Low-temperature sprectrum of correlation lengths of the XXZ chain in the massive antiferromagnetic regime

Frank Göhmann

Bergische Universität Wuppertal
Fachgruppe Physik

Firenze

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1d lattice models

- Quantum many-body systems: Defined by a Hamiltonian $H(L)$ depending on the systems size L
- Simple prototypical class: 'spin models' on a 1d lattice. These are 'fully regularized':
  1. Discrete space, lattice spacing a
  2. Finite number of lattice sites L
  3. Finite local Hilbert space $\cong \mathbb{C}^d$

  1. and 2. imply that the space of states is finite dimensional $\cong \left(\mathbb{C}^d\right)^{\otimes L}$
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- QFTs (relativistic and non-relativistic) as certain scaling limits involving $a \to 0$, $L \to \infty$
- Main example here the integrable XXZ Hamiltonian

$$H(L) = J \sum_{j=-L/2+1}^{L/2} \Big(\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta\big(\sigma_{j-1}^z \sigma_j^z - 1\big)\Big) - \frac{h}{2} \sum_{j=-L/2+1}^{L/2} \sigma_j^z$$

$\Delta = (q + q^{-1})/2 = \mathrm{ch}(\gamma)$, L even, σ_j^α, $\alpha = x, y, z$, Pauli matrices acting on factor j in $\left(\mathbb{C}^d\right)^{\otimes L}$

What needs to be calculated?

- Quantum mechanics
  1. $H(L)|n\rangle = E_n|n\rangle$ spectrum and eigenstates
  2. $E_0 \sim L$ for $L \to \infty$ (thermodynamic limit), $e_0 = \lim_{L\to\infty} E_0/L$ ground state energy per lattice site
  3. Dispersion relation of elementary excitations $\varepsilon(p)$ for $L \to \infty$

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- Statistical mechanics and thermodynamics
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- Statistical mechanics and static correlation functions
  1. Static correlation functions of local operators X, Y

$$\langle X_1 Y_m \rangle = \lim_{L\to\infty} \mathrm{Tr}\{\mathrm{e}^{-H(L)/T} X_1 Y_m\}/Z$$

have been studied by means of the (reduced) density matrix

$$D_m(T) = \lim_{L\to\infty} \mathrm{Tr}_{-L/2+1,\dots,0,m+1,\dots,L/2}\{\mathrm{e}^{-H(L)/T}\}/Z$$

$\in \mathrm{End}\big(\mathbb{C}^d\big)^{\otimes m}$ well defined for every $m \in \mathbb{N}$

What needs to be calculated?

- Static correlation functions of local operators continued
  2. Infinite chain formalism: $X \in \text{End}\big(\bigotimes_{n\in\mathbb{Z}} V_n\big)$, where $V_n \cong \mathbb{C}^d$ and X trivial outside chain segment $[1,\ell]$: $[X, e_{j\,\alpha}^{\,\beta}] = 0$ for $j \in \{1,\dots,\ell\}$, local. Maximal such ℓ is called the length of X
  3. Local operators span a vector space $\mathcal{W}$
  4. Local operators have a natural restriction $X_{[1,m]}$ to $\text{End}\big(\bigotimes_{n\in\{1,\dots,m\}} V_n\big)$ $\forall m \ge \ell$
  5. This allows us to properly define the expectation value of a local operator X on the infinite chain
  $$\langle X\rangle = \text{Tr}_{1,\dots,\ell}\{D_\ell(T) X_{[1,\ell]}\}$$
  where ℓ is the length of X
  6. Construction (in generalized form) nicely compatible with the integrable structure of XXZ [Boos, Jimbo, Miwa, Smirnov, Takeyama 2006-09]

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- Dynamical (= time-dependent) correlation functions $\langle X_1(t) X_{m+1} \rangle$
  1. Most correlation functions encountered in experiments are of this type
  2. By definition the time dependence is
     $$X_1(t) = \mathrm{e}^{\mathrm{i}H(L)t} X_1 \mathrm{e}^{-\mathrm{i}H(L)t}$$
     Problem: $X_1(t)$ is not a local operator, does not fit into the formalism based on reduced density matrix

What needs to be calculated?

- Dynamical correlation functions continued
  3. Alternative: Spectral (or Lehmann) representation. Keep L finite. Then

$$\langle X_1(t) Y_{m+1}\rangle = \sum_k \frac{\mathrm{e}^{-E_k/T}}{Z} \langle k|X_1(t) Y_{m+1}|k\rangle = \sum_{k,\ell} \frac{\mathrm{e}^{-E_k/T}}{Z} \langle k|X_1(t)|\ell\rangle\langle \ell|Y_{m+1}|k\rangle$$

$$= \sum_{k,\ell} \frac{\mathrm{e}^{-E_k/T}}{Z} \mathrm{e}^{-\mathrm{i}\big((E_\ell - E_k)t-(p_\ell-p_k)m\big)} \langle k|X_1|\ell\rangle\langle \ell|Y_1|k\rangle$$

$$\xrightarrow[T\to 0]{} \sum_\ell \mathrm{e}^{-\mathrm{i}\big((E_\ell - E_0)t-(p_\ell-p_0)m\big)} \boxed{\langle 0|X_1|\ell\rangle\langle \ell|Y_1|0\rangle}$$

