Generalized Cauchy determinant and Schur Pfaffian, and Their Applications

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Cauchy determinants

$$\det\left(\frac{1}{1-x_iy_j}\right)_{1\le i,\,j\le n}=\frac{\prod_{1\le i<j\le n}(x_j-x_i)\prod_{1\le i<j\le n}(y_j-y_i)}{\prod_{i,\,j=1}^{n}(1-x_iy_j)},$$

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\le i,\,j\le n}=\frac{\prod_{1\le i<j\le n}(x_j-x_i)\prod_{1\le i<j\le n}(y_j-y_i)}{\prod_{i,\,j=1}^{n}(x_i+y_j)}.$$

Schur Pfaffians

$$\operatorname{Pf}\left(\frac{x_j-x_i}{x_j+x_i}\right)_{1\le i,\,j\le n}=\prod_{1\le i<j\le n}\frac{x_j-x_i}{x_j+x_i},$$

$$\operatorname{Pf}\left(\frac{x_j-x_i}{1-x_ix_j}\right)_{1\le i,\,j\le n}=\prod_{1\le i<j\le n}\frac{x_j-x_i}{1-x_ix_j}.$$

A generalization of Cauchy determinant

$$\det\left(\frac{a_i - b_j}{x_i - y_j}\right)_{1\le i,\, j\le n}$$
$$= \frac{(-1)^{n(n-1)/2}}{\prod_{i,\,j=1}^{n}(x_i - y_j)} \det\left(\begin{array}{ccccc|ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & a_1 & a_1x_1 & a_1x_1^2 & \cdots & a_1x_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & a_n & a_nx_n & a_nx_n^2 & \cdots & a_nx_n^{n-1} \\ \hline 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} & b_1 & b_1y_1 & b_1y_1^2 & \cdots & b_1y_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} & b_n & b_ny_n & b_ny_n^2 & \cdots & b_ny_n^{n-1} \end{array}\right).$$

If we replace

$$x_i \text{ by } x_i^2, \quad y_i \text{ by } y_i^2, \quad a_i \text{ by } x_i, \quad b_i \text{ by } y_i,$$

or

$$x_i \text{ by } x_i, \quad y_i \text{ by } -y_i, \quad a_i \text{ by } 1, \quad b_i \text{ by } 0,$$

then this generalization reduces to the original Cauchy determinant.

A generalization of Cauchy determinant

$$\det\left(\frac{a_i - b_j}{x_i - y_j}\right)_{1\le i,\,j\le n}$$

$$= \frac{(-1)^{n(n-1)/2}}{\prod_{i,\,j=1}^{n}(x_i - y_j)} \det \left(\begin{array}{ccccc|ccccc}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & a_1 & a_1x_1 & a_1x_1^2 & \cdots & a_1x_1^{n-1} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1} & a_n & a_nx_n & a_nx_n^2 & \cdots & a_nx_n^{n-1} \\
\hline
1 & y_1 & y_1^2 & \cdots & y_1^{n-1} & b_1 & b_1y_1 & b_1y_1^2 & \cdots & b_1y_1^{n-1} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & y_n & y_n^2 & \cdots & y_n^{n-1} & b_n & b_ny_n & b_ny_n^2 & \cdots & b_ny_n^{n-1}
\end{array}\right).$$

By replacing

$$x_i \text{ by } x_i^6, \quad y_i \text{ by } y_i^6, \quad a_i \text{ by } x_i^2, \quad b_i \text{ by } y_i^2,$$

this generalization can be used to evaluate the Izergin–Korepin determinant in the enumeration problem of alternating sign matrices.

Plan

- Cauchy determinant and Cauchy formula for Schur functions
- A generalization of Cauchy determinant and restricted Cauchy formula
- Schur Pfaffian and Littlewood formula for Schur functions
- A generalization of Schur Pfaffian and restricted Littlewood formulae
- Application of generalized Schur Pfaffian to Schur's P-functions

Cauchy Determinant
and
Cauchy Formula for Schur Functions

Partitions and Schur functions

A partition is a weakly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots), \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$$

with finitely many nonzero entries. We put

$$|\lambda| = \sum_{i \geq 1} \lambda_i, \quad l(\lambda) = \#\{i : \lambda_i > 0\}.$$

Let n be a positive integer and $\boldsymbol{x} = (x_1, \cdots, x_n)$ be a sequence of n indeterminates. For a partition λ of length $\leq n$, the Schur function $s_\lambda(x_1, \cdots, x_n)$ corresponding to λ is defined by

$$s_\lambda(\boldsymbol{x}) = s_\lambda(x_1, \cdots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1 \leq i, j \leq n}}{\det\left(x_i^{n-j}\right)_{1 \leq i, j \leq n}}.$$

Remark If $l(\lambda) > n$, then we define $s_\lambda(x_1, \cdots, x_n) = 0$.

Cauchy formula for Schur functions

Theorem For $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$, we have

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \frac{1}{\prod_{i=1}^{n} \prod_{j=1}^{n} (1 - x_i y_j)},$$

where λ runs over all partitions.

This theorem can be proved in several ways. For example, it follows from

- Representation thoeretical proof (irreducible decomposition of $\mathbf{GL}_n \times \mathbf{GL}_n$-module $S(M_n)$);
- Combinatorical proof (Robinson–Schensted–Knuth correspondence)
- Linear algebraic proof

Liear algebraic proof uses

- Cauchy–Binet formula: For two $n \times N$ matrices X and Y,

$$\sum_I \det X(I) \cdot \det Y(I) = \det \left(X^t Y \right),$$

where $I = \{i_1 < \cdots < i_n\}$ runs over all n-element subsets of column indices, and $X(I) = \left(x_{p,i_q}\right)_{1 \le p,\, q \le n}$, $Y(I) = \left(x_{p,i_q}\right)_{1 \le p,\, q \le n}$.

