Quasi-invariants of 2-knots and quantum integrable systems

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References

This talk is based on the papers:

- I. Korepanov, G. Sharygin, D.T: ”Cohomologies of *n*-simplex relations”, arXiv:1409.3127
- D.T. ”Zamolodchikov tetrahedral equation and higher Hamiltonians of 2d quantum integrable systems”, arXiv:1505.06579
- ”Cohomology of the tetrahedral complex and quasi-invariants of 2-knots.” in progress

A. Zamolodchikov [1981]

Vector version

Let $\Phi \in End(V^{\otimes 3})$, where V - (f.d) vector space. The tetrahedral equation takes the form

$$\Phi_{123}\Phi_{145}\Phi_{246}\Phi_{356} = \Phi_{356}\Phi_{246}\Phi_{145}\Phi_{123}$$

where both sides are linear operators in $V^{\otimes 6}$ and Φ_{ijk} represents the operator acting in components i, j, k as Φ and trivially in the others.

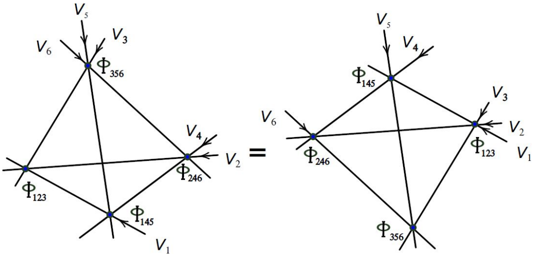


Figure : Tetrahedral equation

Set-theoretic version

Let X be a (f) set. We say that a map

$$X \times X \times X \xrightarrow{R} X \times X \times X,$$

satisfy the s.t. tetrahedral equation if

$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123}$$

where both sides are maps of the Cartesian power $X^{\times 6}$ and the subscripts correspond to components of X.
For example

$$\begin{aligned} R_{356}(a_1, a_2, a_3, a_4, a_5, a_6) &= (a_1, a_2, R_1(a_3, a_5, a_6), a_4, R_2(a_3, a_5, a_6), R_3(a_3, a_5, a_6)) \\ &= (a_1, a_2, a_3', a_4, a_5', a_6'), \end{aligned}$$

where

$$R(x, y, z) = (R_1(x, y, z), R_2(x, y, z), R_3(x, y, z)) = (x', y', z').$$

Functional equation

One distinguishes a functional tetrahedral equation, satisfied by a map on some functional field, in the example below on the field of rational functions. I depict here a famous electric solution:

$$\begin{aligned} \Phi(x,y,z) &= (x_1,y_1,z_1); \\ x_1 &= \frac{xy}{x+z+xyz}, \\ y_1 &= x+z+xyz, \\ z_1 &= \frac{yz}{x+z+xyz}, \end{aligned}$$

related to the so called star-triangle transformation, known in electric circuits

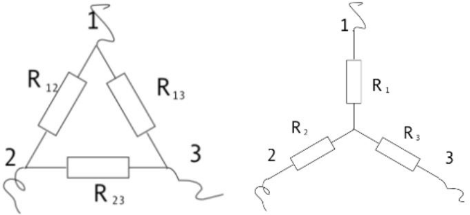


Figure : Star-triangle transformation

Another realization

Let us consider the Euler decomposition of $U \in SO(3)$ and a dual one

$$U = \underbrace{\begin{pmatrix} \cos\phi_1 & \sin\phi_1 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{X_{\alpha\beta}[\phi_1]} \underbrace{\begin{pmatrix} \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 1 & 0 \\ -\sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix}}_{X_{\alpha\gamma}[\phi_2]} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_3 & \sin\phi_3 \\ 0 & -\sin\phi_3 & \cos\phi_3 \end{pmatrix}}_{X_{\beta\gamma}[\phi_3]}$$

$$U = X_{\alpha\beta}[\phi_1]X_{\alpha\gamma}[\phi_2]X_{\beta\gamma}[\phi_3] = X_{\beta\gamma}[\phi_3']X_{\alpha\gamma}[\phi_2']X_{\alpha\beta}[\phi_1']$$

Then the transformation from the Euler angles to the dual Euler angles

$$\begin{aligned} \sin\phi_2' &= \sin\phi_2\cos\phi_1\cos\phi_1 + \sin\phi_1\sin\phi_3 \\ \cos\phi_1' &= \frac{\cos\phi_1\cos\phi_2}{\cos\phi_2'}, \qquad cos\phi_3' = \frac{\cos\phi_2\cos\phi_3}{\cos\phi_2'} \end{aligned}$$

defines a solution of the functional tetrahedral equation.