  4. Integrable case: eigenstates of $H(L)$ are eigenstates of the transfer matrix
  5. Matrix elements of the form $\langle \ell|Y_1|0\rangle$ are often called form factors. They can be calculated by 'integrable methods'
  6. So far form factors are the only way to access dynamical correlation functions of integrable systems
  7. Summation is a problem
  8. Often interest is in asymptotic analysis e.g. $m \to \infty$, $t \to \infty$

Example of a factor series

- XXZ for $\Delta > 1$, $0 < h < h_\ell$ from Dugave, FG, Kozlowski, Suzuki 2015 based on Bethe Ansatz for finite L, then $L \to \infty$

$$\langle \sigma_1^z \sigma_{m+1}^z(t) \rangle = \frac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4} (-1)^m$$

$$+ \sum_{\iota=0}^{1} \sum_{n_h \in 2\mathbb{N}} \frac{(-1)^{\iota m}}{n_h!} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}^{n_h} \nu}{(2\pi)^{n_h}} \mathcal{F}_\iota^{(2)}\big(\{\nu_a\}_1^{n_h}\big) \exp\Big\{ \mathrm{i} \sum_{a=1}^{n_h} \big[\varepsilon^{(0)}(\nu_a) t - 2\pi p(\nu_a) m\big] \Big\}$$

Previous work: Jimbo, Miwa 95, Lashkevich 03

- where

$$\mathcal{F}_\iota^{(2)}\big(\{\nu_a\}_1^{n_h}\big) = \frac{1}{n_\chi!} \oint_{\Gamma_\varepsilon(\{\nu_a\})} \frac{\mathrm{d}^{n_\chi} \psi}{(2\pi \mathrm{i})^{n_\chi}} \cdot \frac{\mathfrak{F}_\iota^{(2)}\big(\{\nu_{h_a}\}_1^{2n_\chi}; \{\psi_a\}_1^{n_\chi}\big)}{\prod_{a=1}^{n_\chi} \mathcal{Y}_0\big(\psi_a \big| \{\psi_b\}_1^{n_\chi}; \{\nu_{h_c}\}_1^{2n_\chi}\big)}$$

and

$$\Gamma_\varepsilon(\{\nu_a\}) = \Big\{ \psi \in \mathbb{C}^{n_\chi} \Big| \mathcal{Y}_0\big(\psi_a \big| \{\psi_b\}_1^{n_\chi}; \{\nu_{h_c}\}_1^{2n_\chi}\big) = \varepsilon,\ a = 1, \dots, n_\chi \Big\}$$

with $\varepsilon > 0$ small enough

Example large-distance asymptotics for equal times

- This (combined with a result of LASHKEVICH 03) allows one to obtain an explicit formula for the next-to-leading term in the asymptotics of the static longitudinal two-point function

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \frac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4}(-1)^m + A \cdot \frac{k(q^2)^m}{m^2}\left((-1)^m - \operatorname{th}^2(\gamma/2)\frac{(q;q^2)_\infty^4}{(-q;q^2)_\infty^4}\right)(1+\mathcal{O}(m^{-1}))$$

  where

$$k(q^2) = \frac{\vartheta_2^2(0|q^2)}{\vartheta_3^2(0|q^2)}, \quad A = \frac{1}{\pi \operatorname{sh}^2(\gamma/2)} \frac{(-q;q^2)_\infty^4}{(q^2;q^2)_\infty^2} \frac{(q^4;q^4,q^4)_\infty^8}{(q^2;q^4,q^4)_\infty^8}$$

  generalizing the result of the correlation length of JOHNSON, KRINSKY AND MCCOY 73
- Time dependent case can be analyzed in a similar way

Generalized reduced density matrix

- Integrability of XXZ chain based on the underlying quantum group $U_q(\widehat{\mathfrak{sl}}_2)$
- From this: R-matrix, transfer matrix, quantum transfer matrix, reduced density matrix $\big($rather than $H(L)$, $\mathrm{e}^{-H(L)/T}\big)$

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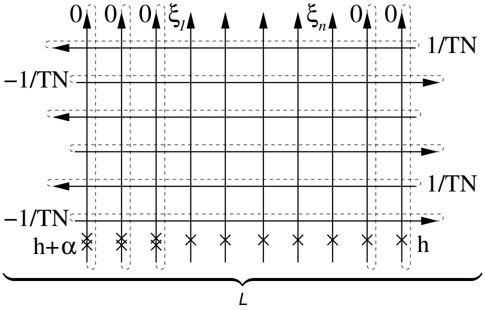