- Cauchy determinant:

$$\det \left(\frac{1}{1 - x_i y_j} \right)_{1 \le i,\, j \le n} = \frac{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})}{\prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j)},$$

where

$$\Delta(\boldsymbol{x}) = \prod_{1 \le i < j \le n} (x_j - x_i), \quad \Delta(\boldsymbol{y}) = \prod_{1 \le i < j \le n} (y_j - y_i).$$

Proof of the Cauchy formula

First we apply the Cauchy–Binet formula (with $N = \infty$) to the matrices

$$X = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & \cdots \end{matrix} \\ \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots \end{pmatrix} \end{array}, \quad Y = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & \cdots \end{matrix} \\ \begin{pmatrix} 1 & y_1 & y_1^2 & y_1^3 & \cdots \\ 1 & y_2 & y_2^2 & y_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & y_n & y_n^2 & y_n^3 & \cdots \end{pmatrix} \end{array}.$$

To a partitions of length $\leq n$, we associate an n-element subsets of $\mathbb{N}$ given by

$$I_n(\lambda) = \{\lambda_1 + n - 1, \lambda_2 + n - 2, \cdots, \lambda_{n-1} + 1, \lambda_n\}.$$

Then the correspondence $\lambda \mapsto I_n(\lambda)$ is a bijection and

$$s_\lambda(\boldsymbol{x}) = \frac{\det X(I_n(\lambda))}{\Delta(\boldsymbol{x})}, \quad s_\lambda(\boldsymbol{y}) = \frac{\det Y(I_n(\lambda))}{\Delta(\boldsymbol{y})}.$$

By applying the Cauchy–Binet formula, we have

$$\begin{aligned}\sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) &= \frac{1}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} \sum_I \det X(I) \cdot \det Y(I) \\ &= \frac{1}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} \det\left(X^t Y\right) \\ &= \frac{1}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} \det\left(\frac{1}{1 - x_i y_j}\right)_{1 \le i,\, j \le n}.\end{aligned}$$

Now we can use the Cauchy determinant to obtain

$$\begin{aligned}\sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) &= \frac{1}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} \cdot \frac{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})}{\prod_{i,\, j=1}^{n} (1 - x_i y_j)} \\ &= \frac{1}{\prod_{i,\, j=1}^{n} (1 - x_i y_j)}.\end{aligned}$$

Generalized Cauchy Determinant
and
Column-length Restricted Cauchy Formula

Theorem (Cauchy formula) For $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$, we have

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \frac{1}{\prod_{i=1}^{n} \prod_{j=1}^{n} (1 - x_i y_j)},$$

where λ runs over all partitions.

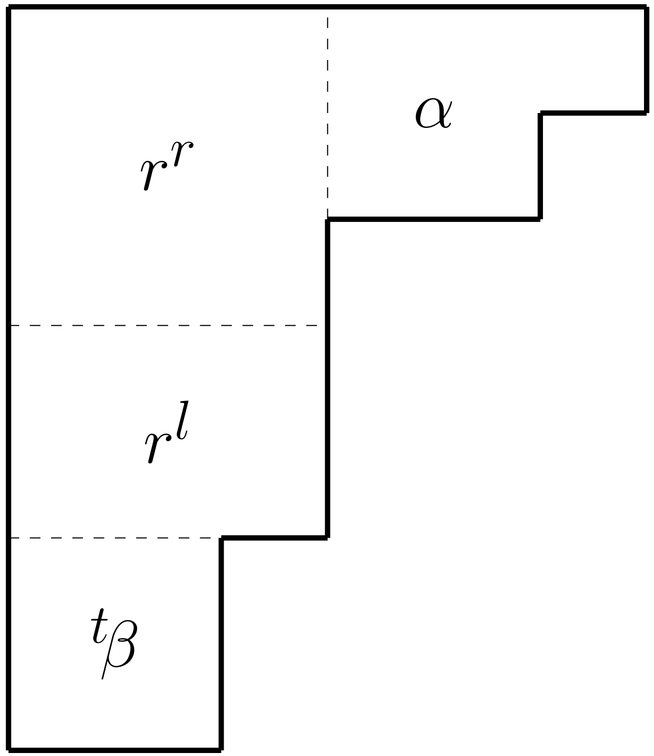
Problem Fix a nonnegative integer l. For $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$, find a formula for

$$\sum_{l(\lambda) \leq l} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}),$$

where λ runs over all partitions of length $l(\lambda) \leq l$.

Let l be a nonnegative integer. To a nonnegative integer r and two partitions α, β with length $\leq r$, we associate a partition

$$\Lambda(r, \alpha, \beta) = (r + \alpha_1, \cdots, r + \alpha_r, \underbrace{r, \cdots, r}_{l}, {}^t\beta_1, {}^t\beta_2, \cdots).$$



Let l be a nonnegative integer. To a nonnegative integer r and two partitions α, β with length $\leq r$, we associate a partition

$$\Lambda(r, \alpha, \beta) = (r + \alpha_1, \cdots, r + \alpha_r, \underbrace{r, \cdots, r}_{l}, {}^t\beta_1, {}^t\beta_2, \cdots).$$

We denote r by $p(\Lambda(r, \alpha, \beta))$. We put

$$\mathcal{C}_l = \text{the set of such partitions } \Lambda(r, \alpha, \beta).$$

Let $\Lambda \mapsto \Lambda^*$ be the involution on $\mathcal{C}_l$ defined by

$$\Lambda(r, \alpha, \beta)^* = \Lambda(r, \beta, \alpha).$$

Note that, if $l = 0$, then

$$\mathcal{C}_0 = \text{the set of all partitions},$$
$$\Lambda^* = {}^t\Lambda \quad \text{(the conjugate partition)}.$$

Theorem (Column-length restricted Cauchy formula; King) For $\boldsymbol{x} = (x_1, \dots, x_m)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$, we have

$$\sum_{l(\lambda) \le l} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \frac{\sum_{\mu \in \mathcal{C}_l} (-1)^{|\mu| + lp(\mu)} s_\mu(\boldsymbol{x}) s_{\mu^*}(\boldsymbol{y})}{\prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)}.$$

