Kac modules and boundary Temperley-Lieb algebras for logarithmic minimal models

Alexi Morin-Duchesne

Université Catholique de Louvain

Florence, 28/05/2015

Joint work with Jørgen Rasmussen and David Ridout

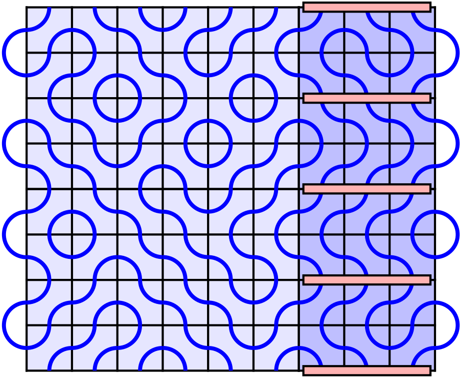
arXiv:1503.07584 [hep-th]

Outline

- Loop models with boundary seams
- Relation with the one-boundary Temperley-Lieb algebra
- Virasoro Kac modules
- Scaling limit of the loop models

Dense loop model

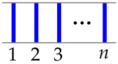
- Configuration of the dense loop model with a **boundary seam:**

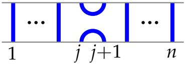


- **Fugacity of closed loops:** $\beta = 2\cos\lambda$
- **Roots of unity:** $\lambda = \frac{\pi(p'-p)}{p'}$ $\quad p, p' \in \mathbb{Z}_+ \quad p < p'$

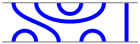
Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$

Generators

$I =$ 

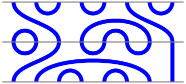
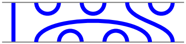
$e_j =$ 

A connectivity

$a =$ 

$= e_1 e_2 e_4 e_3$

- Multiplication is by vertical concatenation:

$a_1 a_2 =$  $= \beta^2$  $= \beta^2 a_3$

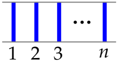
Algebraic definition

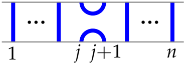
$$\mathsf{TL}_n(\beta) = \langle I, e_j \,;\, j = 1, \ldots, n-1 \rangle$$

$$(e_j)^2 = \beta\, e_j \qquad e_j e_{j\pm1} e_j = e_j \qquad e_i e_j = e_j e_i \qquad (|i-j| > 1)$$

Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$

Generators

$I =$ 

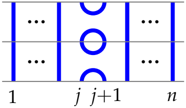
$e_j =$ 

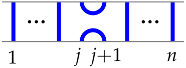
A connectivity

$a =$ 

$= e_1 e_2 e_4 e_3$

- Multiplication is by vertical concatenation:

$(e_j)^2 =$ 

$= \beta$ 

$= \beta\, e_j$

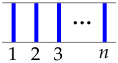
Algebraic definition

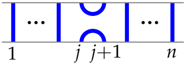
$$\mathsf{TL}_n(\beta) = \langle I, e_j\,;\, j = 1, \ldots, n-1 \rangle$$

$$(e_j)^2 = \beta\, e_j \qquad e_j e_{j\pm1} e_j = e_j \qquad e_i e_j = e_j e_i \qquad (|i-j| > 1)$$

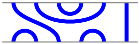
Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$

Generators

$I =$ 

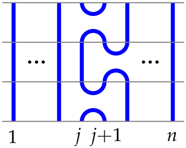
$e_j =$ 

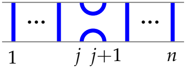
A connectivity

$a =$ 

$= e_1 e_2 e_4 e_3$

- Multiplication is by vertical concatenation:

$e_j\, e_{j+1}\, e_j =$ 

$=$ 

$= e_j$

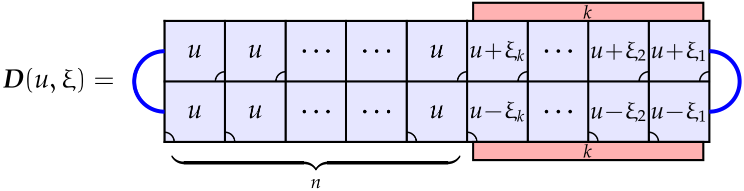
Algebraic definition

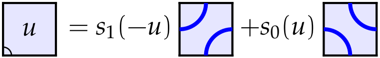
$$\mathsf{TL}_n(\beta) = \langle I,\, e_j\,;\, j = 1, \ldots, n-1 \rangle$$

$$(e_j)^2 = \beta\, e_j \qquad e_j e_{j\pm 1} e_j = e_j \qquad e_i e_j = e_j e_i \qquad (|i-j| > 1)$$