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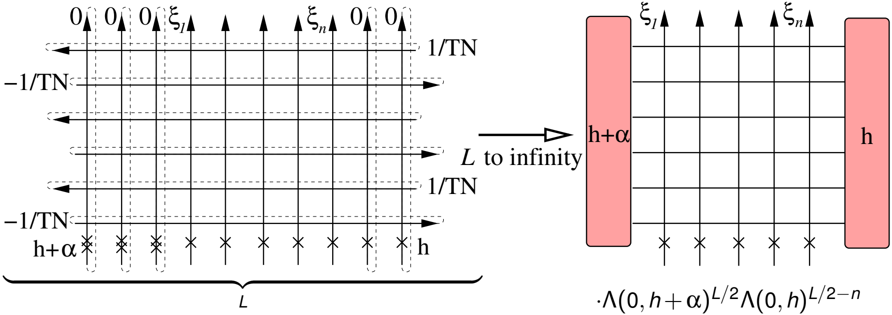

$\cdot\Lambda(0,h+\alpha)^{L/2}\Lambda(0,h)^{L/2-n}$

$$\to D_{[1,n]}(\xi_1,\dots,\xi_n|T,h,\alpha,N) = \frac{\langle h+\alpha|T(\xi_1|h)\otimes\cdots\otimes T(\xi_n|h)|h\rangle}{\langle h+\alpha|\prod_{j=1}^n t(\xi_j|h)|h\rangle}$$

Generalized reduced density matrix

Reduced density matrix and QTM form factor expansion

- Here

$$T(\xi|h) = \mathrm{e}^{\frac{h\sigma^z}{2T}} T(\xi) = \begin{pmatrix} A(\xi|h) & B(\xi|h) \\ C(\xi|h) & D(\xi|h) \end{pmatrix}$$

is the monodromy matrix corresponding to the staggered column-to-column transfer matrix in the picture

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- Using the generalized density matrix we obtain e.g. the transverse two-point functions of the XXZ chain as

$$\begin{aligned} \langle \sigma_1^- \sigma_{m+1}^+ \rangle_N &= \mathrm{Tr}\big\{ D_{[1,m+1]}(0,\dots,0|T,h,0,N)\, \sigma_1^- \sigma_{m+1}^+ \big\} \\ &= \frac{\langle \Psi_0 | B(0|h) t(0|h)^{m-1} C(0|h) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \Lambda_0(0)^{m+1}} = \sum_\ell A_\ell^{-+} \rho_\ell^m \qquad (*) \end{aligned}$$

  where we have used the notation

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- $(*)$ is a large-distance asymptotic expansion for static correlation functions at finite temperature. Expressions for A_ℓ^{-+} in the Trotter limit $N \to \infty$ were obtained in [DUGAVE, FG, KOZLOWSKI 2013]

Hamiltonian and ground state phase diagram of the XXZ chain

- Low-temperature spectrum of correlation lengths of the XXZ chain in the massive antiferromagnetic regime [joint work with M. DUGAVE, K. KOZLOWSKI AND J. SUZUKI, ARXIVE:1504.07923]

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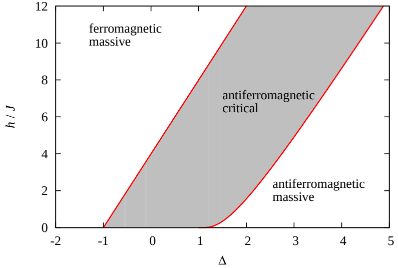

$$H(L) = J \sum_{j=-L/2+1}^{L/2} \left(\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \left(\sigma_{j-1}^z \sigma_j^z - 1 \right) \right) - \frac{h}{2} \sum_{j=-L/2+1}^{L/2} \sigma_j^z$$

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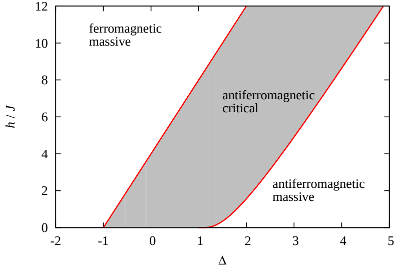

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- Large-distance asymptotics for $T \to 0$ in the critical regime for $|\Delta| < 1$ in [DUGAVE, FG, KOZLOWSKI 2013, 14]

Bethe Ansatz solution for eigenvalue problem of the QTM

- For any finite Trotter number the eigenvalues of the quantum transfer matrix are determined by the algebraic Bethe Ansatz

$$\Lambda(x) = e^{\frac{h}{2T}} \left(\frac{\sin(x + i\beta/N)}{\sin(x + i\beta/N + i\gamma)} \right)^{\frac{N}{2}} \left[\prod_{j=1}^{M} \frac{\sin(x - x_j^r + i\gamma/2)}{\sin(x - x_j^r - i\gamma/2)} \right] \left(1 + \mathfrak{a}(x - i\gamma/2)\right)$$

where the auxiliary function $\mathfrak{a}$ is defined by

$$\begin{aligned} \mathfrak{a}(x) &= \mathfrak{a}\left(x \middle| \{x_k^r\}_{k=1}^M\right) \\ &= e^{-\frac{h}{T}} \left(\frac{\sin(x + i\gamma/2 - i\beta/N)\sin(x + 3i\gamma/2 + i\beta/N)}{\sin(x + i\gamma/2 + i\beta/N)\sin(x - i\gamma/2 - i\beta/N)} \right)^{\frac{N}{2}} \prod_{k=1}^{M} \frac{\sin(x - x_k^r - i\gamma)}{\sin(x - x_k^r + i\gamma)} \end{aligned}$$

