Towards combinatorics of elliptic lattice models

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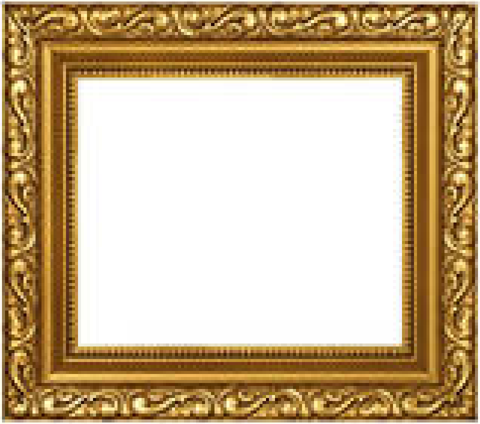
A missing big picture



Outline

1. Introduction
2. Eigenvectors (Mangazeev & Bazhanov 2010, Razumov & Stroganov 2010, Zinn-Justin 2013)
3. Eigenvalues of Q-operator (Bazhanov & Mangazeev 2005, 2006)
4. Three-coloured chessboards (R. 2011)
5. Towards a synthesis (R., to appear)

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Main point

All three contexts

- eigenvectors
- eigenvalues of Q-operator
- domain wall partition functions

lead to polynomials that

- have positive coefficients
- are Painlevé tau functions

We know what these polynomials "are",
but conceptual explanations are still lacking.

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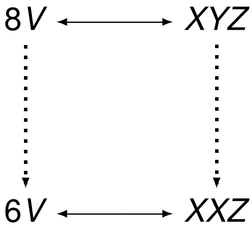
Solvable lattice models

$$6V \longleftrightarrow XXZ$$

Solvable lattice models



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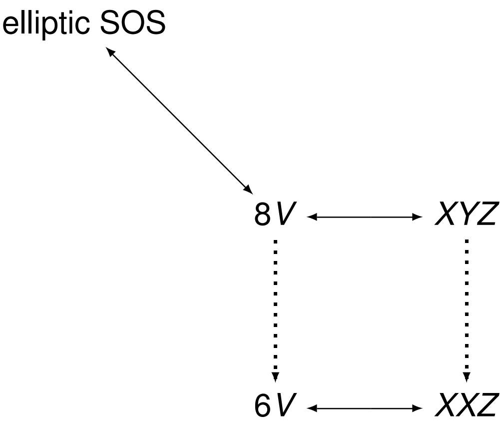

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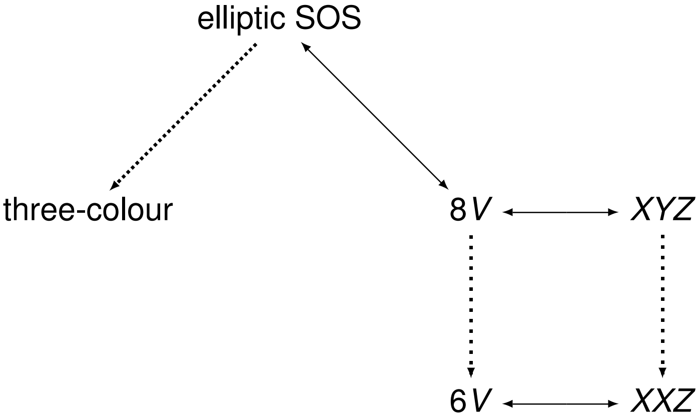

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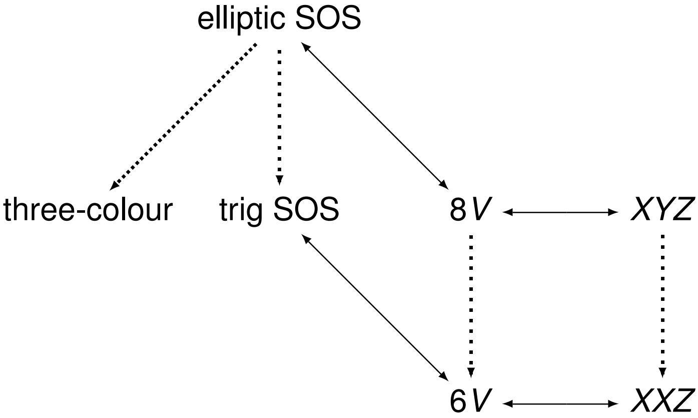