Transfer tangles with boundary seams

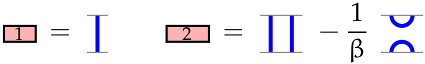
- $\boldsymbol{D}(u,\xi)$ is an element of $\mathsf{TL}_{n+k}(\beta)$: (Pearce, Rasmussen, Zuber 2006)

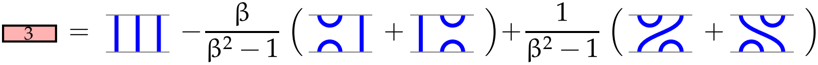




$$s_k(u) = \frac{\sin(u+k\lambda)}{\sin\lambda} \qquad \xi_j = \xi + j\lambda$$

- Projectors:

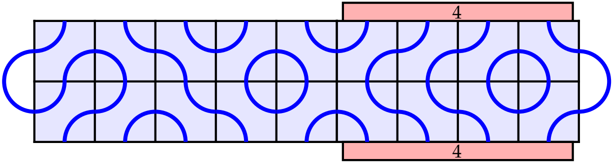


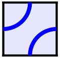
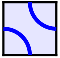


- YBE + BYBE $\quad\rightarrow\quad [\boldsymbol{D}(u,\xi),\boldsymbol{D}(v,\xi)] = 0$

Transfer tangles with boundary seams

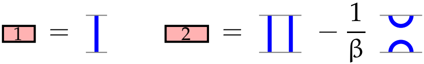
- $\boldsymbol{D}(u,\xi)$ is an element of $\mathsf{TL}_{n+k}(\beta)$: (Pearce, Rasmussen, Zuber 2006)

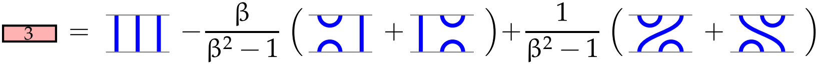


$= s_1(-u)$  $+ s_0(u)$ 

$$s_k(u) = \frac{\sin(u+k\lambda)}{\sin\lambda} \qquad \xi_j = \xi + j\lambda$$

- Projectors:

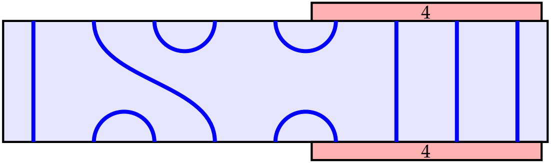


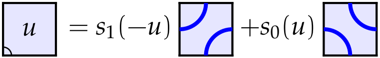


- YBE + BYBE $\quad\to\quad [\boldsymbol{D}(u,\xi),\boldsymbol{D}(v,\xi)] = 0$

Transfer tangles with boundary seams

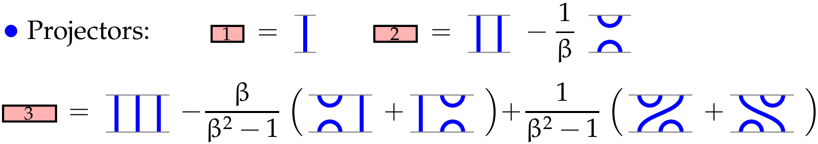
- $\boldsymbol{D}(u,\xi)$ is an element of $\mathsf{TL}_{n+k}(\beta)$: (Pearce, Rasmussen, Zuber 2006)

β^2 



$$s_k(u) = \frac{\sin(u+k\lambda)}{\sin\lambda} \qquad \xi_j = \xi + j\lambda$$

- Projectors:



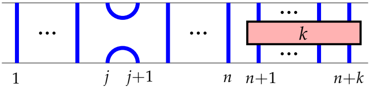
- YBE + BYBE $\rightarrow$ $[\boldsymbol{D}(u,\xi),\boldsymbol{D}(v,\xi)] = 0$

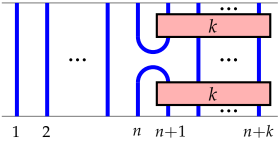
Hamiltonian tangle

- The Hamiltonian tangle $\mathcal{H}$ is obtained by taking $\frac{d\boldsymbol{D}(u,\xi)}{du}\Big|_{u=0}$:

$$\mathcal{H} = -\sum_{j=1}^{n-1} E_j^{(k)} + \frac{1}{s_0(\xi)s_{k+1}(\xi)} E_n^{(k)}$$

where

$E_j^{(k)} =$  $(j = 1, \ldots, n-1)$

$E_n^{(k)} = U_{k-1}(\frac{\beta}{2})$ 

- $U_k(x)$ are Chebyshev polynomials of the second kind:

$$U_0(\tfrac{\beta}{2}) = 1, \quad U_1(\tfrac{\beta}{2}) = \beta, \quad U_2(\tfrac{\beta}{2}) = \beta^2-1, \quad U_3(\tfrac{\beta}{2}) = \beta(\beta^2-2), \quad \ldots$$