4-cube colorings

Let us consider the 4-cube and its projection to a 3-dimensional space. This is a rhombo-dodecahedron divided in two ways into four parallelepipeds, corresponding to the 3-cubes of the border of the 4-cube.

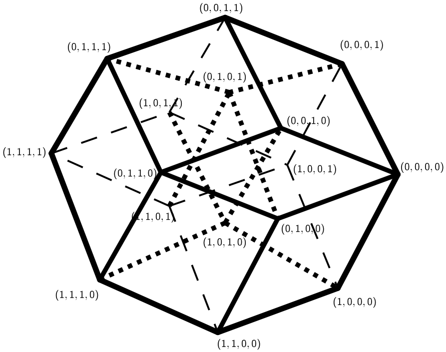
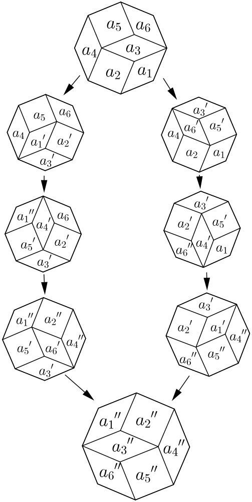


Figure : Tesseract

One may associate to this division a problem of coloring the 2-faces of the 4-cube by elements (called colors) of some set X in such a way that the colors of the faces in each 3-cube are related by some transformation $\Phi : (a_1, a_2, a_3) \to (a_1', a_2', a_3')$.
There is a special way to choose the incoming and outgoing 2-faces of each 3-cube. It appears that the compatibility condition for Φ is nothing but the tetrahedral equation.



Recalling 1-knots

Figure : Trefoil

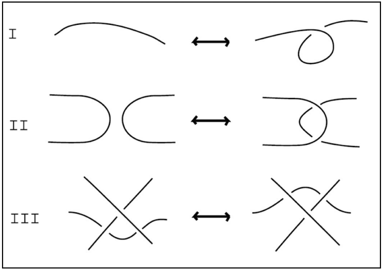


Figure : Reidemeister moves

2-knot

Definition

By a 2-knot we mean an isotopy class of embeddings $S^2 \hookrightarrow \mathbb{R}^4$.

A class of examples of non-trivial 2-knots is given by the Zeeman's [1965] twisted-spun knot, which is a generalization of the Artin spun knot.

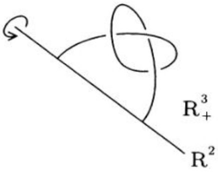


Figure : Example

Diagrams

To obtain a diagram of a 2-knot one takes a generic projection p to the hyperplane P in $\mathbb{R}^4$. The generic position entails that there are singularities only of the following types: double point, triple point and the Whitney point (or branch point)

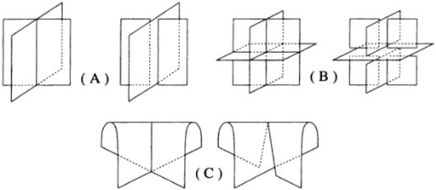


Figure : Singularity types

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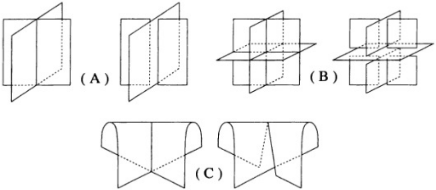
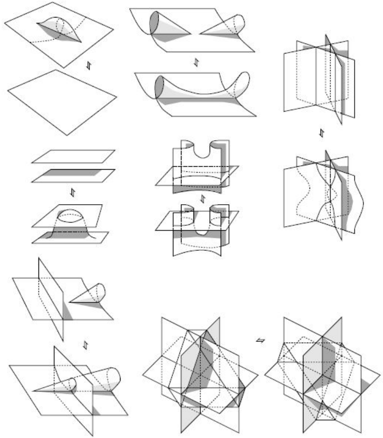


Figure : Singularity types

The diagram of a 2-knot is a singular surface with arcs of double points which end in triple points and branch points. This defines a graph of singular points. The additional information consists of the order of 2-leaves in intersection lines subject to the projection direction. We always work here with oriented surfaces.

