Counting Perfect Matchings In Graphs

with Application in monomer dimer models

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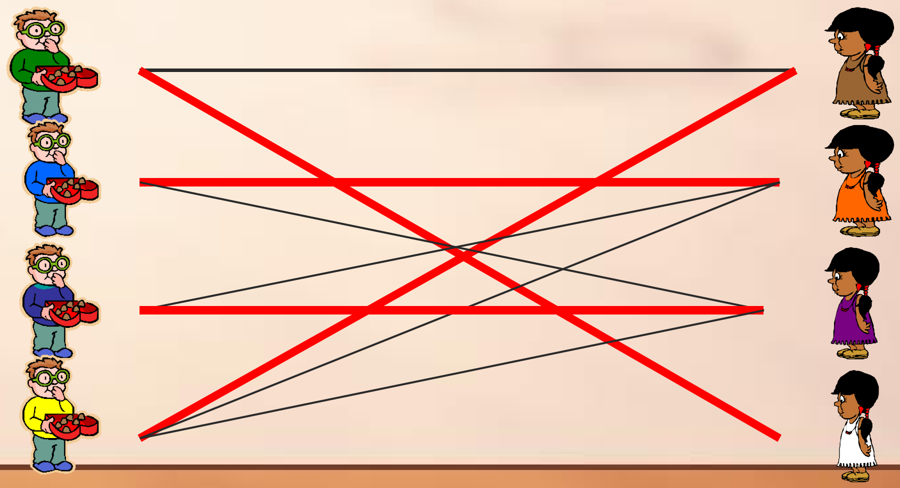
Definition

Å Let G be simple graph. A matching M in G is the set of pair wise non adjacent edges , that is ,no two edges share a common vertex.

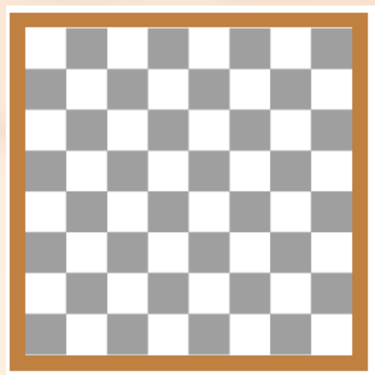
Å Every edges of M is called dimer. If the vertex v not covered by M is called monomer .

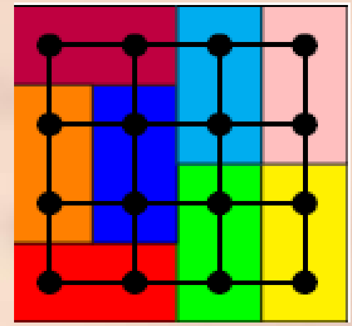
Å If every vertex from G is incident with exactly one edge from M , the matching is perfect. The number of perfect matchings in a given graph is denoted by Pm(G)

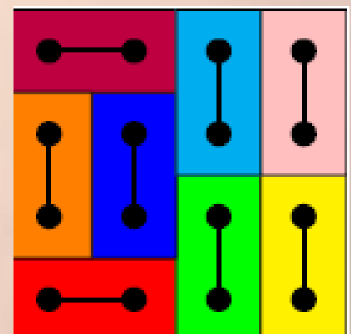


Application

Dominoes problem





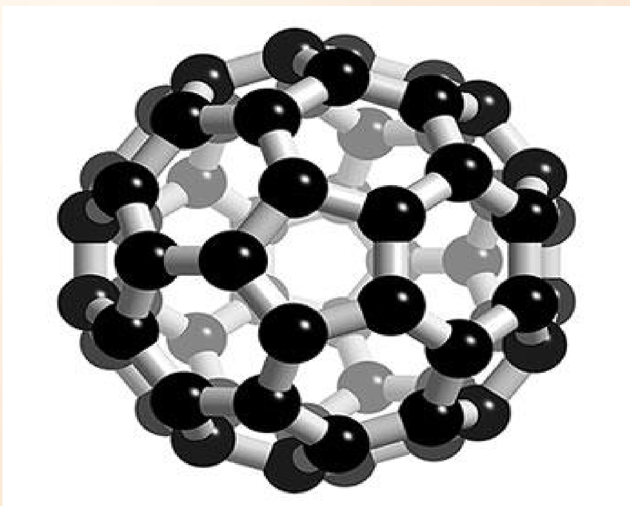


$$\mathrm{Pm}(P_m \times P_n) = 2^{mn/2} \prod_{k=1}^{m} \prod_{l=1}^{n} \left(\cos^2\left(\frac{\pi k}{m+1}\right) + \cos^2\left(\frac{\pi l}{n+1}\right) \right)^{\frac{1}{4}}.$$