Standard modules

- Definition:

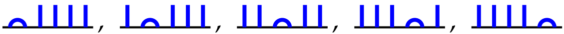
  V_n^d: vector space generated by **link patterns**

  n: number of **nodes**

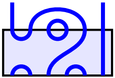
  d: number of **defects** (vertical segments)

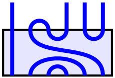
Examples:

$\mathsf{V}_6^0 = \text{span}\{$  $\}$

$\mathsf{V}_6^4 = \text{span}\{$  $\}$

- $\mathsf{TL}_n(\beta)$ action on V_n^d:

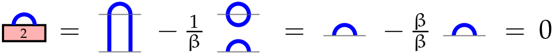
 $= \beta$ 

 $= 0$

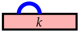
- Defines one representation of $\mathsf{TL}_n(\beta)$ for each d.

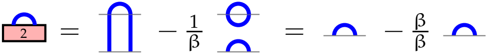
Lattice Kac modules

- Projector – half-arc annihilation relation: 

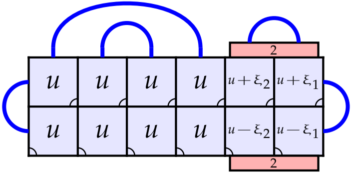
Example: 

Lattice Kac modules

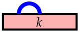
- Projector – half-arc annihilation relation:  $= 0$

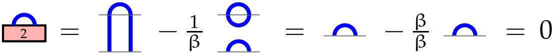
Example:  $= 0$

- $\boldsymbol{D}(u, \xi)$ and $\mathcal{H}$ act trivially on a subspace of V^{d}_{n+k}:

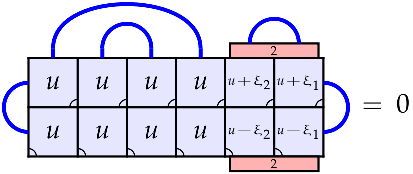
 $= 0$

Lattice Kac modules

- Projector – half-arc annihilation relation:  $= 0$

Example: 

- $\boldsymbol{D}(u,\xi)$ and $\mathcal{H}$ act trivially on a subspace of V^d_{n+k}:



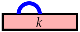
- **Lattice Kac module $\mathsf{K}^d_{n,k}$:** quotient of V^d_{n+k} by the trivial subspace

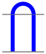
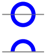
Examples for $k = 2$:

$\mathsf{K}^0_{4,2} = \text{span}\Big\{$  $\Big\} \Big/ \text{span}\Big\{$  $\Big\}$

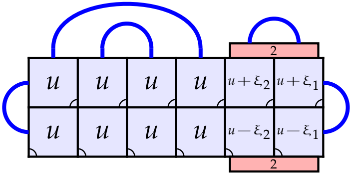
- Hamiltonians = realisations of $\mathcal{H}$ in $\mathsf{K}^d_{n,k}$

Lattice Kac modules

- Projector – half-arc annihilation relation:  $= 0$

Example:  $=$  $- \frac{1}{\beta}$  $=$  $- \frac{\beta}{\beta}$  $= 0$

- $\boldsymbol{D}(u, \xi)$ and $\mathcal{H}$ act trivially on a subspace of V^d_{n+k}:



$= 0$

- **Lattice Kac module $\mathsf{K}^d_{n,k}$:** quotient of V^d_{n+k} by the trivial subspace

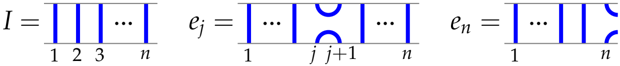
Examples for $k = 2$:

$\mathsf{K}^4_{4,2} = \text{span}\Big\{$ , , , ,  $\Big\} \Big/ \text{span}\Big\{$  $\Big\}$

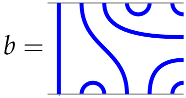
- Hamiltonians = realisations of $\mathcal{H}$ in $\mathsf{K}^d_{n,k}$

One-boundary TL algebra: $\mathsf{TL}_n^{(1)}$

Generators

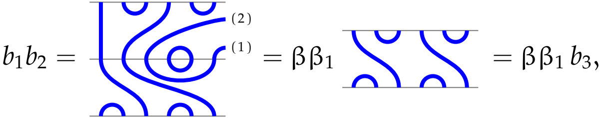


A connectivity



$= e_2e_6e_3e_5e_4e_6$

- Multiplication is again by vertical concatenation:



Algebraic definition

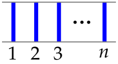
$$\mathsf{TL}_n^{(1)}(\beta, \beta_1, \beta_2) = \langle I, e_j\,;\, j = 1, \ldots, n \rangle$$