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- Goal: analyse above equations in the Trotter limit $N \to \infty$ for small T, for h below the lower critical field, and for fixed value of the 'spin'

$$s = N/2 - M$$

Taking the logarithm

- Logarithms: for every $\delta > 0$

$$K(x|\delta) = \frac{1}{2\pi i}\bigl(\operatorname{ctg}(x - i\delta) - \operatorname{ctg}(x + i\delta)\bigr)$$

$$\theta(x|\delta) = 2\pi i \int_{\Gamma_x} dy\, K(y|\delta)$$

$\theta(x|\delta)$ is defined in the cut complex plane with cuts along the line segments $(-\infty \pm i\delta, -\pi \pm i\delta] \cup [\pm i\delta, \pm i\delta + \infty)$. We write $K(x) = K(x|\gamma)$ and $\theta(x) = \theta(x|\gamma)$

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- We keep in mind the following properties of these functions

$$K(x|\delta) = \frac{1}{2\pi} \frac{\operatorname{sh}(2\delta)}{\operatorname{sh}^2(\delta) + \sin^2(x)} > 0\,, \quad \text{for } x \in \mathbb{R}$$

$$K(x + \pi|\delta) = K(x|\delta)\,, \quad K(-x|\delta) = K(x|\delta)$$

$$\theta(x + \pi|\delta) = \theta(x|\delta) + \begin{cases} 2\pi i & |\operatorname{Im} x| < \delta \\ 0 & |\operatorname{Im} x| > \delta \end{cases}$$

Taking the logarithm

- Using θ we can define the function $\ln \mathfrak{a}$ as

$$\ln \mathfrak{a}(x) = -\frac{\varepsilon_0^{(N)}(x)}{T} - \frac{N}{2}\theta(x + i\gamma/2 + i\beta/N) + \sum_{j=1}^{M} \theta(x - x_k^r)$$

where

$$\varepsilon_0^{(N)}(x) = h - \frac{NT}{2}\big[\theta(x + i\beta/N|\gamma/2) - \theta(x - i\beta/N|\gamma/2)\big]$$

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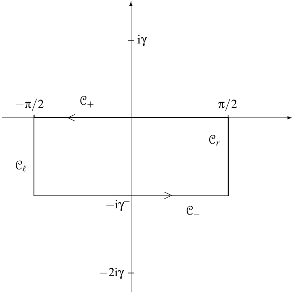
- Contour $\mathcal{C}$



x_k^f	far particle roots
x_k^c	close particle roots
x_k^h	hole roots

Singularities of $\partial_x \ln(1 + \mathfrak{a}(x))$ inside $\mathcal{C}$ are simple poles at x_k^r if x_k^r is not a particle root, at x_k^h and at $-i(\gamma/2 + \beta/N)$

Nonlinear integral equation and subsidiary conditions

- LEMMA. The auxiliary function $\mathfrak{a}$ satisfies the nonlinear integral equation

$$\ln \mathfrak{a}(x) = -\frac{\varepsilon_0^{(N)}(x)}{T} - \sum_{j=1}^{n_h} \theta(x - x_j^h) + \sum_{j=1}^{n_c} \big(\theta(x - x_j^c) + \theta(x - x_j^c + \mathrm{i}\gamma)\big) + \sum_{j=1}^{n_f} \theta(x - x_j^f) + d\theta(x + \pi/2) + \int_{\mathcal{C}} \mathrm{d}y\, K(x-y) \ln_{\mathcal{C}}(1+\mathfrak{a})(y)$$

  This equation determines $\mathfrak{a}$ directly inside the strip $-\gamma < \mathrm{Im}\, x < 0$ and, by analytic continuation, in the entire complex plane. For $x \in \mathcal{C}_\pm$ the integral term should be understood as an appropriate boundary value of a Cauchy-like operator

  The particles and holes have to be determined such that they satisfy the subsidiary conditions

$$1 + \mathfrak{a}\big(x_j^{h,p,f}\big) = 0$$

  with $x_j^{h,p,f}$ in their respective domains of definition

-

$$d = \int_{\mathcal{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \partial_y \ln(1 + \mathfrak{a}(y)) = n_h - 2n_c - n_f - s\,, \quad \ln_{\mathcal{C}} f(x) = \int_{\mathcal{C}_x} \mathrm{d}y\, \partial_y \ln f(y)$$