Roseman moves

Theorem [Roseman 1998]
Two diagrams
represent equivalent knotted surfaces
iff one can be obtained from another
by a finite series of moves from the
list and an isotopy of a diagram in $\mathbb{R}^3$.



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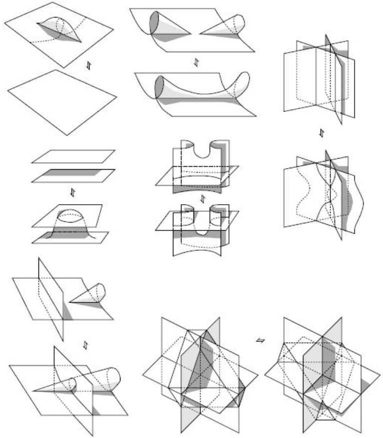
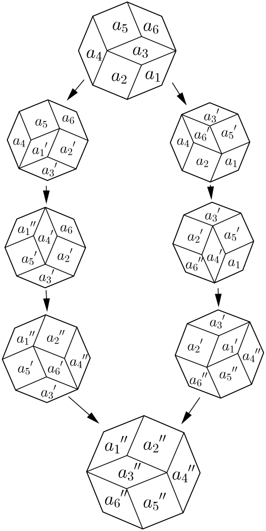
There is an approach due
to Carter, Saito and others (2003) which
produces invariants of 2-knots by means
of the so called quandle cohomology.
Invariants are constructed as some
partition functions on the space of states
which are coloring of the 2-leaves of
a diagram by elements of the quandle.



Cocycles

Electric solution:
$x_1 = xy/(x + z + xyz)$,
$y_1 = x + z + xyz$,
$z_1 = yz/(x + z + xyz)$.



Definition

For a given solution Φ of the set-theoretic tetrahedral equation on the set X and a given field $\mathbb{k}$ we say that a function $\varphi : X^{\times 3} \to \mathbb{k}$ is a 3-cocycle of the tetrahedral complex if

$$\varphi(a_1, a_2, a_3)\varphi(a_1', a_4, a_5)\varphi(a_2', a_4', a_6)\varphi(a_3', a_5', a_6') =$$
$$= \varphi(a_3, a_5, a_6)\varphi(a_2, a_4, a_6')\varphi(a_1, a_4', a_5')\varphi(a_1', a_2', a_3').$$

Definition

For a given solution Φ *of the set-theoretic tetrahedral equation on the set* X *and a given field* $\mathbb{k}$ *we say that a function* $\varphi : X^{\times 3} \to \mathbb{k}$ *is a* 3*-cocycle of the tetrahedral complex if*

$$\varphi(a_1, a_2, a_3)\varphi(a'_1, a_4, a_5)\varphi(a'_2, a'_4, a_6)\varphi(a'_3, a'_5, a'_6) =$$
$$= \varphi(a_3, a_5, a_6)\varphi(a_2, a_4, a'_6)\varphi(a_1, a'_4, a'_5)\varphi(a'_1, a'_2, a'_3).$$

Lemma

Let us consider the electric solution (1) $\Phi : (a_1, a_2, a_3) \mapsto (a'_1, a'_2, a'_3)$. *The following expressions, as like as their product and quotient, provide* 3*-cocycles of the tetrahedral complex*

$$\begin{aligned} c_1(a_1, a_2, a_3) &= a_2 \\ c_2(a_1, a_2, a_3) &= a'_2 \end{aligned}$$

Notations

Let us recall that we consider an oriented 2-surface with prescribed singularities.

1. The overall orientation allows to define an orientation for the arcs of double points of a diagram in such a way the tangent vector, the normal to the top and the bottom leaves constitute a positive triple.
2. The sign of a triple point is defined to be the orientation of the triple of normal vectors to the top, middle and bottom leaves.
3. The order of incoming edges at a triple point is defined by the order of faces transversal to edges.