Two extreme cases:

- If $l \ge \min(m, n)$, then we recover the Cauchy formula:

$$\sum_\lambda s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \frac{1}{\prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)}$$

- If $l = 0$, then we have the dual Cauchy formula:

$$\sum_\mu (-1)^{|\mu|} s_\mu(\boldsymbol{x}) s_{{}^t\mu}(\boldsymbol{y}) = \prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j).$$

Recall the bijection

$$\lambda \longleftrightarrow I_n(\lambda) = \{\lambda_1 + n - 1, \lambda_2 + n - 2, \cdots, \lambda_{n-1} + 1, \lambda_n\}.$$

Then we have

$$l(\lambda) \leq l \iff [0, n-l-1] \subset I_n(\lambda).$$

In this case, we have

$$s_\lambda(\boldsymbol{x}) = \frac{1}{\Delta(\boldsymbol{x})} \det \left(\begin{array}{cccc|ccc} 1 & x_1 & \cdots & x_1^{n-l-1} & x_1^{\lambda_l+n-l} & \cdots & x_1^{\lambda_1+n-1} \\ 1 & x_2 & \cdots & x_2^{n-l-1} & x_2^{\lambda_l+n-l} & \cdots & x_2^{\lambda_1+n-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-l-1} & x_n^{\lambda_l+n-l} & \cdots & x_n^{\lambda_1+n-1} \end{array} \right).$$

Proof of the restricted Cauchy formula

We prove the formula by using

- generalized Cauchy–Binet formula:

$$\sum_I \det X(\{1,\dots,m-l\} \cup \{i_1+(m-l),\dots,i_l+(m-l)\}) \\ \times \det Y(\{1,\dots,n-l\} \cup \{i_1+(n-l),\dots,i_l+(n-l)\})$$

- generalized Cauchy determinant:

$$\det \left(\begin{array}{c|c} \left(\dfrac{a_i - b_j}{x_i - y_j} \right)_{1 \le i \le m,\, 1 \le j \le n} & \left(1, x_i, x_i^2, \cdots, x_i^{q-1} \right)_{1 \le i \le m} \\ \hline -{}^t\left(1, y_j, y_j^2, \cdots, y_j^{p-1} \right)_{1 \le j \le n} & O \end{array} \right)$$

Generalized Cauchy–Binet formula

Let m, n, M be positive integers and l a nonnegative integer such that $l \leq m$ and $l \leq n$. Let X and Y be $m \times (m-l+M)$ and $n \times (n-l+M)$ matrices respectively. Then we have

Proposition

$$\sum_I \det X(\{1,\ldots,m-l\} \cup \{i_1+(m-l),\ldots,i_l+(m-l)\}) \times \det Y(\{1,\ldots,n-l\} \cup \{i_1+(n-l),\ldots,i_l+(n-l)\}) = (-1)^{mn+l^2} \det \begin{pmatrix} F\,{}^tG & D \\ {}^tE & O \end{pmatrix},$$

where $I = \{i_1 < \cdots < i_l\}$ runs over all l-element subsets of $[M] = \{1,\ldots,M\}$, and

$$D = X(\{1,\cdots,m-l\}), \quad F = X(\{m-l+1,\cdots,m-l+M\}),$$
$$E = Y(\{1,\cdots,n-l\}), \quad G = Y(\{n-l+1,\cdots,n-l+M\}).$$

We apply the generalized Cauchy–Binet identity to

$$X = \left(x_i^j \right)_{1 \le i \le m, j \ge 0}, \quad Y = \left(y_i^j \right)_{1 \le i \le n, j \ge 0}.$$

Then we have

$$\sum_{l(\lambda) \le l} s_\lambda(x_1, \cdots, x_m) s_\lambda(y_1, \cdots, y_n)$$

$$= \frac{(-1)^{l^2+mn}}{\Delta(x)\Delta(y)} \det \left(\begin{array}{c|c} \left(\dfrac{x_i^{m-n}}{1 - x_i y_j} \right)_{1 \le i \le m, 1 \le j \le n} & \left(1 \ \ x_i \ \ \cdots \ \ x_i^{m-l-1} \right)_{1 \le i \le m} \\ \hline {}^t\left(1 \ \ y_j \ \ \cdots \ \ y_j^{n-l-1} \right)_{1 \le j \le n} & O \end{array} \right)$$

This determinant is evaluated by using the following generalized Cauchy determinant.

Theorem A (Generalized Cauchy determinant)

If $m+p=n+q$ and $l=m-q=n-p\geq 0$, then we have

$$\det\left(\begin{array}{c|c} \left(\dfrac{a_i-b_j}{x_i-y_j}\right)_{1\leq i\leq m,1\leq j\leq n} & \begin{pmatrix}1 & x_i & \cdots & x_i^{q-1}\end{pmatrix}_{1\leq i\leq m} \\ \hline {}^t\begin{pmatrix}1 & y_j & \cdots & y_j^{p-1}\end{pmatrix}_{1\leq j\leq n} & O \end{array}\right)$$

$$= \frac{(-1)^{l(l+1)/2}}{\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i-y_j)}$$

$$\times \det\left(\begin{array}{ccccc|ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{m+n-l} & a_1 & a_1x_1 & a_1x_1^2 & \cdots & a_1x_1^{l-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m+n-l} & a_m & a_mx_m & a_mx_m^2 & \cdots & a_mx_m^{l-1} \\ \hline 1 & y_1 & y_1^2 & \cdots & y_1^{m+n-l} & b_1 & b_1y_1 & b_1y_1^2 & \cdots & b_1y_1^{l-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{m+n-l} & b_n & b_ny_n & b_ny_n^2 & \cdots & b_ny_n^{l-1} \end{array}\right).$$