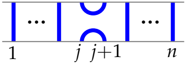
$$(e_j)^2 = \beta\, e_j \qquad e_je_{j\pm1}e_j = e_j \qquad e_ie_j = e_je_i \quad (|i-j| > 1)$$

$$e_n^2 = \beta_2\, e_n \qquad e_{n-1}e_ne_{n-1} = \beta_1 e_{n-1} \qquad e_ie_n = e_ne_i \quad (i < n-1)$$

One-boundary TL algebra: $\mathsf{TL}_n^{(1)}$

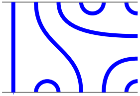
Generators

$I =$ 

$e_j =$ 

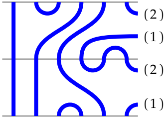
$e_n =$ 

A connectivity

$b =$ 

$= e_2 e_6 e_3 e_5 e_4 e_6$

- Multiplication is again by vertical concatenation:

$b_1 b_2 =$ 

$= \beta_2$  $= \beta_2\, b_3,$

Algebraic definition

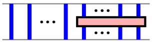
$$\mathsf{TL}_n^{(1)}(\beta, \beta_1, \beta_2) = \langle I, e_j \,;\, j = 1, \ldots, n \rangle$$

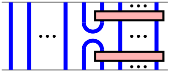
$$(e_j)^2 = \beta\, e_j \qquad e_j e_{j\pm1} e_j = e_j \qquad e_i e_j = e_j e_i \quad (|i-j| > 1)$$

$$e_n^2 = \beta_2\, e_n \qquad e_{n-1} e_n e_{n-1} = \beta_1 e_{n-1} \qquad e_i e_n = e_n e_i \quad (i < n-1)$$

Boundary seam algebras

- **Definition:** $\mathsf{B}_{n,k} = \langle I^{(k)}, E_j^{(k)}; j = 1, \ldots, n \rangle$

$I^{(k)} =$  $\qquad E_j^{(k)} =$ 

$E_n^{(k)} = U_{k-1}(\frac{\beta}{2})$ 

- $\boldsymbol{D}(u, \xi)$ and $\mathcal{H}$ are elements of $\mathsf{B}_{n,k}$.
- Algebraic relations, with $\beta_1 = U_k(\frac{\beta}{2})$, $\beta_2 = U_{k-1}(\frac{\beta}{2})$:

$$(E_j^{(k)})^2 = \beta\, E_j^{(k)} \qquad E_j^{(k)} E_{j\pm1}^{(k)} E_j^{(k)} = E_j^{(k)} \qquad E_i^{(k)} E_j^{(k)} = E_j^{(k)} E_i^{(k)} \quad (|i-j|>1)$$

$$(E_n^{(k)})^2 = \beta_2\, E_n^{(k)} \qquad E_{n-1}^{(k)} E_n^{(k)} E_{n-1}^{(k)} = \beta_1 E_{n-1}^{(k)} \qquad E_i^{(k)} E_n^{(k)} = E_n^{(k)} E_i^{(k)} \quad (i < n-1)$$

- These algebraic relations are well-defined for all β.
- $\mathsf{B}_{n,k}$ is a **quotient** of $\mathsf{TL}_n^{(1)}$. **Its generators satisfy more relations.**

Extra relations: generic case

- Extra relations for **β generic:**
  - $k = 1$: $(e_n e_{n-1} - 1)\, e_n = 0$
  - $k = 2$: $(e_n e_{n-1} e_{n-2} - \beta\, e_{n-2} + 1)(e_n e_{n-1} - \beta)\, e_n = 0$
  - $k = 3$: $e_n e_{n-1} e_{n-2} e_{n-3} e_n e_{n-1} e_{n-2} e_n e_{n-1} e_n +$ lower order terms $= 0$
  - Any k: $\Big[$**Polynomial of degree $\frac{(k+1)(k+2)}{2}$** in the $e_j\Big] = 0$
- These are the **full set of algebraic relations** defining $\mathsf{B}_{n,k}$.
- The lattice Kac modules $\mathsf{K}^d_{n,k}$ are really modules over $\mathsf{B}_{n,k}$.

Extra relations: roots of unity

- For roots of unity, **the diagrammatic algebra is not well-defined.**
- The **algebraic relations are well-defined** in the limit $\beta \to \beta_c$.
- We define $\mathsf{B}_{n,k}$ through its algebraic relations only.
- The extra relation is different than in the generic case:
  - Any k: $\left[\textbf{Polynomial of degree } \frac{(k'+1)(k'+2)}{2} \text{ in the } e_j\right] = 0$

$$\left(\beta = 2\cos\frac{\pi(p'-p)}{p'} \qquad k' = k \bmod p' \qquad 1 \leq k' \leq p'\right)$$

Example: $(p, p') = (1, 2)$ $\quad k = 3$: $\qquad (e_n e_{n-1} + 1)\, e_n = 0$

$(k' = 1 \to \text{degree } 3 \text{ instead of } 10)$

- Lattice Kac modules have **no singularities** in the limit $\beta \to \beta_c$.