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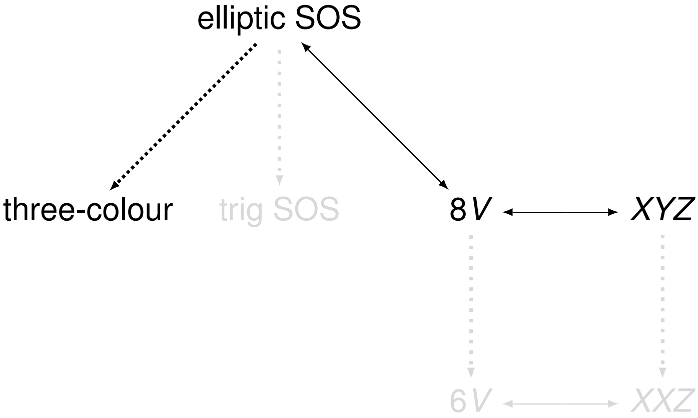

elliptic models

"Combinatorial" parameter values

$\Delta = 1/2$:
ASM enumeration, three-colourings etc.

$\Delta = -1/2$: supersymmetry
Magic in spectra, Razumov–Stroganov etc.

$\Delta = 0$: free fermions
Domino tilings, arctic circle etc.

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XYZ spin chain

Hamiltonian acting on $(\mathbb{C}^2)^{\otimes N}$,

$$\mathbf{H} = -\frac{1}{2}\sum_{j=1}^{N}\left(J_x\sigma_x^j\sigma_x^{j+1} + J_y\sigma_y^j\sigma_y^{j+1} + J_z\sigma_z^j\sigma_z^{j+1}\right);$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Periodic boundary conditions: $\sigma^{N+1} = \sigma^1$.

If N is odd and

$$J_xJ_y + J_xJ_z + J_yJ_z = 0$$

$(\Delta = -1/2)$ then $\mathbf{H}$ has lowest eigenvalue

$$-\frac{N}{2}(J_x + J_y + J_z).$$

Observed by Stroganov (2001), proved by Hagendorf (2013).

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Ground state eigenvectors

Consider cyclically symmetric eigenvector Ψ
in sector $e_\pm \otimes \cdots \otimes e_\pm$ with even number of plus signs.
Unique up to normalization.

Razumov & Stroganov observed that if

$$J_x = 1 + \zeta, \qquad J_y = 1 - \zeta, \qquad J_z = \frac{\zeta^2 - 1}{2},$$

then

$$\Psi = \sum_{k_1 \cdots k_N \in \{\pm\}} \Psi_{k_1 \cdots k_N} e_{k_1} \otimes \cdots \otimes e_{k_N},$$

where $\Psi_{k_1 \cdots k_N}$ seem to be polynomials in ζ with positive integer coefficients.

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Example: N=7

$$\begin{aligned}\Psi_{-+-+-++} &= 7+\zeta^2,\\ \Psi_{--+-+++} &= 3+5\zeta^2,\\ \Psi_{---++++} &= 1+5\zeta^2+2\zeta^4,\\ \Psi_{--++-++} &= 4+3\zeta^2+\zeta^4.\end{aligned}$$

All other components are equal to one of these four, up to multiplication by ζ or ζ^2.

Conjectures

There are polynomials s_n, $\bar{s}_n$, given by explicit recursions, such that

$$\Psi_{--\cdots-} = \zeta^{n(n+1)/2} s_n(\zeta^{-2}), \qquad \Psi_{+\cdots+-} = N^{-1}\zeta^{n(n-1)/2}\bar{s}_n(\zeta^{-2}),$$

where $N = 2n+1$.