By applying the generalized Cauchy determinant with

$$x_i = x_i^{-1}, \quad a_i = x_i^{-(n-l)}, \quad b_i = 0,$$

we see that

$$\begin{aligned}
&\sum_{l(\lambda)\le l} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) \\
&= \frac{(-1)^{mn+m(m-1)/2}}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y}) \prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)} \\
&\qquad \times \det \left(\begin{array}{ccc|ccc|ccc|ccc}
x_1^{m+n-l-1} & \cdots & x_1^m & 0 & \cdots & 0 & x_1^{m-1} & \cdots & x_1^{m-l} & x_1^{m-l-1} & \cdots & 1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
x_m^{m+n-l-1} & \cdots & x_m^m & 0 & \cdots & 0 & x_m^{m-1} & \cdots & x_m^{m-l} & x_m^{m-l-1} & \cdots & 1 \\
\hline
1 & \cdots & y_1^{n-l-1} & y_1^{n-l} & \cdots & y_1^{n-1} & 0 & \cdots & 0 & y_1^n & \cdots & y_1^{m+n-l-1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
1 & \cdots & y_n^{n-l-1} & y_n^{n-l} & \cdots & y_n^{n-1} & 0 & \cdots & 0 & y_n^n & \cdots & y_n^{m+n-l-1}
\end{array} \right)
\end{aligned}$$

Finally we use the Laplace expansion to obtain the desired restricted Cauchy formula.

Application to generating function of plane partitions

A plane partition is an array of non-negative integers

$$\pi = \left(\pi_{i,j}\right)_{i,\,j\ge 1} = \begin{matrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

satisfying

$$\pi_{i,j} \ge \pi_{i,j+1}, \quad \pi_{i,j} \ge \pi_{i+1,j}, \quad |\pi| = \sum_{i,\,j\ge 1} \pi_{i,j} < \infty.$$

Theorem (MacMahon)

$$\sum_{\pi} q^{|\pi|} = \frac{1}{\prod_{k\ge 1}(1-q^k)^k},$$

where π runs over all plane partitions.

The MacMahon theorem is proved by using the Cauchy formula for Schur functions.

A shifted plane partition is a triangular array of non-negative integers

$$\sigma = (\sigma_{i,j})_{1 \le i \le j} = \begin{array}{cccc} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \cdots \\ & \sigma_{2,2} & \sigma_{2,3} & \cdots \\ & & \sigma_{3,3} & \cdots \\ & & & \ddots \end{array}$$

satisfying

$$\sigma_{i,j} \ge \sigma_{i,j+1}, \quad \sigma_{i,j} \ge \sigma_{i+1,j}, \quad |\sigma| = \sum_{i \le j} \sigma_{i,j} < \infty.$$

The partition $(\sigma_{1,1}, \sigma_{2,2}, \dots)$ is called the profile of σ.

Proposition For a partition λ,

$$\sum_{\sigma} q^{|\sigma|} = q^{|\lambda|} s_{\lambda}(1, q, q^2, \cdots),$$

where the summation is taken over all shifted plane partitions σ with profile λ.

A plane partition π is decomposed into two shifted plane partitions

$$\pi^{+} = (\pi_{i,j})_{1 \le i \le j}, \quad \text{and} \quad \pi^{-} = (\pi_{j,i})_{1 \le i \le j}$$

with the same profile. Hence we have

$$\sum_{\pi} q^{|\pi|} = \sum_{\lambda} q^{|\lambda|} s_{\lambda}(1, q, q^2, \cdots)^2 = \sum_{\lambda} s_{\lambda}(q^{1/2}, q^{3/2}, q^{5/2}, \cdots)^2$$

$$= \frac{1}{\prod_{i,\, j \ge 1} (1 - q^{i+j-1})} = \frac{1}{\prod_{k \ge 1} (1 - q^k)^k}.$$

Similarly, by using the restricted Cauchy formula, we obtain

Theorem

$$\sum_{\pi:\pi_{l+1,l+1}=0} q^{|\pi|} = \frac{\sum_{\mu\in\mathcal{C}_l}(-1)^{|\mu|}q^{|\mu|}s_\mu(1,q,q^2,\dots)s_{\mu^*}(1,q,q^2,\dots)}{\prod_{k\geq 1}(1-q^k)^k}.$$

where π runs over all plane partitions with $\pi_{l+1,l+1} = 0$, i.e., plane partitions whose shapes are contained in a hook of width l.

Remark Mutafyan and Feign proved that

$$\sum_{\pi:\pi_{l+1,l+1}=0} q^{|\pi|} = \frac{\sum_{\nu:l(\nu)\leq l}(-1)^{|\nu|}q^{n({}^t\nu)-n(\nu)}s_\nu(1,q,\dots,q^{l-1})^2}{\prod_{k=1}^{\infty}(1-q^k)^{2\min(k,l)}},$$

which was conjectured by Feigin–Jimbo–Miwa–Mukhin.

Schur Pfaffian
and
Littlewood Formulae

Schur–Littlewood formula

Theorem (Schur, Littlewood) For $\boldsymbol{x} = (x_1, \cdots, x_n)$, we have

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^{n}(1 - x_i) \prod_{1 \le i < j \le n}(1 - x_i x_j)},$$

where λ runs over all partitions.

A linear algebraic proof uses

- Minor-summation formula (Ishikawa–Wakayama), and
- Schur Pfaffian (Laksov–Lascoux–Thorup, Stembridge):

$$\mathrm{Pf}\left(\frac{x_j - x_i}{1 - x_i x_j} \right)_{1 \le i, j \le n} = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{1 - x_i x_j}.$$

Pfaffian

Let $A = (a_{ij})_{1 \le i, j \le 2m}$ be a $2m \times 2m$ skew-symmetric matrix. The Pfaffian of A is defined by

$$\mathrm{Pf}\, A = \sum_{\pi \in \mathfrak{F}_{2m}} \operatorname{sgn}(\pi) a_{\pi(1),\pi(2)} a_{\pi(3),\pi(4)} \cdots a_{\pi(2m-1),\pi(2m)},$$

where $\mathfrak{F}_{2m}$ is the subset of the symmetric group $\mathfrak{S}_{2m}$ given by

$$\mathfrak{F}_{2m} = \left\{ \pi \in \mathfrak{S}_{2m} : \begin{array}{ccccccc} \pi(1) & < & \pi(3) & < & \cdots & < & \pi(2m-1) \\ \wedge & & \wedge & & & & \wedge \\ \pi(2) & & \pi(4) & & & & \pi(2m) \end{array} \right\},$$

and $\operatorname{sgn}(\pi)$ denotes the signature of π.