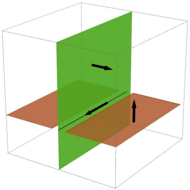


Figure : Edges orientation

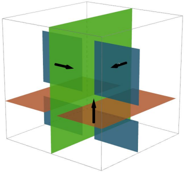


Figure : Positive triple point

Let us now fix a solution for the set-theoretic tetrahedral equation Φ on the set X and a 3-cocycle ϕ. We say that a map $C : E \to X$ is a coloring of the edges set of a diagram if in each triple point $\tau \in T$ the colors of incoming edges are related with the colors of the outgoing ones by the formula:

$$(x', y', z') = \Phi(x, y, z)$$

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Definition

The partition function corresponding to the chosen diagram D, TE solution Φ and an element $\phi \in H^3(X, \Phi)$ is defined by an equation:

$$Z(s) = \sum_{Col} \prod_{\tau \in T} \phi(x_\tau, y_\tau, z_\tau)^s$$

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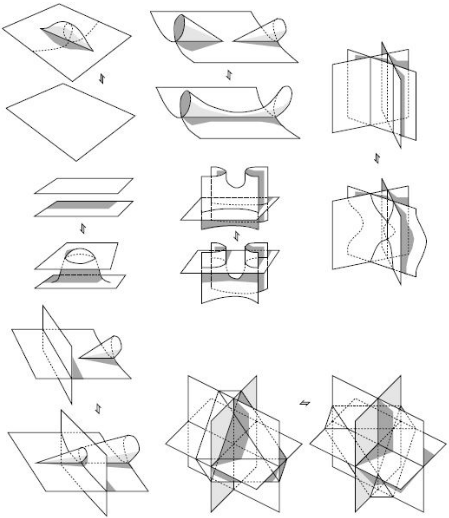
Theorem

The partition function $Z(s)$ is invariant with respect to the 3-th and 7-th Roseman moves. Moreover the choice $\phi = c_2/c_1$ from lemma 1 guaranties the invariance with respect to 6-th Roseman moves.

Roseman moves



Statistical model

Let us consider a 3d periodic oriented lattice with $K \times L \times M$ sites. We denote the edges incoming to the site (i, j, k) as $x_{i,j,k}, y_{i,j,k}, z_{i,j,k}$.
We suppose some periodicity condition in all directions. For example in the 1-st direction this means $*_{N+1,j,k} = *_{1,j,k}$.

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Let us consider a statistical model those Boltzmann weights at the sites are defined by the 3-cocycle ϕ of the tetrahedral complex and the admissible states of the system are defined by the colorings subject to the relations:

$$\Phi(x_{i,j,k}, y_{i,j,k}, z_{i,j,k}) = (x_{i+1,j,k}, y_{i,j+1,k}, z_{i,j,k+1}).$$

at each triple point.

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at each triple point.
The partition function of such a model is calculated by an expression:

$$Z(s) = \sum_{Col} \prod_{i,j,k} \phi(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})^s.$$

Transfer-matrix

1. A solution for the s-t TE Φ and a 3-cocycle ϕ provides a solution for the vector TE. Let V be the vector space generated by elements of the set X. Then we define a linear operator A in $V^{\otimes 3}$ by the image of basis elements. We say that

$$A(s)(e_x \otimes e_y \otimes e_z) = \phi(x,y,z)^s (e_{x'} \otimes e_{y'} \otimes e_{z'})$$

   if $\Phi(x,y,z) = (x',y',z')$.
2. We correspond a copy of the space V to each line in the lattice, for convenience we denote the vertical spaces by V_{ik} and the horizontal ones by E_i and N_k.

We define the transfer-matrix by the layer product:

$$T(s) = Tr \prod_{\alpha} \prod_{\beta} A_{\alpha\beta}(s)$$

which is an operator in the tensor product of vertical vector spaces. Here $A_{\alpha\beta}(s)$ is an operator in the space $E_\alpha \otimes V_{\alpha\beta} \otimes N_\beta$, the product and trace is taken over horizontal spaces.