Sum rule

$$\sum_k \Psi^2_{k_1\cdots k_N} = (4/3)^n \zeta^{n(n+1)} s_n(\zeta^{-2}) s_{-n-1}(\zeta^{-2}),$$

where s_n is naturally extended to $n < 0$.
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More conjectures

There are polynomials q_n, r_n, given by explicit recursions, such that for n even ($N = 2n + 1$)

$$\Psi_{-+-+-\cdots+-} = \mathrm{Const}_n(\zeta(3+\zeta))^{\frac{n(n-2)}{4}} r_{\frac{n-2}{2}}\left(\frac{1-\zeta}{3+\zeta}\right) q_{\frac{n-2}{2}}(\zeta^{-1}).$$

and for n odd

$$\Psi_{+-+-\cdots+-+} = \mathrm{Const}_n(\zeta(3+\zeta))^{\frac{n^2-1}{4}} r_{\frac{n-1}{2}}\left(\frac{1-\zeta}{3+\zeta}\right) q_{\frac{n-3}{2}}(\zeta^{-1}).$$

Factorizations

$$s_{2n+1}(y^2) = \mathrm{Const}_n\, r_n(y) r_n(-y),$$

$$s_{2n}(y^2) = \mathrm{Const}_n(1+3y)^{n(n+1)} r_{-n-1}\left(\frac{y-1}{3y+1}\right) q_{n-1}(y).$$

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Properties

All the polynomials seem to have positive integer coefficients (for all $n \in \mathbb{Z}$ or $n \in \mathbb{Z}_{\geq 0}$):

$$
\begin{aligned}
s_3(y) &= 1 + 3y + 4y^2, \\
\bar{s}_3(y) &= 7(5 + 3y), \\
q_3(y) &= 1 + 15y^2 + 112y^4 + 518y^6 + 1257y^8 + 1547y^{10} \\
&\quad + 646y^{12}, \\
r_3(y) &= 1 + 3y + 15y^2 + 35y^3 + 105y^4 + 195y^5 + 435y^6 \\
&\quad + 555y^7 + 840y^8 + 710y^9 + 738y^{10} + 294y^{11} + 170y^{12}.
\end{aligned}
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All the polynomials are tau functions of Painlevé VI (explained later).

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Q-operator

Same setting: periodic XYZ chain of odd length N.
Hamiltonian $\mathbf{H}$ commutes with transfer matrix $\mathbf{T}(z)$ of the eight-vertex model and with Q-operators $\mathbf{Q}(z)$.

$$\mathbf{T}(z)\mathbf{Q}(z) = \phi(z-\eta)\mathbf{Q}(z+2\eta) + \phi(z+\eta)\mathbf{Q}(z-2\eta),$$

$\phi(z) = \theta_1(z|e^{2\pi i\tau})^N$, τ and η are parameters.
$\Delta = -1/2$ means $\eta = \pi/3$.
Evaluate at ground state eigenvector Ψ.
$\mathbf{T}(z)$ has (conjecturally?) eigenvalue $\phi(z)$.
Eigenvalue $Q(z)$ of $\mathbf{Q}(z)$ satisfies

$$\phi(z)Q(z) = \phi(z-\eta)Q(z+2\eta) + \phi(z+\eta)Q(z-2\eta)$$

or equivalently

$$(\phi Q)(z) + (\phi Q)(z+2\eta) + (\phi Q)(z-2\eta) = 0.$$

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Solution space of the *TQ* relation

Solution space of *TQ*-relation, with appropriate analytic properties, is two-dimensional. Basis $Q(z)$, $Q(z+\pi)$.

Writing $\Psi(z) = \phi(2\pi z)Q(2\pi z)$,

- Ψ is entire,
- $\Psi(z+1) = \Psi(z)$, $\quad \Psi(z+\tau) = e^{-6\pi i N(2z+\tau)}\Psi(z)$
  $\Psi(-z) = \Psi(z)$,
- $\Psi(z) + \Psi(z+1/3) + \Psi(z-1/3) = 0$,
  i.e. $\Psi(z) = \sum_{n \equiv \pm 1 \mod 3} \psi_n e^{2\pi i n z}$
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The space of functions satisfying these conditions is one-dimensional.

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Uniformization

Up to elementary multiplier, Ψ is meromorphic function on

$$(\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}))/(z=-z).$$

This is a sphere. Thus, up to elementary factor,
Ψ is polynomial in some variable $x = x(z,\tau)$.