Example If $2m = 4$, then

$$\mathrm{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Minor-summation Formula

Let $A = (a_{ij})_{1 \le i,\, j \le N}$ be an $N \times N$ skew-symmetric matrix, and $T = (t_{ij})_{1 \le i \le n,\, 1 \le j \le N}$ an $n \times N$ matrix. For an n-element subset $J = \{j_1 < \cdots < j_n\}$ of $[N]$, we put

$$A_J = \left(a_{j_p, j_q} \right)_{1 \le p,\, q \le n}, \quad T(J) = \left(t_{p, j_q} \right)_{1 \le p,\, q \le n}.$$

Theorem (Ishikawa–Wakayama) If n is even, then we have

$$\sum_J \mathrm{Pf}\, A_J \cdot \det T(J) = \mathrm{Pf} \left(T A \, {}^t T \right),$$

where J runs over all n-element subsets of $[N]$.

Remark The minor-summation formula is a Pfaffian version of Cauchy–Binet formula.

Proof of Schur–Littlewood formula

It is enough to consider the case where n is even.
We apply the minor-summation formula to the matrices

$$A = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & \cdots \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ & 0 & 1 & 1 & \cdots \\ & & 0 & 1 & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix} \end{array}, \quad T = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & \cdots \end{matrix} \\ \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots \end{pmatrix} \end{array}.$$

For a partition λ of length $\leq n$, we have

$$\operatorname{Pf} A_{I_n(\lambda)} = 1, \quad s_\lambda(\boldsymbol{x}) = \frac{\det T(I_n(\lambda))}{\Delta(\boldsymbol{x})},$$

where $I_n(\lambda) = \{\lambda_n, \lambda_{n-1} + 1, \cdots, \lambda_1 + n - 1\}$. Hence we have

$$
\begin{aligned}
\sum_{\lambda} s_\lambda(\boldsymbol{x}) &= \frac{1}{\Delta(\boldsymbol{x})} \sum_J \operatorname{Pf} A_J \cdot \det T(J) = \frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \left(T A {}^t T \right) \\
&= \frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \left(\frac{x_j - x_i}{(1 - x_i)(1 - x_j)(1 - x_i x_j)} \right)_{1 \le i,\, j \le n} \\
&= \frac{1}{\Delta(\boldsymbol{x})} \cdot \frac{1}{\prod_{i=1}^n (1 - x_i)} \operatorname{Pf} \left(\frac{x_j - x_i}{1 - x_i x_j} \right)_{1 \le i,\, j \le n} .
\end{aligned}
$$

Now we can use the Schur Pfaffian (Laksov–Lascoux–Thorup, Stembridge) to obtain

$$
\begin{aligned}
\sum_{\lambda} s_\lambda(\boldsymbol{x}) &= \frac{1}{\Delta(\boldsymbol{x})} \cdot \frac{1}{\prod_{i=1}^n (1 - x_i)} \cdot \prod_{1 \le i < j \le n} \frac{x_j - x_i}{1 - x_i x_j} \\
&= \frac{1}{\prod_{i=1}^n (1 - x_i) \prod_{1 \le i < j \le n} (1 - x_i x_j)}.
\end{aligned}
$$

Variation

For a partition λ, we define

$$r(\lambda) = \text{the number of odd parts in } \lambda.$$

Theorem (cf. Macdonald)

$$\sum_{\lambda} u^{r(\lambda)} s_\lambda(\boldsymbol{x}) = \frac{\prod_{i=1}^n (1 + u x_i)}{\prod_{i=1}^n (1 - x_i^2) \prod_{1 \le i < j \le n} (1 - x_i x_j)},$$

where λ runs over all partitions.

If we put $u = 1$, we recover Theorem 1 (Schur–Littlewood formula). If we put $u = 0$, then we have

Corollary (Littlewood)

$$\sum_{\lambda:\text{even}} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^n (1 - x_i^2) \prod_{1 \le i < j \le n} (1 - x_i x_j)},$$

where λ runs over all even partitions (i.e., partitions with only even parts).

Generalized Schur Pfaffian
and
Column-length Restricted Littlewood Formulae

Column-length Restricted Littlewood Formula

Theorem (Schur, Littlewood)

$$\sum_{\lambda} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)},$$

where λ runs over all partitions.

Theorem (King; Conj. by Lievens–Stoilova–Van der Jeugt)

$$\sum_{l(\lambda)\le l} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)} \times \frac{\det\left(x_i^{n-j} - (-1)^l \chi[j>l] x_i^{n-l+j-1}\right)_{1\le i,\,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,\,j\le n}},$$

where λ runs over all partitions of length $\le l$, and $\chi[j>l]=1$ if $j>l$ and 0 otherwise.

Theorem (King; Conj. by Lievens–Stoilova–Van der Jeugt)

$$\sum_{l(\lambda)\le l} s_\lambda(\boldsymbol{x}) = \frac{1}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(1-x_ix_j)} \times \frac{\det\left(x_i^{n-j} - (-1)^l \chi[j>l] x_i^{n-l+j-1}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}},$$

where λ runs over all partitions of length $\le l$, and $\chi[j>l]=1$ if $j>l$ and 0 otherwise.

We give another proof by using

- another type of minor-summation formula (Ishikawa–Wakayama), and
- generalized Schur Pfaffian.