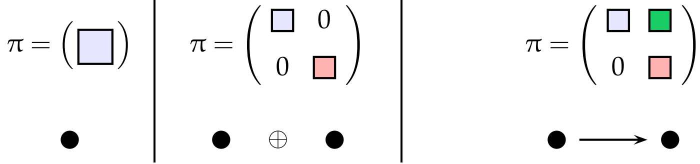
Virasoro algebra and modules

- Defining relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n=0}$$

- Describes the scaling limit of critical statistical models
- Admits a large spectrum of representations

irreducible | **fully reducible** | **reducible yet indecomposable**



- Rational conformal field theories are well understood.
- Logarithmic conformal field theories are less understood.

Verma modules

- Definition of $\mathcal{V}_\Delta$: $\quad L_n|\Delta\rangle = 0 \text{ for } n > 0, \quad L_0|\Delta\rangle = \Delta|\Delta\rangle$
- **Character:** $\quad \mathrm{ch}(\mathcal{V}_\Delta) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}}{\prod_i (1 - q^i)}$
- **Central charge** and **conformal dimensions**:

$$c = 1 - \frac{6(p'-p)^2}{pp'} \qquad \Delta_{r,s} = \frac{(p'r-ps)^2-(p'-p)^2}{4pp'} \qquad p, p' \in \mathbb{Z}_+ \quad r, s \in \mathbb{Z}_+$$

- **Extended Kac table** for percolation:

$(p, p') = (2, 3)$

$c = 0$

$r \downarrow$ \ $s \rightarrow$	1	2	3	4	5	6	7	8	9	
1	0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	5	7	$\frac{28}{3}$	…
2	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	$\frac{21}{8}$	$\frac{33}{8}$	$\frac{143}{24}$	…
3	2	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	…
4	$\frac{33}{8}$	$\frac{21}{8}$	$\frac{35}{24}$	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	…
5	7	5	$\frac{10}{3}$	2	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Verma modules

- Definition of $\mathcal{V}_\Delta$: $L_n|\Delta\rangle = 0$ for $n > 0$, $L_0|\Delta\rangle = \Delta|\Delta\rangle$
- **Character:** $\mathrm{ch}(\mathcal{V}_\Delta) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}}{\prod_i (1-q^i)}$
- **Central charge** and **conformal dimensions**:

$$c = 1 - \frac{6(p'-p)^2}{pp'} \qquad \Delta_{r,s} = \frac{(p'r-ps)^2-(p'-p)^2}{4pp'} \qquad p, p' \in \mathbb{Z}_+ \quad r, s \in \mathbb{Z}_+$$

- **Extended Kac table** for the Ising model:

$(p, p') = (3, 4)$

$c = \frac{1}{2}$

r \ s	1	2	3	4	5	6	7	8	9	
1	0	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{21}{16}$	$\frac{5}{2}$	$\frac{65}{16}$	6	$\frac{133}{16}$	11	…
2	$\frac{1}{2}$	$\frac{1}{16}$	0	$\frac{5}{16}$	1	$\frac{33}{16}$	$\frac{7}{2}$	$\frac{85}{16}$	$\frac{15}{2}$	…
3	$\frac{5}{3}$	$\frac{35}{48}$	$\frac{1}{6}$	$-\frac{1}{48}$	$\frac{1}{6}$	$\frac{35}{48}$	$\frac{5}{3}$	$\frac{143}{48}$	$\frac{14}{3}$	…
4	$\frac{7}{2}$	$\frac{33}{16}$	1	$\frac{5}{16}$	0	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{21}{16}$	$\frac{5}{2}$	…
5	6	$\frac{65}{16}$	$\frac{5}{2}$	$\frac{21}{16}$	$\frac{1}{2}$	$\frac{1}{16}$	0	$\frac{5}{16}$	1	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Verma modules

- Definition of $\mathcal{V}_\Delta$: $L_n|\Delta\rangle = 0$ for $n > 0$, $L_0|\Delta\rangle = \Delta|\Delta\rangle$

- **Character:** $\mathrm{ch}(\mathcal{V}_\Delta) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}}{\prod_i (1-q^i)}$