As a function of τ, Ψ can be normalized to live on modular curve $\Gamma_0(6)$.
This is a sphere, so Ψ is also polynomial in $\zeta = \zeta(\tau)$.

Can express eigenvalue $Q(z)$ in terms of polynomial $\mathcal{P}_n(x,\zeta)$.

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The polynomials $\mathcal{P}_n(x,\zeta)$

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\begin{aligned}
\mathcal{P}_0 &= 1,\\
\mathcal{P}_1 &= x+3,\\
\mathcal{P}_2 &= (\zeta+1)x^2+5(3\zeta+1)x+10,\\
\mathcal{P}_3 &= (4\zeta^2+3\zeta+1)x^3+7(18\zeta^2+5\zeta+1)x^2\\
&\quad +7(18\zeta^2+19\zeta+3)x+7(3\zeta+5),\\
&\dots
\end{aligned}
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$\mathcal{P}_n$ seems to have positive coefficients.

$\mathcal{P}_n$ satisfies a quantization of Painlevé VI
(non-stationary Lamé equation). Explained below.

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A remarkable coincidence?

$$\mathcal{P}_3 = (4\zeta^2 + 3\zeta + 1)x^3 + 7(18\zeta^2 + 5\zeta + 1)x^2 + 7(18\zeta^2 + 19\zeta + 3)x + 7(3\zeta + 5)$$

The highest and lowest coefficients in $\mathcal{P}_n$ are $s_n(\zeta)$ and $\bar{s}_n(\zeta)$. Not clear why the same polynomials appear also in the eigenvector and in the sum rule.

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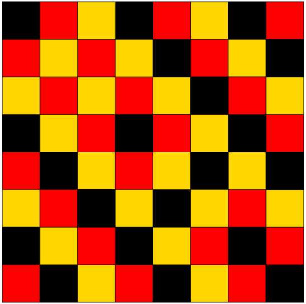
Three-coloured chessboards



Rules

0	1	2	0
1	2	1	2
2	1	2	1
0	2	1	0

Chessboard of size $(n+1) \times (n+1)$.
Paint squares with three colours
$0, 1, 2 \mod 3$.

- Adjacent squares have distinct colour.
- “Domain wall boundary conditions” (DWBC). Read entries mod 3.

Rules

0	1	2	0
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0	1	2	$\cdots$	n
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2				$\vdots$
$\vdots$				2
				1
n	$\cdots$	2	1	0

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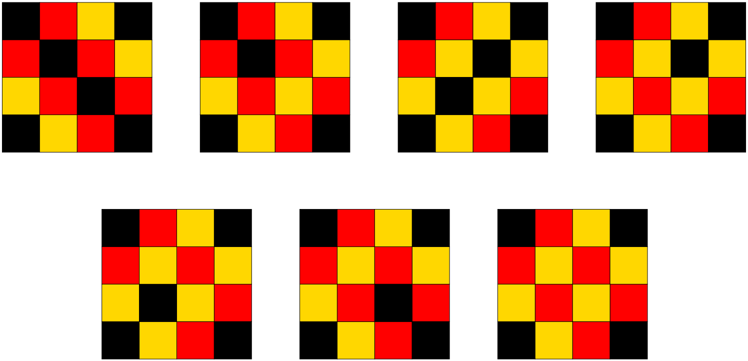
Example

When $n = 3$ there are seven chessboards.
$0 =$ black, $1 =$ red, $2 =$ yellow.



Bijection to “square ice" (=Alternating sign matrices)

0	1	2	0
1	2	1	2
2	1	2	1
0	2	1	0

Put arrows between adjacent entries.
Larger entry to the right,
$0 < 1 < 2 < 0$.

"Rock – Paper – Scissors"

- Each vertex has two incoming and two outgoing edges.
- Domain wall boundary conditions.

Vertex = oxygen, incoming edge = hydrogen.