Minor Summation Formula

Theorem (Ishikawa–Wakayama) Suppose that $n+r$ is even and $0 \leq n-r \leq N$. For an $n \times (r+N)$ matrix $T = \left(t_{ij}\right)_{1 \leq i \leq n,\, 1 \leq j \leq r+N}$ and a $N \times N$ skew-symmetric matrix $A = \left(a_{ij}\right)_{r+1 \leq i,\, j \leq r+N}$, we have

$$\sum_J \operatorname{Pf} A_J \cdot \det T(\{1, \dots, r\} \cup \{j_1, \dots, j_{n-r}\})$$

$$= (-1)^{r(r-1)/2} \operatorname{Pf} \begin{pmatrix} K A {}^tK & H \\ -{}^tH & O \end{pmatrix},$$

where $J = \{j_1 < \cdots < j_{n-r}\}$ runs over all $(n-r)$-element subsets of $[r+1, r+N]$ and

$$A_J = \left(a_{j_p, j_q}\right)_{1 \leq p,\, q \leq n-r},$$

$$H = T(\{1, \dots, r\}), \quad K = T(\{r+1, \dots, r+N\}).$$

Proof of the restricted Littlewood formula

For simplicity, we consider the case where l is even.
We apply the minor-summation formula above to the matrices

$$T = \begin{array}{c} \begin{matrix} 0 & 1 & \cdots & r-1 & r & r+1 & \cdots \end{matrix} \\ \begin{pmatrix} 1 & x_1 & \cdots & x_1^{r-1} & x_1^{r} & x_1^{r+1} & \cdots \\ 1 & x_2 & \cdots & x_2^{r-1} & x_2^{r} & x_2^{r+1} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 1 & x_n & \cdots & x_n^{r-1} & x_n^{r} & x_n^{r+1} & \cdots \end{pmatrix} \end{array}, \quad A = \begin{array}{c} \begin{matrix} r & r+1 & r+2 & r+3 & \cdots \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ & 0 & 1 & 1 & \cdots \\ & & 0 & 1 & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{pmatrix} \end{array},$$

where $r = n - l$. If $l(\lambda) \leq l$ and $J = I_n(\lambda) \setminus [0, n-l-1]$, then we have

$$s_\lambda(\boldsymbol{x}) = \frac{\det X(\{0, \dots, r-1\} \cup J)}{\Delta(\boldsymbol{x})}, \quad \operatorname{Pf} A_J = 1.$$

Hence, by applying the minor-summation formula, we have

$$\sum_{l(\lambda) \leq l} s_\lambda(x_1, \cdots, x_n) = \frac{(-1)^{r(n-r)}}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} KA^tK & H \\ -^tH & O \end{pmatrix}.$$

By explicitly computing the entries of KA^tK, we have

$$\sum_{l(\lambda)\leq l} s_\lambda(\boldsymbol{x}) = \frac{(-1)^{r(n-r)}}{\Delta(\boldsymbol{x})}$$
$$\times \operatorname{Pf}\left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{(1-x_i)(1-x_j)(1-x_ix_j)}\right)_{i,j} & \left(1, x_i, x_i^2, \cdots, x_i^{r-1}\right)_i \\ \hline -{}^t\left(1, x_i, x_i^2, \cdots, x_i^{r-1}\right)_i & O \end{array}\right).$$

We need to evaluate this resulting Pfaffian.

Note that

$$\frac{x_j - x_i}{(1-x_i)(1-x_j)(1-x_ix_j)} = \frac{1}{1-x_ix_j}\left(\frac{x_j}{1-x_j} - \frac{x_i}{1-x_i}\right).$$

Now the proof is reduced to the following generalization of Schur Pfaffian.

Generalizations of Schur Pfaffians

Theorem B If $n+r=2m$ is even and $n \geq r$, then we have

$$\operatorname{Pf}\left(\begin{array}{c|c} \left(\dfrac{a_j-a_i}{1-x_ix_j}\right)_{1\leq i,j\leq n} & \left(1,x_i,x_i^2,\cdots,x_i^{r-1}\right)_{1\leq i\leq n} \\ \hline -{}^t\left(1,x_i,x_i^2,\cdots,x_i^{r-1}\right)_{1\leq i\leq n} & O \end{array}\right)$$

$$= \frac{(-1)^{\binom{m}{2}+\binom{r}{2}}}{\prod_{1\leq i<j\leq n}(1-x_ix_j)}$$

$$\times \det\Big(\underbrace{x_i^{m-1}, x_i^m+x_i^{m-2}, x_i^{m+1}+x_i^{m-3}, \cdots, x_i^{2m-2}+1}_{m},$$

$$\underbrace{a_ix_i^{m-1}, a_i\left(x_i^m+x_i^{m-2}\right), \cdots, a_i\left(x_i^{n-2}+x_i^r\right)}_{m-r}\Big)_{1\leq i\leq n}.$$

Example If $n = 3$ and $r = 1$, then we have

$$\operatorname{Pf}\left(\begin{array}{ccc|c} 0 & \dfrac{a_2 - a_1}{1 - x_1 x_2} & \dfrac{a_3 - a_1}{1 - x_1 x_3} & 1 \\ -\dfrac{a_2 - a_1}{1 - x_1 x_2} & 0 & \dfrac{a_3 - a_2}{1 - x_2 x_3} & 1 \\ -\dfrac{a_3 - a_1}{1 - x_1 x_3} & -\dfrac{a_3 - a_2}{1 - x_2 x_3} & 0 & 1 \\ \hline -1 & -1 & -1 & 0 \end{array}\right)$$

$$= \frac{(-1)^1}{\prod_{1 \le i < j \le 3} (1 - x_i x_j)} \det \begin{pmatrix} x_1 & x_1^2 + 1 & a_1 x_1 \\ x_2 & x_2^2 + 1 & a_2 x_2 \\ x_3 & x_3^2 + 1 & a_3 x_3 \end{pmatrix}.$$

Example If $r = 0$ and $a_i = x_i$ $(1 \le i \le n)$, then we recover Laksov–Lascoux–Thorup–Stembridge Pfaffian.