Introduction-Fullerene graphs:

Å A fullerene graph is a 3-regular 3-connected planar graph with pentagon or hexagon faces.

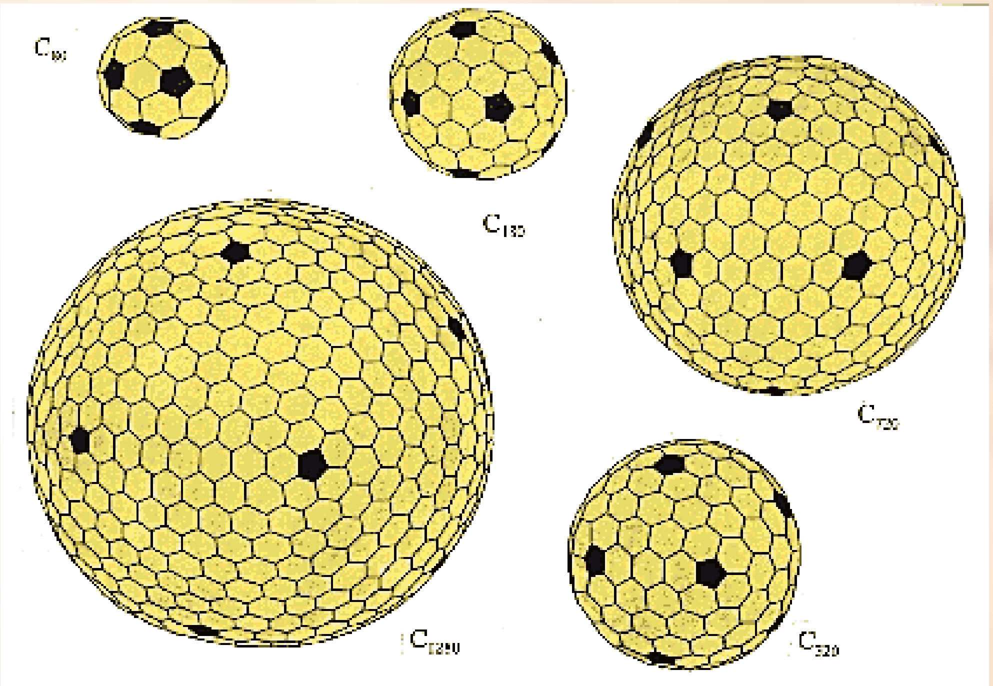
Å In chemistry, fullerene is a molecule consisting entirely of carbon atoms. Each carbon is three-connected to other carbon atoms by one double bond and two single bonds.

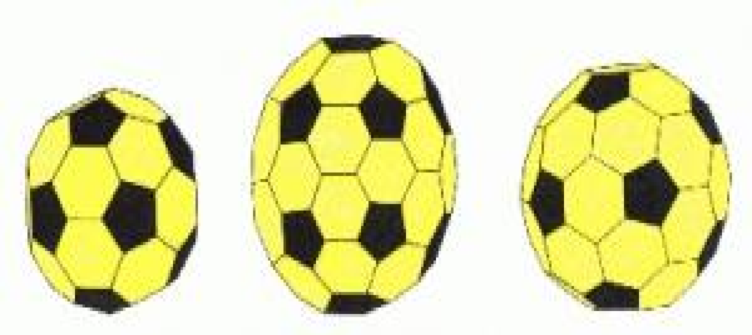


Introduction-Fullerene graphs:

Å By the Eulerôs formula n ī m + f = 2, one can deduce that :

$p = 12$, $v = 2h + 20$ and $m = 3h + 30$.