Low-T solution on the contour

- Send $N \to \infty$ and switch notation

$$u(x) = -T \ln \mathfrak{a}(x)$$

- LEMMA. In the strip $-\gamma < \mathrm{Im}\, x \le 0$ the nonlinear integral equations have self-consistent low-temperature solutions of the form

$$u(x) = u_1(x) + \mathcal{O}(T^\infty)$$

where

$$u_1(x) = \varepsilon(x) + T\Big\{ \mathrm{i}\pi k + \sum_{j=1}^{n_h} \varphi(x, x_j^h) - \sum_{j=1}^{n_c} \big(\varphi(x, x_j^c) + \varphi(x, x_j^c - \mathrm{i}\gamma)\big) - \sum_{j=1}^{n_f} \varphi(x, x_j^f) \Big\}$$

and where the numbers of particles and holes are related to the spin by the condition

$$n_h - 2n_c - 2n_f = 2s$$

Low-T solution on the contour

- Send $N \to \infty$ and switch notation

$$u(x) = -T \ln \mathfrak{a}(x)$$

- LEMMA. In the strip $-\gamma < \mathrm{Im}\, x \le 0$ the nonlinear integral equations have self-consistent low-temperature solutions of the form

$$u(x) = u_1(x) + \mathcal{O}(T^\infty)$$

where

$$u_1(x) = \varepsilon(x) + T\left\{\mathrm{i}\pi k + \sum_{j=1}^{n_h} \varphi(x, x_j^h) - \sum_{j=1}^{n_c} \left(\varphi(x, x_j^c) + \varphi(x, x_j^c - \mathrm{i}\gamma)\right) - \sum_{j=1}^{n_f} \varphi(x, x_j^f)\right\}$$

and where the numbers of particles and holes are related to the spin by the condition

$$n_h - 2n_c - 2n_f = 2s$$

- This being valid in the antiferromagnetic massive regime

$$0 < h < h_\ell = \frac{1}{\pi} 8JK \,\mathrm{sh}\left(\frac{\pi K'}{K}\right) \mathrm{dn}(K|k)$$

Momentum, dressed energy and dressed phase

- Momentum

$$p(x) = \frac{1}{4} + \frac{x}{2\pi} + \frac{1}{2\pi \mathrm{i}} \ln\left(\frac{\vartheta_4(x + \mathrm{i}\gamma/2|q^2)}{\vartheta_4(x - \mathrm{i}\gamma/2|q^2)}\right)$$

- Dressed energy

$$\varepsilon(x) = \frac{h}{2} - \frac{4JK\,\mathrm{sh}(\gamma)}{\pi}\,\mathrm{dn}\left(\frac{2Kx}{\pi}\middle|k\right)$$

- Dressed phase

$$\varphi(x_1, x_2) = \mathrm{i}\left(\frac{\pi}{2} + x_{12}\right) + \ln\left\{\frac{\Gamma_{q^4}\left(1 + \frac{\mathrm{i}x_{12}}{2\gamma}\right)\Gamma_{q^4}\left(\frac{1}{2} - \frac{\mathrm{i}x_{12}}{2\gamma}\right)}{\Gamma_{q^4}\left(1 - \frac{\mathrm{i}x_{12}}{2\gamma}\right)\Gamma_{q^4}\left(\frac{1}{2} + \frac{\mathrm{i}x_{12}}{2\gamma}\right)}\right\}$$

where $x_{12} = x_1 - x_2$ and $|\mathrm{Im}\, x_2| < \gamma$. For $|\mathrm{Im}\, z| > \gamma$ we have the explicit representation

$$\mathrm{e}^{\varphi(x,z)} = \begin{cases} \dfrac{\sin(x - z)}{\sin(x - z + \mathrm{i}\gamma)} & \text{if } \mathrm{Im}\, z > \gamma \\[2ex] \dfrac{\sin(x - z - \mathrm{i}\gamma)}{\sin(x - z)} & \text{if } \mathrm{Im}\, z < -\gamma. \end{cases}$$

Functional equations

- Γ_q is defined by the infinite product

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{n+x-1}}$$

  q-numbers are defines as

$$[x]_q = \frac{1-q^x}{1-q}$$

  Using q-numbers the fundamental recursion relation of the q-Γ functions becomes

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x)\,, \quad \Gamma_q(1) = 1$$

  It implies that the dressed phase obeys the functional equation

$$\mathrm{e}^{\varphi(x_1,x_2)+\varphi(x_1+\mathrm{i}\gamma,x_2)} = \frac{\sin(x_1-x_2)}{\sin(x_1-x_2+\mathrm{i}\gamma)}$$

- The dressed energy obeys the simpler relation

$$\varepsilon(x) + \varepsilon(x+\mathrm{i}\gamma) = h$$

Auxiliary function in the entire complex plane

- LEMMA. Low-temperature form of the auxiliary function in the complex plane.

$$\mathfrak{a}(x) = \begin{cases} e^{-\frac{1}{T}(u_1(x)+u_1(x-i\gamma))} & \operatorname{Im} x > \gamma \\ e^{-\frac{1}{T}u_1(x)} + e^{-\frac{1}{T}(u_1(x)+u_1(x-i\gamma))} & 0 < \operatorname{Im} x < \gamma \\ e^{-\frac{1}{T}u_1(x)} & -\gamma < \operatorname{Im} x < 0 \\ \left[e^{\frac{1}{T}u_1(x)} + e^{\frac{1}{T}(u_1(x)+u_1(x+i\gamma))}\right]^{-1} & -2\gamma < \operatorname{Im} x < -\gamma \\ e^{-\frac{1}{T}(u_1(x)+u_1(x+i\gamma))} & \operatorname{Im} x < -2\gamma \end{cases}$$

  up to multiplicative corrections of the form $1 + \mathcal{O}(T^\infty)$ (in front of each exponent)