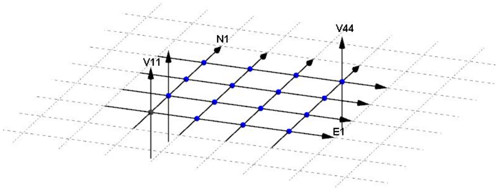

Figure : 1-layer configuration

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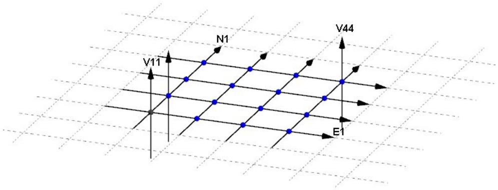

Figure : 1-layer configuration

Then the partition function takes the form

$$Z(s) = Tr_{V_{\alpha\beta}} T(s)^L.$$

Integrability

By integrability here we mean an existence of a ”sufficiently large” commutative family which includes the transfer-matrix.

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Commutative family $\rightarrow$ Spectrum $\rightarrow$ Asymptotic properties of the partition function.

Let us recall some results from the Yang-Baxter equation theory. Let R be a solution of the YB equation in the form:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

and L be a so-called L-operator:

$$RL \otimes L = L \otimes LR$$

Then one constructs a commutative family by the formula [Maillet 1990]

$$I_k = Tr_{1\ldots k} \underbrace{L \otimes \ldots \otimes L}_{k} R_{12}R_{23}\ldots R_{k-1,k}.$$

2d-Generalization

Let us introduce some notations

$$\Phi_{(i)*(j)} = \Phi_{(i_1\dots i_k)*(j_1\dots j_m)} = \prod_{\alpha=\overrightarrow{1,\dots k}}^{\beta=\overrightarrow{1,\dots m}} \Phi_{i_\alpha l_{\alpha\beta} j_\beta}$$

The transfer-matrix can be represented as the trace

$$T = I_1 = Tr_{(i)(j)} \Phi_{(i)*(j)}.$$

We also make use of the twisted elements

$$\Phi^L_{123} = P_{12}\Phi_{123}, \qquad \Phi^R_{123} = \Phi_{123}P_{23}.$$

A simple consequence of the Maillet result gives us a

Lemma

For a generic solution of the tetrahedral equation there are two commutative families

$$I_{0,k} = Tr_{i_l,j_l,s_m} \prod_{\overrightarrow{l=1,\ldots,k}} \Phi_{(i_m)*(j_m)} \prod_{\overrightarrow{m=1,\ldots,k-1}} \Phi^R_{s_m(j_m)(j_{m+1})}$$

and

$$I_{n,0} = Tr_{i_l,j_l,t_m} \prod_{\overrightarrow{l=1,\ldots,n}} \Phi_{(i_m)*(j_m)} \prod_{\overrightarrow{m=1,\ldots,n-1}} \Phi^L_{(i_m)(i_{m+1})t_m}$$

both of them containing the transfer-matrix.

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both of them containing the transfer-matrix.

The main result is the following

Theorem

For a generic solution Φ for the tetrahedral equation the families $I_{n,0}$ and $I_{0,k}$ commute between themselves.

Summary and perspectives

- It is presented a construction of a statistical model on graphs with 6-valent notes with some additional orientation structures, which specializes to a quasi-invariant of 2-knots if one considers the graph of double points of a diagram of a 2-knot.
- This statistical model being considered on a regular 3-d lattice is demonstrated to be integrable in the sense that there exists a commutative family of operators which include a 1-layer transfer-matrix.
- I expect that this family may be organized into the generating function defining a quantum spectral surface of the model, and that the 2-dimensional Bethe ansatz could be applied in this case.
- I also hope that there is a close relation of this subject with topological quantum field theories in $d = 4$ (like the BF-theory), which allows to interpret our quasi-invariants as some quantum observables.

2-faces coloring

One describes the n-faces of N-cube by sequence of symbols $(\tau_1, \dots, \tau_N)$ which take values $0, 1, *$, where $*$ corresponds to a coordinate varying in the interval $[0, 1]$. Let us also denote by $\{j_k\}$ a set of indices of symbols $*$ in a sequence. A subface of codimension 1 is defined by a substitution of some $*$ by one of the numbers 0 or 1. Let us fix the index j_k of the corresponding symbol. We define an alternating sequence:

$$\varkappa_1 = 0, \varkappa_2 = 1, \varkappa_3 \dots .$$

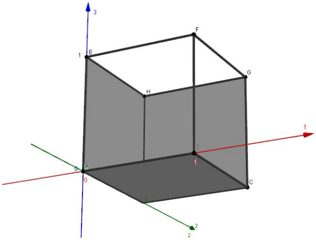

Figure : Incoming(black) and outgoing(white) faces of a standart 3-cube

Definition

A subface is called incoming if the j_k-th coordinate coincides with $\varkappa_k$ and outgoing otherwise.