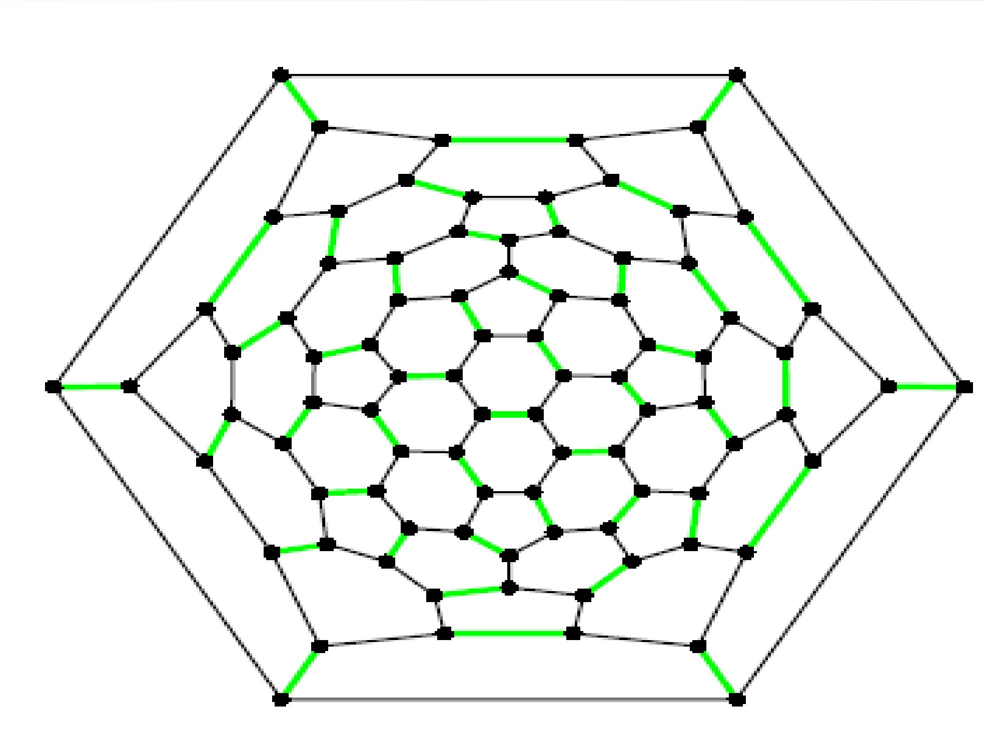




Introduction- matchings in molecular graph:

- If two fullerene graphs G and H have the same vertices then:

  pm(G)≥ pm(H) => G is more stable than H



Matching in fullerene

- Doslic in 1998 prove that every fullerene graph have at least n/2+1 perfect matching.

- H Zhang &F Zhang in 2001 prove that every fullerene graph have at least [3(n+2)/4] perfect matching.

- Theorem (Kardos, Kral', Miskuf and Sereni, 2008).
  Every fullerene graph with p vertices has at least $2^{(p-380)/61}$ perfect matching

pfaffian of matrices:

- Let A be n×n skew symmetric matrix. It is well known in linear algebra that if n is odd then:

det(A)=0

- For skew symmetric matrix of size 4 we have:

$$\det(A) = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2$$

- In general case we have this theorem from Cayley:

Theorem 1.1: for any n×n skew symmetric matrix A, we have:

$$\det(A) = (pf(A))^2$$

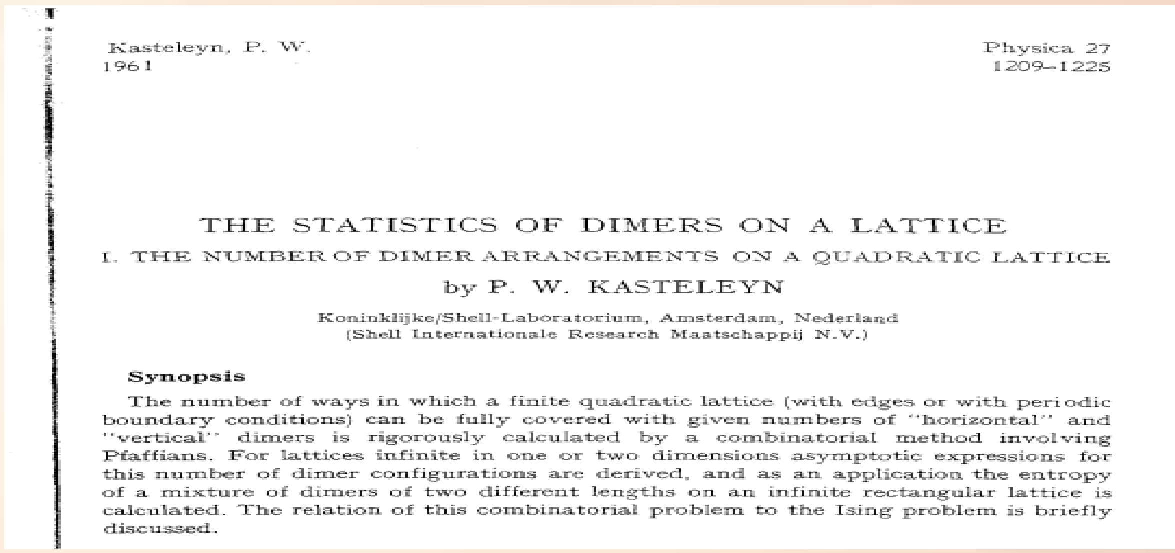
Where pfaffian of A is defined as:

$$\mathrm{Pf}(A) = \sum \mathrm{sign}\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix} a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n}$$

pfaffian and matchings:

- We say that the graph G has a **pfaffian oriention** , if there exists an oriention for edges of G such that :

$$|pf(A)|=pm(G)$$



Kasteleyn, P. W.
1961

Physica 27
1209–1225

THE STATISTICS OF DIMERS ON A LATTICE

I. THE NUMBER OF DIMER ARRANGEMENTS ON A QUADRATIC LATTICE

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(Shell Internationale Research Maatschappij N.V.)

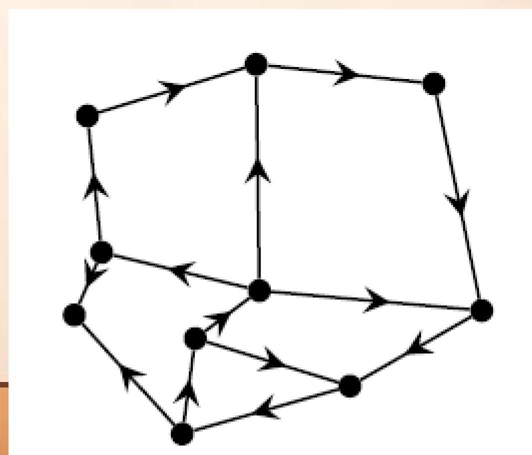
Synopsis

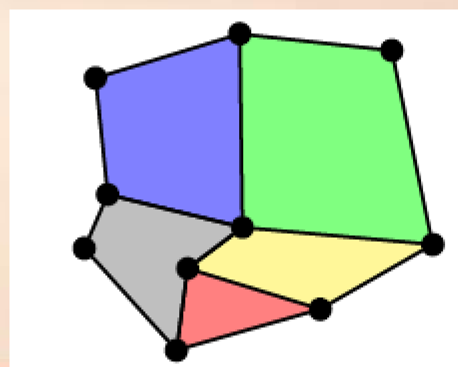
The number of ways in which a finite quadratic lattice (with edges or with periodic boundary conditions) can be fully covered with given numbers of "horizontal" and "vertical" dimers is rigorously calculated by a combinatorial method involving Pfaffians. For lattices infinite in one or two dimensions asymptotic expressions for this number of dimer configurations are derived, and as an application the entropy of a mixture of dimers of two different lengths on an infinite rectangular lattice is calculated. The relation of this combinatorial problem to the Ising problem is briefly discussed.