Bijection to “square ice"
(=Alternating sign matrices)

$$\begin{array}{ccccccc}
0 & \uparrow & 1 & \uparrow & 2 & \uparrow & 0 \\
\rightarrow & & \rightarrow & & \leftarrow & & \leftarrow \\
1 & \uparrow & 2 & \downarrow & 1 & \uparrow & 2 \\
\rightarrow & & \leftarrow & & \rightarrow & & \leftarrow \\
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\end{array}$$

Put arrows between adjacent entries.
Larger entry to the right, $0 < 1 < 2 < 0$.

"Rock – Paper – Scissors"

- Each vertex has two incoming and two outgoing edges.
- Domain wall boundary conditions.

Vertex = oxygen, incoming edge = hydrogen.

Bijection to “square ice" (=Alternating sign matrices)

$$
\begin{array}{ccccccc}
0 & \uparrow & 1 & \uparrow & 2 & \uparrow & 0 \\
\rightarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow \\
1 & \uparrow & 2 & \downarrow & 1 & \uparrow & 2 \\
\rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow \\
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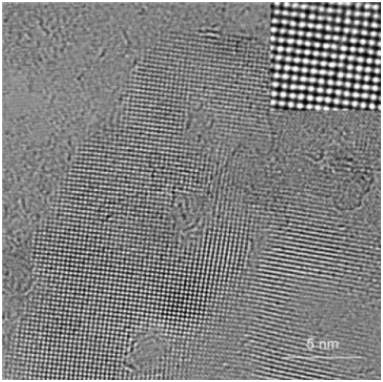
Square ice exists!

G. Algara-Siller et al., *Square ice in graphene nanocapillaries*, Nature (2015).



ASM Theorem

Three-coloured chessboards are in bijection with alternating sign matrices. Their number is

$$Z_n(1,1,1) = \frac{1!\,4!\,7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}.$$

Kuperberg found a proof of this using the six-vertex model.

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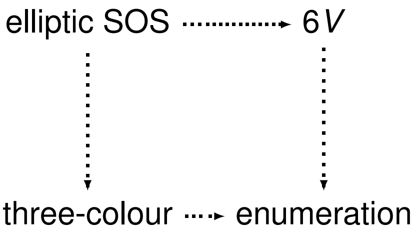

Three-colour model

Domain wall partition functions
= Generating function for colours:

$$Z_n^{3C}(t_0, t_1, t_2) = \sum_{\substack{\text{chessboards}\\ \text{of size } (n+1)\times(n+1)}} t_0^{\#\text{ squares coloured } 0}\, t_1^{\#\text{ squares coloured } 1}\, t_2^{\#\text{ squares coloured } 2}$$

Elliptic SOS model

Inhomogeneous domain wall partition function

$$Z_n^{\mathrm{SOS}}(x_1,\dots,x_n;y_1,\dots,y_n;p,q,\lambda),$$

x_j, y_j spectral parameters,
λ parameter of face weight,
$p=e^{2\pi \mathrm{i}\tau}$, $q=e^{2\pi \mathrm{i}\eta}$ further parameters.

With $\omega=e^{2\pi \mathrm{i}/3}$,

$$Z_n^{\mathrm{SOS}}(\omega,\dots,\omega;1,\dots,1;p,\omega,\lambda)$$
$$= \text{elementary factor} \times Z_n^{3C}(t_0,t_1,t_2),$$

$$t_j=\frac{1}{\theta_1(\lambda+2\pi j/3;p)^3}.$$

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Specialized SOS partition function

Keep x_1 free, but specialize other parameters as above.
As function of x_1, Z_n^{SOS} satisfies similar analytic conditions as eigenvalue $Q(z)$.

Specialized Z_n^{SOS} can be expressed in terms of $\mathcal{P}_n(x, \zeta)$.
Z_n^{3C} can be expressed in terms of $\mathcal{P}_n(x, \zeta)$ for special x.

Relates 8V model with $\Delta = -1/2$ on chain of length $2N + 1$ to SOS model with $\Delta = +1/2$ on $(N + 1) \times (N + 1)$ square.