Theorem B follows from the following Theorem C with $k = l$ or $k = l + 1$ by replacing x_i by $x_i + x_i^{-1}$ and b_i by x_i.

Theorem C If $n + k + l = 2m$ is even and $n \geq k + l$, then we have

$$\begin{aligned}
&\mathrm{Pf} \begin{pmatrix} \widetilde{S}_n(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}) & \widetilde{V}_n^{k,l}(\boldsymbol{x}; \boldsymbol{b}) \\ -{}^t\widetilde{V}_n^{k,l}(\boldsymbol{x}; \boldsymbol{b}) & O \end{pmatrix} \\
&\qquad = \frac{(-1)^{\binom{k-l}{2}+(m-k)l}}{\Delta(\boldsymbol{x})} \det \widetilde{V}_n^{m,m-k-l}(\boldsymbol{x}; \boldsymbol{a}) \det \widetilde{V}_n^{m-l,m-k}(\boldsymbol{x}; \boldsymbol{b}),
\end{aligned}$$

where

$$\widetilde{S}_n(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}) = \left(\frac{(a_j - a_i)(b_j - b_i)}{x_j - x_i} \right)_{1 \leq i, j \leq n},$$

$$\widetilde{V}_n^{p,q}(\boldsymbol{x}; \boldsymbol{a}) = \Big(\underbrace{1, x_i, x_i^2, \cdots, x_i^{p-1}}_{p}, \underbrace{a_i, a_i x_i, a_i x_i^2, \cdots, a_i x_i^{q-1}}_{q} \Big)_{1 \leq i \leq n}.$$

Variation

Recall

$$r(\lambda) = \text{the number of odd parts in } \lambda.$$

And we put

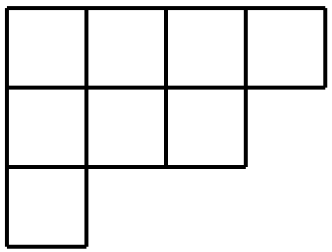
$$p(\lambda) = \#\{i : \lambda_i \geq i\}, \quad \alpha_i = \lambda_i - i, \quad \beta_i = {}^t\lambda_i - i,$$

where ${}^t\lambda$ is the conjugate partition of λ, and write

$$\lambda = (\alpha_1, \cdots, \alpha_{p(\lambda)} | \beta_1, \cdots, \beta_{p(\lambda)}).$$

We call it the Frobenius notation of λ.

Example If $\lambda = (4, 3, 1)$, then $r(\lambda) = 2$, $p(\lambda) = 2$, and λ is written as $(3, 1|2, 0)$.



Theorem

$$\sum_{l(\lambda)\le l} u^{r(\lambda)} s_\lambda(\boldsymbol{x}) = \frac{\sum_\mu f_{l,\mu}(u) s_\mu(\boldsymbol{x})}{\prod_{i=1}^n (1-x_i^2) \prod_{1\le i<j\le n}(1-x_i x_j)},$$

where λ runs over all partitions of length $\le l$, μ runs over all partitions $\mu = (\alpha_1, \cdots, \alpha_r | \beta_1, \cdots, \beta_r)$ satisfying

- if $\alpha_i > 0$, then $\alpha_i + l = \beta_i + 1$;
- if $\alpha_i = 0$, then $\alpha_i + l \ge \beta_i + 1$,

and, for such μ, we define

$$f_{l,\mu}(u) = (-1)^{|\alpha|} \times \begin{cases} u^{l-\beta_r-1} & \text{if } r \text{ is even and } \alpha_r = 0, \\ 1 & \text{if } r \text{ is even and } \alpha_r > 0, \\ u^{\beta_r+1} & \text{if } r \text{ is odd and } \alpha_r = 0, \\ u^l & \text{if } r \text{ is odd and } \alpha_r > 0. \end{cases}$$

Theorem

$$\sum_{l(\lambda)\le l} u^{r(\lambda)} s_\lambda(\boldsymbol{x}) = \frac{\sum_\mu f_{l,\mu}(u) s_\mu(\boldsymbol{x})}{\prod_{i=1}^n (1-x_i^2) \prod_{1\le i<j\le n}(1-x_i x_j)}.$$

By substituting $u=0$, we have

Corollary (King)

$$\sum_{\lambda:\text{even},\, l(\lambda)\le l} s_\lambda(\boldsymbol{x}) = \frac{\sum_\mu (-1)^{(|\mu|-lp(\mu))/2} s_\mu(\boldsymbol{x})}{\prod_{i=1}^n (1-x_i^2) \prod_{1\le i<j\le n}(1-x_i x_j)},$$

where λ runs over all even partitions (i.e., partitions with only even parts) of length $\le l$, and μ runs over all partitions $\mu=(\alpha_1,\cdots,\alpha_r|\beta_1,\cdots,\beta_r)$ satisfying the conditions

- $r=p(\mu)$ is even;
- $\alpha_i + l = \beta_i + 1$ for $1\le i\le r$.

Proof

If we consider the skew-symmetric matrix

$$A = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} 0 & 1 & u & 1 & u & \cdots \\ & 0 & u^2 & u & u^2 & \cdots \\ & & 0 & 1 & u & \cdots \\ & & & 0 & u^2 & \cdots \\ & & & & 0 & \cdots \\ & & & & & \ddots \end{pmatrix} \end{array},$$

then we have

$$\operatorname{Pf} A_{I_l(\lambda)} = u^{r(\lambda)},$$

and we obtain an expression of $\sum_{l(\lambda) \le l} u^{r(\lambda)} s_\lambda(\boldsymbol{x})$ in terms of a Pfaffian. However the resulting Pfaffian cannot be converted into a determinant.