Let us fix a set X and a solution of the set-theoretic tetrahedral equation $\Phi : X \times X \times X \to X \times X \times X$.

Definition

A coloring of 2-faces of an N-cube $C : I^N \to X$ is called admissible if for any 3-face the colors of the incoming 2-faces (x, y, z) and the colors of the outgoing 2-faces (x', y', z') are related by

$$(x', y', z') = \Phi(x, y, z).$$

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Let us consider a complex $C_*(X) = \bigoplus_{n\geq 2} C_n(X)$ where

$$C_n(X) = C_n(X,\, \mathbb{k}) = \mathbb{k} \cdot C^2(n,\, X),$$

here $C_n(X)$ is a free $\mathbb{k}$-module generated by the set of 2-face colorings of the n-cube. The differential $d_n : C_n \to C_{n-1}(X)$ is defined by the formula

$$d_n(c) = \sum_{k=1}^{n} \left(d_k^i c - d_k^o c \right),$$

where $d_i^f c$ ($d_k^o c$) is the restriction of the coloring c to the k-th incoming (resp. outgoing) $(n-1)$-face of the cube I^n. Denote by $H_*(X,\, \mathbb{k})$ the corresponding homologies.

Absolutely incoming faces

Definition

We call an n-face of an N-cube absolutely incoming if it is not outgoing of any $n+1$*-face.*

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A coloring of 2-faces of an N-cube is uniquely defined by a coloring of absolutely incoming 2-faces.

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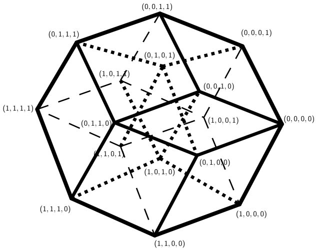
The number of absolutely incoming 2-faces is equal to C_N^2. Hence in low dimension the complex is represented by

$$C_2(X) = \mathbb{k} \cdot X,$$
$$C_3(X) = \mathbb{k} \cdot X^{\times 3},$$
$$C_4(X) = \mathbb{k} \cdot X^{\times 6}.$$

We will denote a coloring by colors of absolutely incoming faces.


(0, 0, 1, 1)
(0, 1, 1, 1)
(0, 0, 0, 1)
(0, 1, 0, 1)
(1, 0, 1, 1)
(0, 0, 1, 0)
(1, 1, 1, 1)
(0, 1, 1, 0)
(1, 0, 0, 1)
(0, 0, 0, 0)
(1, 1, 0, 1)
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(1, 0, 0, 0)
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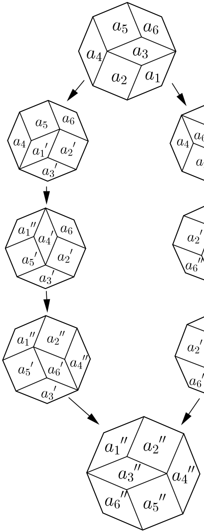

Differential

In the case $n = 3$ the differential is given by:

$$d_3((a,b,c)) = (a) + (b) + (c) - (\Phi_1(a,b,c)) - (\Phi_2(a,b,c)) - \Phi_3(a,b,c)).$$

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The next example in $n = 4$ is

$$\begin{aligned} d_4((a_1, a_2, a_3, a_4, a_5, a_6)) = (a_1, a_2, a_3) - (a_3, a_5, a_6) \\ -(\Phi_1(a_1, \Phi_2(a_2, a_4, \Phi_3(a_3, a_5, a_6)), \Phi_2(a_3, a_5, a_6)), \Phi_1(a_2, a_4, \Phi_3(a_3, a_5, a_6)), \Phi_1(a_3, a_5, a_6)) \\ +(\Phi_3(a_1, a_2, a_3), \Phi_3(\Phi_1(a_1, a_2, a_3), a_4, a_5), \Phi_3(\Phi_2(a_1, a_2, a_3), \Phi_2(\Phi_1(a_1, a_2, a_3), a_4, a_5), a_6)) \\ -(a_1, \Phi_2(a_2, a_4, \Phi_3(a_3, a_5, a_6)), \Phi_2(a_3, a_5, a_6)) - (a_2, a_4, \Phi_3(a_3, a_5, a_6)) \\ +(\Phi_2(a_1, a_2, a_3), \Phi_2(\Phi_1(a_1, a_2, a_3), a_4, a_5), a_6) + (\Phi_1(a_1, a_2, a_3), a_4, a_5). \end{aligned}$$