- This lemma allows us to discuss the subsidiary conditions for $T \to 0$

Using the functional equations

- There are only two independent functions occurring on the right hand side. These can be expressed in terms of special functions. For this purpose we split the far roots into two sets $\{x_j^f\}_{j=1}^{n_f} = \{x_j^+\}_{j=1}^{n_+} \cup \{x_j^-\}_{j=1}^{n_-}$, where the x_j^+ have imaginary parts greater than γ while the x_j^- have imaginary parts less than $-\gamma$. Then

$$\mathfrak{a}^{(+)}(x) = \mathrm{e}^{-\frac{1}{T}(u_1(x)+u_1(x-\mathrm{i}\gamma))} = \mathrm{e}^{-\frac{h}{T}} \left[\prod_{j=1}^{n_h} \frac{\sin(x-x_j^h)}{\sin(x-x_j^h-\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_c} \frac{\sin(x-x_j^c-\mathrm{i}\gamma)}{\sin(x-x_j^c+\mathrm{i}\gamma)}\right]$$
$$\times \left[\prod_{j=1}^{n_+} \frac{\sin(x-x_j^+-\mathrm{i}\gamma)}{\sin(x-x_j^++\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_-} \frac{\sin(x-x_j^--2\mathrm{i}\gamma)}{\sin(x-x_j^-)}\right]$$

and

$$\mathfrak{a}^{(0)}(x) = \mathrm{e}^{-\frac{1}{T}u_1(x)} = (-1)^k \mathrm{e}^{-\frac{\varepsilon(x)}{T} - \sum_{j=1}^{n_h} \varphi(x,x_j^h)} \left[\prod_{j=1}^{n_c} \frac{\sin(x-x_j^c)}{\sin(x-x_j^c+\mathrm{i}\gamma)}\right]$$
$$\times \left[\prod_{j=1}^{n_+} \frac{\sin(x-x_j^+)}{\sin(x-x_j^++\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_-} \frac{\sin(x-x_j^--\mathrm{i}\gamma)}{\sin(x-x_j^-)}\right]$$

Root patterns for $h > 0$

- Based on the previous Lemma and on the explicit form of $\mathfrak{a}^{(+)}$ and $\mathfrak{a}^{(0)}$
  1. Far roots x_j^+ cannot exist
  2. Far roots x_j^- do not exist
  3. Close particles couple to holes in particle-hole strings

$$x_j^c = x_j^h + \mathrm{i}\gamma + \mathrm{i}\delta_j\,, \quad j = 1, \dots, n_c$$

where $\delta_j = \mathcal{O}(\mathrm{e}^{-h/T})$

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- The validity of 2 and 3 rests on the technical assumption that two roots of the same type cannot come exponentially close to each other. In other words: For all solutions satifying the above technical assumptions points 1 - 3 hold true

The higher-level Bethe Ansatz equations for $h > 0$

- LEMMA. Up to corrections of the order T^∞ the independent holes x_j, $j = 1, \dots, n_c + 2s$ and the particles in particle-hole strings y_ℓ, $\ell = 1, \dots, n_c$ are determined by the higher-level Bethe Ansatz equations

$$\frac{\varepsilon(x_j)}{T} = \pi \mathrm{i} n + \sum_{k=1}^{n_c} \varphi(x_j, y_k) - \sum_{k=1}^{n_c+2s} \varphi(x_j, x_k)$$

$$\frac{\varepsilon(y_\ell)}{T} = \pi \mathrm{i} m + \sum_{k=1}^{n_c} \varphi(y_\ell, y_k) - \sum_{k=1}^{n_c+2s} \varphi(y_\ell, x_k)$$

where n, m are even if k is odd, while n, m are odd if k is even, and where $-\gamma < \mathrm{Im}\, x_j < 0$, $0 < \mathrm{Im}\, y_\ell < \gamma$ by definition.

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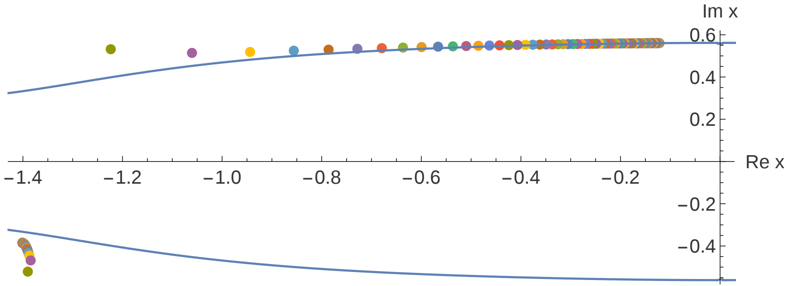
- For the calculation of correlation lengths we need to calculate integrals over $\mathcal{C}$ that involve the auxiliary function $\mathfrak{a}$. Our low-temperature picture implies