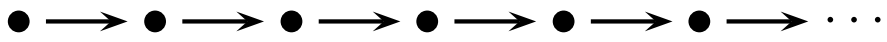
- **Central charge** and **conformal dimensions**:

$$c = 1 - \frac{6(p'-p)^2}{pp'} \qquad \Delta_{r,s} = \frac{(p'r-ps)^2-(p'-p)^2}{4pp'} \qquad p, p' \in \mathbb{Z}_+ \quad r, s \in \mathbb{Z}_+$$

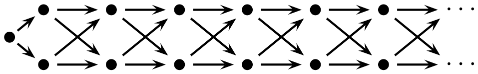
- **Module structures for $\mathcal{V}_\Delta$:**

Not in the Kac table :

Boundary and corner entries :



Interior entries :



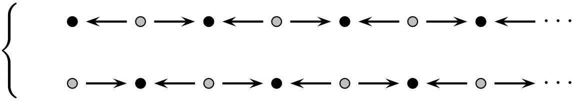
Feigin-Fuchs modules

- Arise in the **Coulomb gas realisation** of the Virasoro algebra
- **Character:** $$\mathrm{ch}(\mathcal{F}_\Delta) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \frac{q^{\Delta - \frac{c}{24}}}{\prod_i (1 - q^i)}$$
- **Module structures for $\mathcal{F}_\Delta$:** (Feigin, Fuchs 1982)

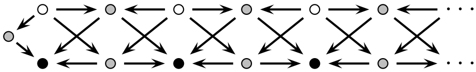
Not in the Kac table :

Corner entries :

Boundary entries :



Interior entries :

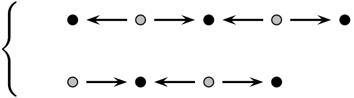


Virasoro Kac modules

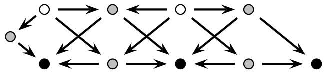
- Only defined for **conformal dimensions in the Kac table**
- **Character:** $$\mathrm{ch}(\mathcal{K}_{r,s}) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \frac{q^{\Delta - \frac{c}{24}}(1 - q^{rs})}{\prod_i (1 - q^i)}$$
- Definition: $\mathcal{K}_{r,s}$ is the **submodule of $\mathcal{F}_{\Delta_{r,s}}$ generated by all states with levels less than rs.**
- **Module structures for $\mathcal{K}_{r,s}$:**

Corner entries :

Boundary entries :

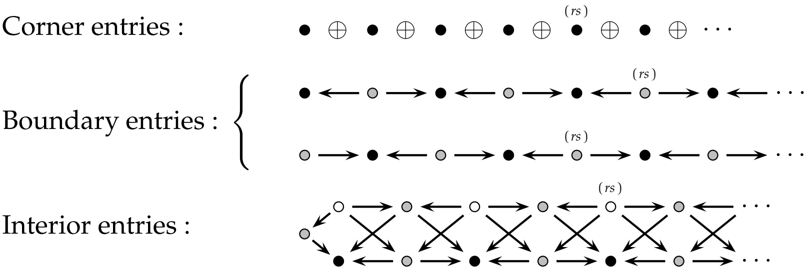


Interior entries :



Virasoro Kac modules

- Only defined for **conformal dimensions in the Kac table**
- **Character:** $\mathrm{ch}(\mathcal{K}_{r,s}) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}(1-q^{rs})}{\prod_i (1-q^i)}$
- Definition: $\mathcal{K}_{r,s}$ is the **submodule of $\mathcal{F}_{\Delta_{r,s}}$ generated by all states with levels less than rs.**
- **Module structures for $\mathcal{F}_\Delta$:**



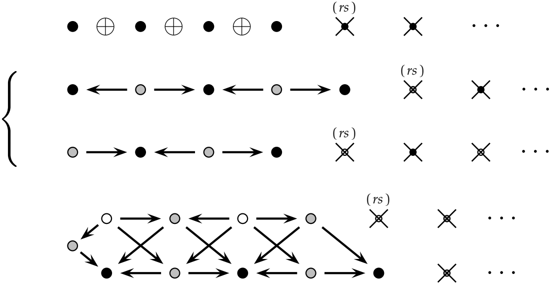
Virasoro Kac modules

- Only defined for **conformal dimensions in the Kac table**
- **Character:** $\mathrm{ch}(\mathcal{K}_{r,s}) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}(1 - q^{rs})}{\prod_i (1 - q^i)}$
- Definition: $\mathcal{K}_{r,s}$ is the **submodule of $\mathcal{F}_{\Delta_{r,s}}$ generated by all states with levels less than rs.**
- **Module structures for $\mathcal{K}_{r,s}$:**

Corner entries :

Boundary entries :

Interior entries :