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The dual differential implies the following equation for the 3-cocycle:

$$\begin{aligned} & f(a_1,a_2,a_3) + f(a_1',a_4,a_5) + f(a_2',a_4',a_6) + f(a_3',a_5',a_6') = \\ & = f(a_3,a_5,a_6) + f(a_2,a_4,a_6') + f(a_1,a_4',a_5') + f(a_1',a_2',a_3'). \end{aligned}$$

Quandles

Definition (Matveev 1982)

A set X with a binary operation $(a,b) \mapsto a*b$ is a quandle if

$$\begin{array}{ll} i) & \forall a \in X \quad a*a=a \\ ii) & \forall a,b \in X \quad \exists! c \in X : c*b=a \\ iii) & \forall a,b,c \in X \quad (a*b)*c=(a*c)*(b*c) \end{array}$$

Example

The group quandle is the set of group elements G with the operation $a*b=b^{-n}ab^n$ for any fixed n.

Example

The Alexander quandle is a Λ-module M, where $\Lambda=\mathbb{Z}[t,t^{-1}]$, with the operation $a*b=ta+(1-t)b$.

Quandle cohomologies

S. Carter, S. Kamada, M. Saito [2000-...]
Let us define a complex $C_n^R(X)$ whose components are free abelian groups generated by n-tuples of elements of X $(x_1, \ldots, x_n)$. Then the differential $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ is:

$$\begin{aligned} \partial_n(x_1, \ldots, x_n) &= \sum_{i=2}^{n} (-1)^i \{(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &- (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\} \end{aligned}$$

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We also consider a subcomplex $C_n^D(X)$, whose components are generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some i and $n \geq 2$. We construct a quotient complex $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ and the induced differential. Then the homologies and cohomologies of a quandle with coefficients in a group G are determined by the complexes:

$$C_*^Q(X, G) = C_*^Q(X) \otimes G, \qquad \partial = \partial \otimes id$$
$$C_Q^*(X, G) = Hom(C_*^Q(X), G), \qquad \delta = Hom(\partial, id)$$

Coloring

Let us firstly define a notion of a diagram coloring. We denote by L the set of 2-leaves of a diagram after cutting. One says that there is a coloring C of a diagram D with elements of a quandle Q if there is a map $C : L \to Q$ satisfying the coherence conditions near the intersections of the diagram illustrated by the picture:

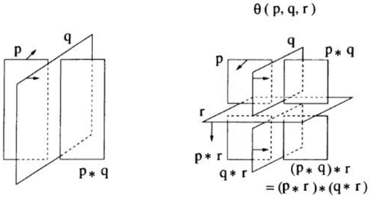

Figure : Coloring

Invariant

Let us fix a 3-cocycle $\theta \in Z^3_Q(Q, A)$. This implies a condition

$$\theta(p, r, s) + \theta(p * r, q * r, s) + \theta(p, q, r) = \theta(p * q, r, s) + \theta(p, q, s) + \theta(p * s, q * s, r * s)$$

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One attributes a following Boltzmann weight to a triple point τ

$$B(\tau, C) = \theta(x, y, z)^{\epsilon(\tau)}$$

here $\epsilon(\tau)$ is the sign of τ, x, y, z - colors of the top, middle and bottom leaves in outgoing octant, i.e. such that it is negative for normals of all leaves. The sign $\epsilon(\tau)$ is defined by the orientation of normals. Then one defines a partition function

$$S(D, \theta, A) = \sum_C \prod_\tau B(\tau, C). \tag{6}$$

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Theorem (Carter,... 03)

The partition function 6 is invariant with respect to the Roseman moves and hence is an invariant of a 2-knot.