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Polynomials p_n

The domain wall three-colour partition function Z_n^{3C} can be expressed in terms of polynomials p_{n-1}.

n	$p_n(\zeta)$
0	1
1	$3\zeta + 1$
2	$5\zeta^3 + 15\zeta^2 + 7\zeta + 1$
3	$\frac{1}{2}(35\zeta^6 + 231\zeta^5 + 504\zeta^4 + 398\zeta^3 + 147\zeta^2 + 27\zeta + 2)$
4	$\frac{1}{2}(63\zeta^{10} + 798\zeta^9 + 4122\zeta^8 + 11052\zeta^7 + 16310\zeta^6 + 13464\zeta^5 + 6636\zeta^4 + 2036\zeta^3 + 387\zeta^2 + 42\zeta + 2)$

Seem to have positive coefficients.

Known to be Painlevé tau functions (see below).

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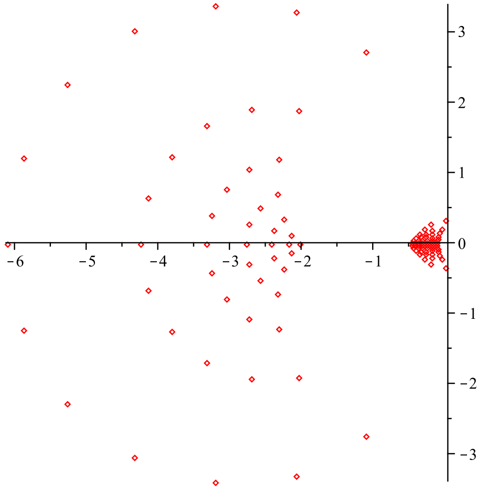
The 105 complex zeroes of p_{14}.



Relation between Z_n^{3C} and p_{n-1}

Suppose $n \equiv 0 \mod 6$ and

$$\frac{(t_0t_1+t_0t_2+t_1t_2)^3}{(t_0t_1t_2)^2}=\frac{2(\zeta^2+4\zeta+1)^3}{\zeta(\zeta+1)^4}$$

$$Z_n^{3C}(t_0,t_1,t_2)=(t_0t_1t_2)^{\frac{n(n+2)}{3}}\left(\frac{2}{\zeta(\zeta+1)^4}\right)^{\frac{n^2}{12}}$$

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$$\left.-\frac{t_0t_1t_2(\zeta^2+4\zeta+1)}{t_0t_1+t_0t_2+t_1t_2}\frac{p_{n-1}(\zeta)-\zeta^{\frac{n^2}{2}}p_{n-1}(1/\zeta)}{1-\zeta^2}\right)$$

Doesn't explain why p_n has positive coefficients.

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Outline

1. Introduction
2. Eigenvectors (Mangazeev & Bazhanov 2010, Razumov & Stroganov 2010, Zinn-Justin 2013)
3. Eigenvalues of Q-operator (Bazhanov & Mangazeev 2005, 2006)
4. Three-coloured chessboards (R. 2011)
5. Towards a synthesis (R., to appear)

A space of theta functions

Consider space V of functions that are analytic except for possible poles at $(1/6)\mathbb{Z} + (\tau/2)\mathbb{Z}$, such that

- $f(z+1) = f(z), \qquad f(z+\tau) = e^{-6\pi i n(2z+\tau)} f(z),$
  $f(-z) = -f(z),$
- $f(z) + f(z+1/3) + f(z-1/3) = 0,$
- $\lim_{z\to\gamma_j} (z-\gamma_j)^{1-2k_j} f(z) = 0,$
  $\lim_{z\to\gamma_j} (z-\gamma_j)^2 \left(f(z+1/3) + f(-z+1/3)\right) = 0.$

Here,

$$\gamma_0 = 0, \quad \gamma_1 = \frac{\tau}{2}, \quad \gamma_2 = \frac{\tau+1}{2}, \quad \gamma_3 = \frac{1}{2}.$$

The integers n, k_0, k_1, k_2, k_3 can be negative, but we assume $m = 2n - \sum_j k_j \geq 0$. Then, $\dim V = m$.

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Uniformization

Uniformizing the one-dimensional space $V^{\wedge m}$, we obtain functions

$$T_n^{(k_0,k_1,k_2,k_3)}(x_1,\dots,x_m;\zeta).$$

Can normalize them to be symmetric polynomials in x_j and polynomials in ζ.