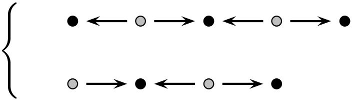


Virasoro Kac modules

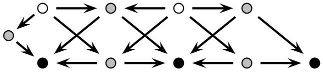
- Only defined for **conformal dimensions in the Kac table**
- **Character:** $\mathrm{ch}(\mathcal{K}_{r,s}) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}(1-q^{rs})}{\prod_i (1-q^i)}$
- Definition: $\mathcal{K}_{r,s}$ is the **submodule of $\mathcal{F}_{\Delta_{r,s}}$ generated by all states with levels less than rs.**
- **Module structures for $\mathcal{K}_{r,s}$:**

Corner entries :

Boundary entries :

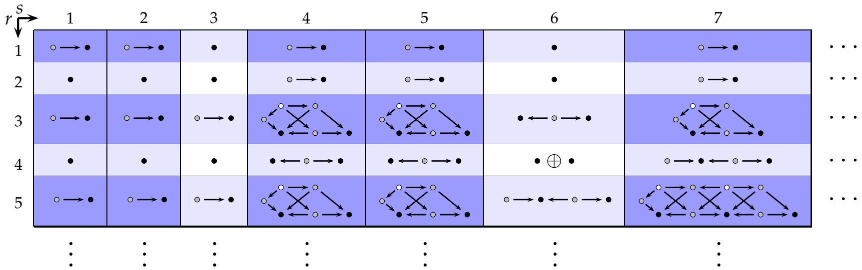


Interior entries :



Virasoro Kac modules

- Only defined for **conformal dimensions in the Kac table**
- **Character:** $\mathrm{ch}(\mathcal{K}_{r,s}) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}}) = \dfrac{q^{\Delta - \frac{c}{24}}(1-q^{rs})}{\prod_i (1-q^i)}$
- Definition: $\mathcal{K}_{r,s}$ is the **submodule of $\mathcal{F}_{\Delta_{r,s}}$ generated by all states with levels less than rs.**
- **Examples for percolation:** $(p,p') = (2,3) \qquad c = 0$



Scaling limit and conformal structure

- **Scaling limit:** define sequences of eigenstates of $\mathcal{H}$ of eigenvalue H_n^i in $\mathsf{K}_{n,k}^d$ for increasing n. Retain those for which

$$\lim_{n\to\infty} n\,(H_n^i - H_n^0) = \kappa \quad \text{for some} \quad \kappa < \infty$$

(H_n^0 is the **ground-state eigenvalue**)

- The surviving sequences give rise to the **states of a Virasoro module**.
- In this limit, **$\mathcal{H}$ "becomes" $L_0 - \frac{c}{24}$** in some Virasoro module:

$$\frac{n}{\pi v_s}\left(\mathcal{H} - n f_{bulk} - f_{bdy}\right) \xrightarrow{n\to\infty} L_0 - \frac{c}{24} \qquad \left(v_s = \frac{\pi \sin\lambda}{\lambda}\right)$$

Scaling limit and conformal structure

- **Scaling limit:** define sequences of eigenstates of $\mathcal{H}$ of eigenvalue H_n^i in $\mathsf{K}_{n,k}^d$ for increasing n. Retain those for which

$$\lim_{n\to\infty} n\,(H_n^i - H_n^0) = \kappa \quad \text{for some} \quad \kappa < \infty$$

(H_n^0 is the **ground-state eigenvalue**)

- The surviving sequences give rise to the **states of a Virasoro module**.
- In this limit, **$\mathcal{H}$ "becomes" $L_0 - \frac{c}{24}$** in some Virasoro module:

$$\frac{n}{\pi v_s}\Big(\mathcal{H} - n f_{bulk} - f_{bdy}\Big) \xrightarrow{n\to\infty} L_0 - \frac{c}{24} \qquad \Big(v_s = \frac{\pi \sin\lambda}{\lambda}\Big)$$

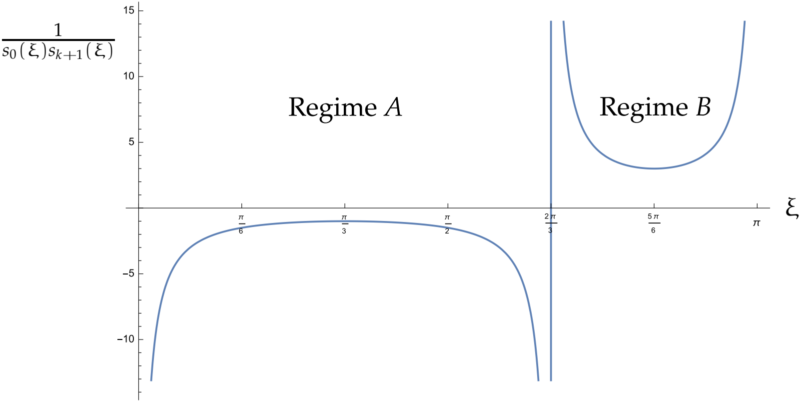
- **Conjecture:** in regime A, **lattice Kac modules become Virasoro Kac modules** in the scaling limit:

$$\mathsf{K}_{n,k}^d \xrightarrow{n\to\infty} \mathcal{K}_{r,s} \qquad r = \left\lceil \frac{(k+1)p}{p'} \right\rceil \qquad s = d+1$$