$$\mathfrak{a}(x) = (-1)^k \mathrm{e}^{-\frac{\varepsilon(x)}{T} + \sum_{k=1}^{n_c} \varphi(x, y_k) - \sum_{k=1}^{n_c+2s} \varphi(x, x_k)} \big(1 + \mathcal{O}(T^\infty)\big)$$

for $x \in \mathcal{C}$

Example particle-hole pairs



Single particle-hole pair excitations ($s = 0$, $n_c = 1$) according to the higher-level Bethe Ansatz equations. $T/J = 0.1$, $h/h_\ell = 2/3$, $\Delta = 1.7$, $h_\ell/J = 0.76$. Shown are particle-hole pairs for $n = 1$ fixed and m running from -1 to -70. The interaction with the particles slightly influences the hole position. The blue lines are the curves $\mathrm{Re}\,\varepsilon(x) = 0$.

Loci of particles and holes for T = 0

- In the limit $T \to 0_+$ at finite s and n_c the higher-level Bethe Ansatz equations decouple, $\mathrm{i}\pi n T$ and $\mathrm{i}\pi m T$ turn into independent continuous variables, and the particles and holes become free parameters on the curves

$$\mathrm{Re}\,\varepsilon(y) = 0\,, \quad 0 < \mathrm{Im}\, y < \gamma, \qquad \mathrm{Re}\,\varepsilon(x) = 0\,, \quad -\gamma < \mathrm{Im}\, x < 0$$

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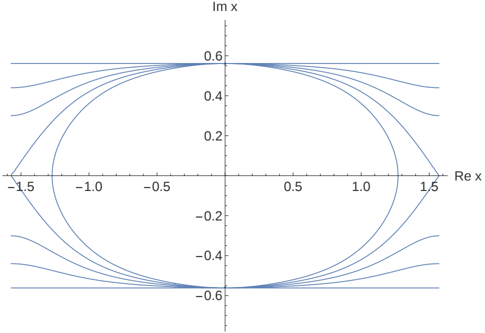
- 

Curves $\mathrm{Re}\,\varepsilon(x) = 0$ for $h/h_\ell = 1.34, 1, 2/3, 1/3, 0$ and $\Delta = 1.7$, $h_\ell/J = 0.76$. The massive regime is distinguished from the massless regime by the opening of a 'band gap' at the critical field h_ℓ.

Dominant state

- For $s = n_h = 0$ there are no higher-level Bethe Ansatz equations, and the auxiliary function is $\mathfrak{a}(x) = \pm \mathrm{e}^{-\varepsilon(x)/T}$. Corresponding Bethe roots are determined by $\mathfrak{a}(x) = \mp 1$ for $-\gamma < \mathrm{Im}\, x < 0$, or

$$\varepsilon(x) = \mathrm{i}\pi n T\,, \quad -\gamma < \mathrm{Im}\, x < 0$$

where the n are odd integers if $\mathfrak{a}(x) = \mathrm{e}^{-\varepsilon(x)/T}$ and even integers else

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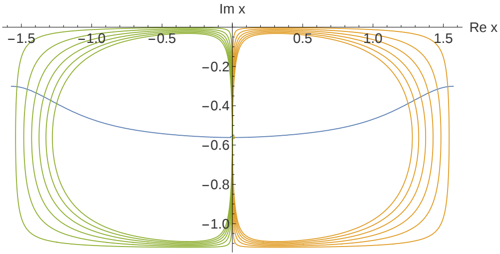
- 

  Bethe roots of the dominant state depicted as the intersections of the curves $\operatorname{Re}\varepsilon(x) = 0$ and $\operatorname{Im}\varepsilon(x) = n\pi T$ for $T/J = 0.01$, $n = \pm 1, \pm 3, \dots, \pm 11$, $h/h_\ell = 2/3$, $\Delta = 1.7$, $h_\ell/J = 0.76$

Eigenvalues

- Starting from the equation that expresses the quantum transfer matrix eigenvalues in terms of Bethe roots and employing a similar reasoning as in the derivation of the nonlinear integral equations we obtain the representation

$$\Lambda(x) = \left(\frac{\cos(\mathrm{i}\gamma/2 + x)}{\cos(\mathrm{i}\gamma/2 - x)}\right)^d \left[\prod_{j=1}^{n_h} \frac{\sin(x - x_j^h - \mathrm{i}\gamma/2)}{\sin(x - x_j^h + \mathrm{i}\gamma/2)}\right] \left[\prod_{j=1}^{n_c} \frac{\sin(x - x_j^c + 3\mathrm{i}\gamma/2)}{\sin(x - x_j^c - \mathrm{i}\gamma/2)}\right] \times \exp\left\{\frac{h}{2T} - \int_{\mathcal{C}} \mathrm{d}y\, K(x - y|\gamma/2) \ln_{\mathcal{C}}(1 + \mathfrak{a}(y))\right\}$$

valid for $-\gamma/2 < \operatorname{Im} x < \gamma/2$. This is the general expression for Λ, still valid for any temperature and magnetic field $h \geq 0$, in the case that there are no far roots