Instead we prove

$$\sum_{l(\lambda)\le l} \left(u^{r(\lambda)} \pm u^{l-r(\lambda)}\right) s_\lambda(\boldsymbol{x}) = \frac{\sum_\mu \left(f_{l,\mu}(u) \pm u^l f_{l,\mu}(u^{-1})\right) s_\mu(\boldsymbol{x})}{\prod_{i=1}^n (1-x_i^2) \prod_{1\le i<j\le n}(1-x_ix_j)},$$

The argument is similar to that in the proof of restricted Littlewood formula.

- Step 1 : Apply the minor-summation formula to express the LHS in terms of a Pfaffian,
- Step 2 : Use Theorem A to convert the resulting Pfaffian into a determinant,
- Step 3 : Evaluate the resulting determinant.

The key is the Pfaffian expression of the weight $u^{r(\lambda)} \pm u^{l-r(\lambda)}$

Lemma Let

$$A = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots \end{matrix} \\ \begin{pmatrix} 0 & 1+u^2 & 2u & 1+u^2 & 2u & \cdots \\ & 0 & 1+u^2 & 2u & 1+u^2 & \cdots \\ & & 0 & 1+u^2 & 2u & \cdots \\ & & & 0 & 1+u^2 & \cdots \\ & & & & 0 & \cdots \\ & & & & & \ddots \end{pmatrix} \end{array}$$

and l an even integer. For a partition λ of length $\leq l$, we have

$$\operatorname{Pf} A_{I_l(\lambda)} = 2^{l/2-1}\left(u^{r(\lambda)} + u^{l-r(\lambda)}\right).$$

Application of Generalized Schur Pfaffian to Schur's P functions

Schur's P-functions

Schur's P-functions $P_\lambda(\boldsymbol{x})$ (or Q-functions $Q_\lambda(\boldsymbol{x})$) are symmetric functions, which play a fundamental role in the theory of projective representations of the symmetric groups, similar to that of Schur functions $s_\lambda(\boldsymbol{x})$ in the theory of linear representations.

Nimmo gave a formula for $P_\lambda(x_1, \cdots, x_n)$ in terms of a Pfaffian. Let λ be a strict partition of length l, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. If $n+l$ is even, then we have

$$P_\lambda(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{x_i + x_j}{x_i - x_j} \cdot \mathrm{Pf} \left(\begin{array}{c|c} \left(\dfrac{x_i - x_j}{x_i + x_j} \right)_{1 \le i, j \le n} & \left(x_i^{\lambda_l}, x_i^{\lambda_{l-1}}, \cdots, x_i^{\lambda_1} \right)_{1 \le i \le n} \\ \hline * & O \end{array} \right).$$

A similar formula holds in the case where $n+l$ is odd.

Recall

Theorem C If $n+p+q=2m$ is even and $n \geq p+q$, then we have

$$\operatorname{Pf}\begin{pmatrix} \widetilde{S}_n(\boldsymbol{x};\boldsymbol{a},\boldsymbol{b}) & \widetilde{V}_n^{p,q}(\boldsymbol{x};\boldsymbol{b}) \\ -{}^t\widetilde{V}_n^{p,q}(\boldsymbol{x};\boldsymbol{b}) & O \end{pmatrix} = \frac{(-1)^{\binom{p-q}{2}+(m-p)q}}{\Delta(\boldsymbol{x})} \det \widetilde{V}_n^{m,m-p-q}(\boldsymbol{x};\boldsymbol{a}) \det \widetilde{V}_n^{m-q,m-p}(\boldsymbol{x};\boldsymbol{b}),$$

where

$$\widetilde{S}_n(\boldsymbol{x};\boldsymbol{a},\boldsymbol{b}) = \left(\frac{(a_j-a_i)(b_j-b_i)}{x_j-x_i}\right)_{1\leq i,j\leq n},$$

$$\widetilde{V}_n^{p,q}(\boldsymbol{x};\boldsymbol{a}) = \Big(\underbrace{1, x_i, x_i^2, \cdots, x_i^{p-1}}_{p}, \underbrace{a_i, a_i x_i, a_i x_i^2, \cdots, a_i x_i^{q-1}}_{q}\Big)_{1\leq i\leq n}.$$

By replacing x_i by x_i^2, a_i by x_i, and b_i by x_i, the left hand side of the Pfaffian formula in Theorem C reads

$$\operatorname{Pf}\left(\begin{array}{c|c} \left(\dfrac{x_j - x_i}{x_j + x_i}\right)_{1\le i,j\le n} & \left(1, x_i^2, x_i^4, \cdots, x_i^{2(p-1)}, x_i, x_i^3, x_i^5, \cdots, x_i^{2(q-1)+1}\right)_{1\le i\le n} \\ \hline * & O \end{array}\right).$$

Comparing this with Nimmo's formula, we obtain an algebraic proof of

Theorem (Worley; Conj. by Stanley) We put

$$\rho_k = (k, k-1, \cdots, 2, 1).$$

Then we have

$$P_{\rho_k+\rho_l}(\boldsymbol{x}) = s_{\rho_k}(\boldsymbol{x}) s_{\rho_l}(\boldsymbol{x}).$$

In particular, we have

$$P_{\rho_k}(\boldsymbol{x}) = s_{\rho_k}(\boldsymbol{x}).$$

Similarly, by replacing

$$x_i \text{ by } x_i^2, \quad a_i \text{ by } \frac{x_i}{1+tx_i}, \quad b_i \text{ by } x_i$$

in Theorem C, and equating the coefficients of t^l, we can prove

Theorem (Worley) We put

$$\rho_k = (k, k-1, \cdots, 2, 1), \quad \text{and} \quad (1^l) = (\underbrace{1, \cdots, 1}_{l}).$$

If $0 \le l \le k+1$, then we have

$$P_{\rho_k + (1^l)}(\boldsymbol{x}) = \sum_{\lambda} s_\lambda(\boldsymbol{x}),$$

where λ runs over all partitions satisfying $\rho_k \subset \lambda \subset \rho_{k+1}$ and $|\lambda| - |\rho_k| = l$.