- **Theorem(Kasteleyn-1963).** An orientation of a graph G is Pfaffian if every even cycle C such that G - V(C) has a perfect matching has an odd number of edges directed in either direction of the cycle.

Pfaffian and planar graph

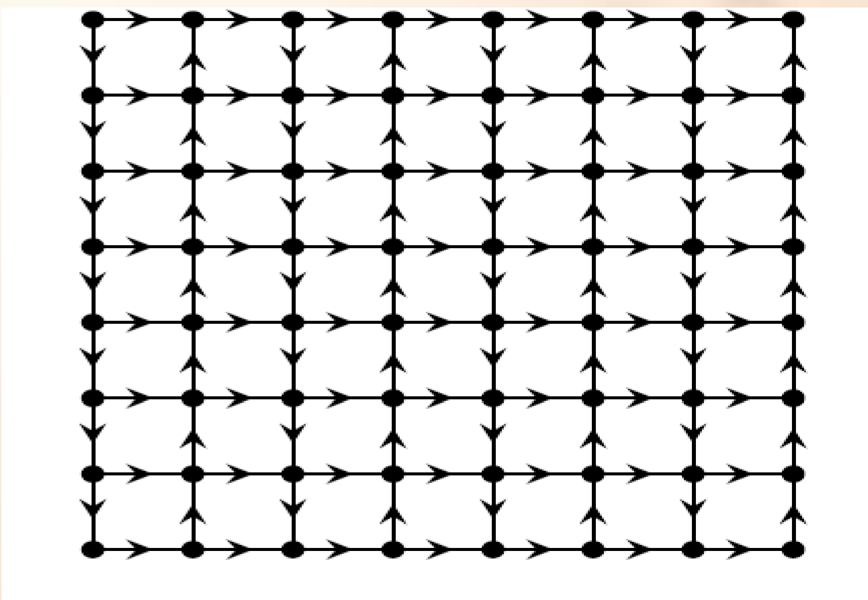
- **Theorem(kasteleyn-1963)** every planar graphs has pfaffian orientation.
- Orient edges such that each boundary cycle of even length has an odd number of edges oriented clockwise .





Solving domino problem by pfaffian

- **Theorem(Kasteleyn-1963).** Every planar graphs has pfaffian orientation.
- Orient edges such that each boundary cycle of even length has an odd number of edges oriented clockwise .



$$= 2^{mn/2} \prod_{k=1}^{m} \prod_{l=1}^{n} \left(\cos^2\left(\frac{\pi k}{m+1}\right) + \cos^2\left(\frac{\pi l}{n+1}\right)\right)^{\frac{1}{4}} .$$

Perfect matching

Results in pfaffian and planar graphs

- **Bergman inequality:** let G=(A,B) is bipartite graph and let r_i are the degree of A then we have:

$$pm(G) \leq \prod (r_i!)^{\frac{1}{r_i}}$$

- Theorem (Friedland & Alon, 2008). Let G be graph with degree d_i ,then for the perfect matching of G we have:

$$pm(G) \leq \prod (d_i!)^{\frac{1}{2d_i}}$$

- Equlity holds if and only if G is a union of complete bipartite regular graphs

Results in pfaffian and planar graphs

- **Theorem** (Behmaram,Friedland) Let G is pfaffian graph with degrees d_i ,then for the number of perfect matching in this graph we have:

$$pm(G) \le \prod d_i^{\frac{1}{4}}$$

- Lemma : For d>2 we have:

$$(d!)^{\frac{1}{2d}} \ge d^{\frac{1}{4}}$$

- **Corollary** . $K_{r,r}$ is not pfaffian graph for r>2.
- **Corollary.** If g is girth of the planar graph G then we have:
- $$pm(G) \le \left(\frac{2g}{g-2}\right)^{\frac{n}{4}}$$
- especially if G is triangle free then : $pm(G) \le 2^{\frac{n}{2}}$
-