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- At low T

$$\Lambda(x) = \left(\frac{\cos(\mathrm{i}\gamma/2 + x)}{\cos(\mathrm{i}\gamma/2 - x)}\right)^d \left[\prod_{j=1}^{n_c+2s} \frac{\sin(x - x_j - \mathrm{i}\gamma/2)}{\sin(x - x_j + \mathrm{i}\gamma/2)}\right] \left[\prod_{j=1}^{n_c} \frac{\sin(x - y_j + \mathrm{i}\gamma/2)}{\sin(x - y_j - \mathrm{i}\gamma/2)}\right] \\ \times \exp\left\{\frac{h}{2T} + \int_{-\pi/2}^{\pi/2} \mathrm{d}y\, K(x-y|\gamma/2) \ln \mathfrak{a}(y)\right\} \left(1 + \mathcal{O}(T^\infty)\right)$$

Main result

- THEOREM. For small T the following eigenvalue ratios ρ_ℓ occurr in the antiferromagnetic massive regime at finite magnetic field

$$\rho_\ell = (-1)^k \exp\Big\{2\pi i\Big[\sum_{j=1}^{n_c} p(y_j) - \sum_{j=1}^{n_c+2s} p(x_j)\Big]\Big\}$$

$$= (-1)^k \Bigg[\prod_{j=1}^{n_c} \frac{\vartheta_1(y_j - i\gamma/2|q^2)}{\vartheta_4(y_j - i\gamma/2|q^2)}\Bigg]\Bigg[\prod_{j=1}^{n_c+2s} \frac{\vartheta_4(x_j - i\gamma/2|q^2)}{\vartheta_1(x_j - i\gamma/2|q^2)}\Bigg]$$

this being valid up to multiplicative corrections of the order $\big(1 + \mathcal{O}(T^\infty)\big)$. Here y_j and x_k are particle and holes determined by the higher-level Bethe Ansatz equations

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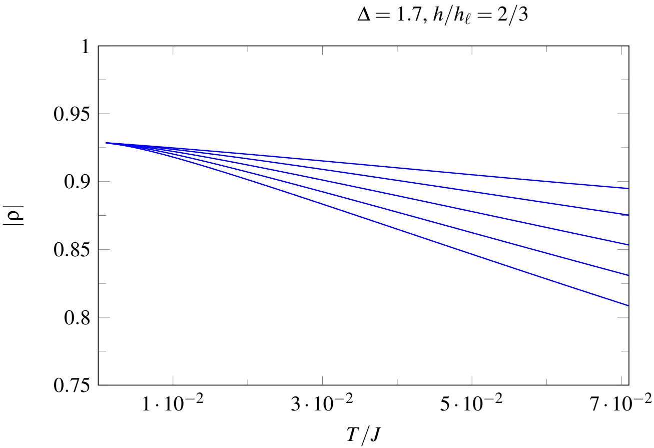
  this being valid up to multiplicative corrections of the order $\big(1 + \mathcal{O}(T^\infty)\big)$. Here y_j and x_k are particle and holes determined by the higher-level Bethe Ansatz equations

- CONJECTURE. All eigenvalue ratios are of this form, i.e. all of them are parameterized by particle-hole excitations

Large-distance asymptotics



The behaviour of $|\rho|$ as a function of temperature for one hole and one particle. The particle and hole roots are obtained from the higher-level Bethe Ansatz for $n = 1$, $m = -1, -3, -5, -7, -9$ (from top to bottom). The parameters are chosen as $h/h_\ell = 2/3$, $\Delta = 1.7$, $h_\ell/J = 0.76$, $\alpha = 0$

Summary and further outlook

- Summary
  - We have analyzed the spectrum of the quantum transfer matrix of the XXZ chain in the massive antiferromagnetic regime at finite magnetic field and for small temperature
  - We have obtained an explicit formula for an infinite set of correlation lengths, parameterized by solutions of a set of higher level Bethe Ansatz equations
  - These solutions habe been interpreted as particle-hole excitations
  - For $T \to 0$ the particle and hole parameters become free on two curves in the complex plane (no higher leves Bethe equations remain!)
  - We conjecture that for $h > 0$ ALL correlation lengths are of this form

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  - If $h = 0$ the following equations remain for $T \to 0$

$$-1 = \left[\prod_{k=1}^{n_h} \frac{\sin(\chi_j - x_k^h + i\gamma/2)}{\sin(\chi_j - x_k^h - i\gamma/2)}\right] \left[\prod_{j=1}^{n_\chi} \frac{\sin(\chi_j - \chi_k - i\gamma)}{\sin(\chi_j - \chi_k + i\gamma)}\right]$$

$j = 1, \dots, n_\chi$, where

$$n_\chi = n_c + n_f = \frac{n_h}{2} - s$$

and where we performed the following change of variables

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- Outlook
  - Calculate amplitudes and analyze form-factor series