\delta z_2\mathcal{O}(z_2)+e^{-i\phi_r/N}\delta z_3\mathcal{O}(z_3)+e^{-i\phi_s/N}\delta z_4\mathcal{O}(z_4)=0$$

can be described precisely in this way as discrete version of

$$\bar{\partial}\mathcal{O}(z,\bar{z})=\pi\Big(\delta_+\ \chi_+(z,\bar{z})+\delta_-\ \chi_-(z,\bar{z})+\tau\ \chi_0(z,\bar{z})\Big)$$

with $\chi_\pm$ and χ_0 identified in terms of correct-spin lattice operators and parameters (δ_+,δ_-,τ) given in terms of (r,s).

CFT Interpretation . . .

3.The Ising Case

- In general case, we find parafermions associated with $\bar{e}_1$ also gives DH condition
- Those associated with e_0 and e_1 give parafermionic currents with are discretely *antiholomorphic*

CFT Interpretation . . .

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- Combining DH relations for $\bar{e}_0$ and $\bar{e}_1$ in Ising case gives a discrete version of

$$\bar{\partial}\Psi = -im\bar{\Psi}$$

where Ψ and $\bar{\Psi}$ are two spin $\pm 1/2$ components of Ising fermions

CFT Interpretation . . .

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- Combing DAH relations for e_0 and e_1 gives discrete version of

$$\partial\bar{\Psi} = im\Psi$$

- Together = Dirac eqn - seen in Ising by [Riva & Cardy (06)]

Conclusions

- Quantum group currents give operators with DH or DAH relations
- Works for a range of models: dilute and dense loop models [IWWZ (13)] and CP [IW (15)]
- DH conditions tell us about underlying CFT *and* the perturbations of CFT our lattice model corresponds to
- Hopefully useful in establishing rigourous scaling limits to CFT (i.e., the Smirnov programme)