Results in fullerene graphs-upper bound

Å **Theorem 3.1.** If G is a cubic pfaffian graph with no 4 ï cycle then we have:

$$pm(G) \leq 8^{\frac{n}{12}} 3^{\frac{n}{12}}$$

Å **Theorem 3.2**. For every fullerene graphs F, we have the following inequality:

$$pm(F) \leq 20^{\frac{n}{12}}$$

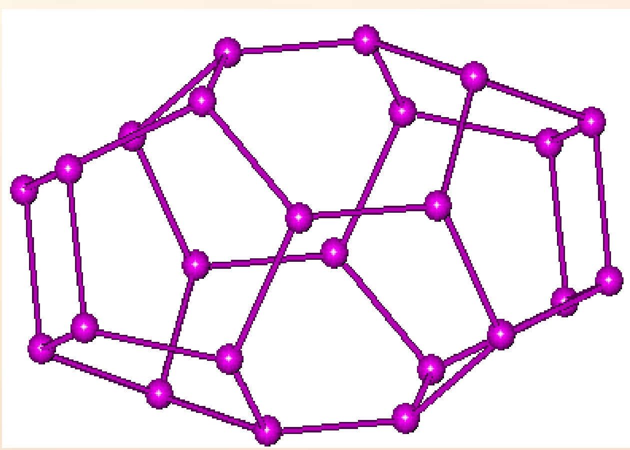
m-Generalized Fullerene

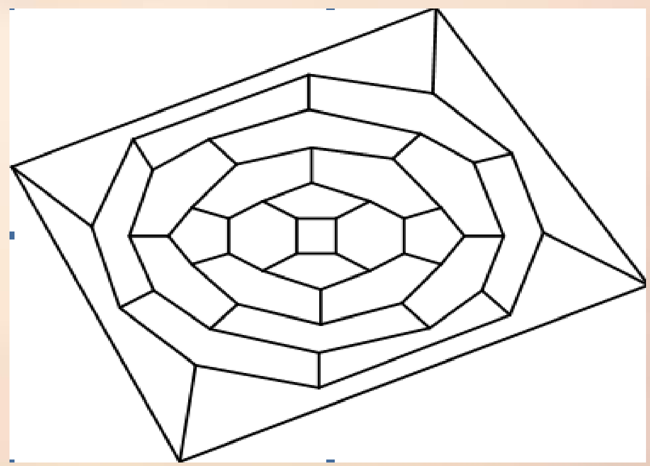
Å A connected 3-regular planar graph G = (V , E) is called m-generalized fullerene if it has the following types of faces:

two m-gons and all other pentagons and hexagons.

Å Lemma. Let m Ó3 be an integer different from 5. Assume that G = (V,E) is an m-generalized fullerene. Then the faces of G have exactly 2m pentagons.

Å For m=5,6 , a m-generalized fullerene graph is an ordinary fullerene



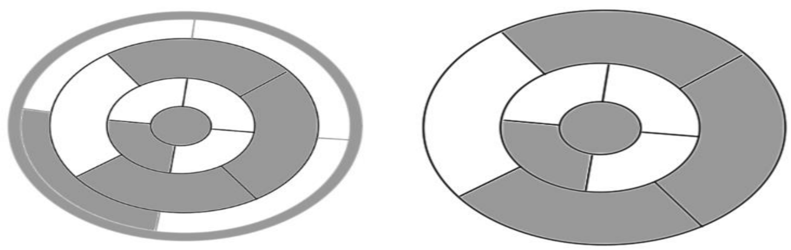


m-Generalized Fullerene (circular latice)

- The Family of m-generalized fullerene F(m,k):

The first circle is an m-gon. Then m-gon is bounded by m pentagons. After that we have additional k layers of hexagon. At the last circle m-pentagons connected to the second m-gon.

- Theorem. F(m,k) is Hamiltonian graphs.



m-Generalized Fullerene

- Theorem. The diameter of F(m,k) is: $\lfloor \frac{m}{2} \rfloor + 2k + 2$

- Theorem. For the perfect matchings in F(m,k) we have the following results:

$$pm(F(3,k)) = 3^{k+1} + 1,$$

$$5^{k+1} + 5 \cdot 3^k + 1 \leq pm(F(5,k)) \leq 5^{k+1} + 5 \cdot 4^k + 1$$

The End

Thanks your attention