Increasing $k_j \mapsto k_j + 1$ corresponds to specializing one of the variables to γ_j.

Permuting k_j corresponds to rational transformation of variables.

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Special cases

The following polynomials agree
(up to elementary prefactor and change of variables)

		m
$\mathcal{P}_n$	$T_n^{(n,n,0,-1)}$	1
p_n	$T_n^{(n+1,n,0,-1)}$	0
s_n	$T_n^{(n,n,0,0)}$	0
$\bar{s}_n$	$T_n^{(n,n,1,-1)}$	0
q_n	$T_{n+1}^{(0,2n+2,0,0)}$	0
r_n	$T_n^{(-1,2n+1,0,0)}$	0

Example: N=7 eigenvectors

$$\Psi_{-+-+-++} = 7 + \zeta^2, \qquad \Psi_{--+-+++} = 3 + 5\zeta^2,$$
$$\Psi_{---++++} = 1 + 5\zeta^2 + 2\zeta^4, \qquad \Psi_{--++-++} = 4 + 3\zeta^2 + \zeta^4.$$

If $\zeta^2 = 2(y + y^{-1}) + 5$,

$$3 + 5\zeta^2 \sim T_3^{(3,3,1,-1)}(y),$$
$$4 + 3\zeta^2 + \zeta^4 \sim T_3^{(3,3,0,0)}(y).$$

If $\zeta = (y + 2)/y$,

$$7 + \zeta^2 \sim T_1^{(0,3,-1,0)}(y)T_1^{(0,2,0,0)}(y),$$
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Properties of $T_n^{(k_0,k_1,k_2,k_3)}$

- Explicit Izergin–Korepin-type determinant formulas.
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- $T_n^{(k_0,k_1,k_2,k_3)}$ gives solution to Schrödinger equation

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Case $m = 1$ appears in several contexts:

- KZB heat equation from CFT (Bernard, Etingof & Kirillov).
- Radial part of $\widehat{\mathfrak{sl}}(2)$ Casimir operator (Kolb).
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Painlevé VI

Painlevé VI is the most general second order ODE such that all movable singularities are poles.

Painlevé VI for $q = q(t)$ is equivalent to Hamiltonian system

$$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}, \qquad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q},$$

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Bäcklund transformations

Painlevé VI has a group of symmetries
(Bäcklund transformations) containing $\mathbb{Z}^4$.

Knowing one solution, we can create $\mathbb{Z}^4$ lattice of solutions.

Tau functions

Tau functions satisfy

$$\frac{\tau'}{\tau} = H(p(t), q(t), t),$$

where $p(t), q(t)$ solve Painlevé VI.

Formally

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- The case $m = 0$ (depending only on ζ) are tau functions of Painlevé VI.

  They are precisely the solutions obtained from a known algebraic solution of Picard, acting with the $\mathbb{Z}^4$ lattice of Bäcklund transformations.

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Proof that case $m = 0$ are Painlevé tau functions (sketch)

Jacobi–Desnanot relation for determinants
$\Longrightarrow$ Differential recursions for $T_n^{(k_0,k_1,k_2,k_3)}$.
Involve derivatives in x_j.

Schrödinger equation $\Longrightarrow$
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Application of Painlevé connection

Each of the systems p_n, q_n, r_n, s_n, $\bar{s}_n$ satisfies a bilinear recursion like

$$p_{n+1}(\zeta)p_{n-1}(\zeta) = A_n(\zeta)p_n(\zeta)^2 + B_n(\zeta)p_n(\zeta)p_n'(\zeta) + C_n(\zeta)p_n'(\zeta)^2 + D_n(\zeta)p_n(\zeta)p_n''(\zeta),$$

with explicit coefficients.

For p_n, this gives a fast way of computing Z_n^{3C}.

Easily gives conjecture for the free energy $\lim_{n\to\infty} \log(Z_n^{3C})/n^2$. Can probably be used to prove this conjecture.

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- Tau functions for one very special solution of Painlevé VI are specialized affine Lie algebra characters.
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