(Rasmussen 2011; Pearce, Rasmussen, Villani 2013; AMD, Rasmussen, Ridout 2015)

Regimes A and B

- Example for $(p, p') = (2, 3), k = 3$:



- The structure of the Virasoro modules is different in regimes A and B, but is generally **unchanged within a given regime.**

Evidence from character approximations

- Recall that:

$$\mathcal{H} \xrightarrow{n\to\infty} L_0 - \frac{c}{24} \qquad \mathrm{ch}(\mathcal{K}_{r,s}) = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} = \frac{q^{\Delta - \frac{c}{24}}(1-q^{rs})}{\prod_i (1-q^i)}$$

- For given $\mathsf{K}^d_{n,k}$, Δ can be estimated numerically from H^0_n for small n.

(Pearce, Rasmussen, Zuber 2006; Pearce, Tartaglia, Couvreur 2014)

- **Character approximations:**
  - find the eigenvalues H^i_n of $\mathcal{H}$ using a computer
  - compute the ratios $R^i_n = \dfrac{H^i_n - H^0_n}{H^1_n - H^0_n}$ and the sum $\sum_i q^{R^i_n}$
  - compare with $\widehat{\mathrm{ch}}(\mathcal{K}_{r,s}) = \dfrac{1-q^{rs}}{\prod_i (1-q^i)}$

Evidence from character approximations

- Example for $(p, p') = (1, 3)$ $k = 0$ $d = 1$:

$n = 13$	$1 + q + q^{2.05} + q^{2.96} + q^{3.15} + q^{3.85} + q^{4.15} + q^{4.31} + q^{4.57} + q^{4.78} + \cdots$
$n = 15$	$1 + q + q^{2.04} + q^{2.97} + q^{3.11} + q^{3.89} + q^{4.11} + q^{4.24} + q^{4.68} + q^{4.83} + \cdots$
$n = 17$	$1 + q + q^{2.03} + q^{2.98} + q^{3.09} + q^{3.91} + q^{4.09} + q^{4.20} + q^{4.76} + q^{4.87} + \cdots$
$n = 19$	$1 + q + q^{2.02} + q^{2.98} + q^{3.07} + q^{3.93} + q^{4.07} + q^{4.16} + q^{4.81} + q^{4.90} + \cdots$
$\widehat{\mathrm{ch}}(\mathcal{K}_{1,2})$	$1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 6q^6 + \cdots$

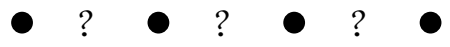
Evidence from character approximations

- Example for $(p,p') = (1,3) \quad k=0 \quad d=1$:

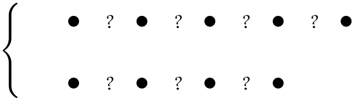
$n=13$	$1+q+q^{2.05}+q^{2.96}+q^{3.15}+q^{3.85}+q^{4.15}+q^{4.31}+q^{4.57}+q^{4.78}+\cdots$
$n=15$	$1+q+q^{2.04}+q^{2.97}+q^{3.11}+q^{3.89}+q^{4.11}+q^{4.24}+q^{4.68}+q^{4.83}+\cdots$
$n=17$	$1+q+q^{2.03}+q^{2.98}+q^{3.09}+q^{3.91}+q^{4.09}+q^{4.20}+q^{4.76}+q^{4.87}+\cdots$
$n=19$	$1+q+q^{2.02}+q^{2.98}+q^{3.07}+q^{3.93}+q^{4.07}+q^{4.16}+q^{4.81}+q^{4.90}+\cdots$
$\widehat{\mathrm{ch}}(\mathcal{K}_{1,2})$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6+\cdots$

- The character only provides **partial information:**

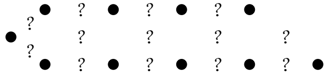
Corner entries :



Boundary entries :



Interior entries :



Evidence from character approximations

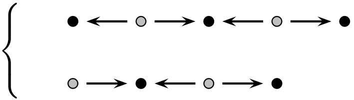
- Example for $(p,p') = (1,3) \quad k=0 \quad d=1$:

$n=13$	$1+q+q^{2.05}+q^{2.96}+q^{3.15}+q^{3.85}+q^{4.15}+q^{4.31}+q^{4.57}+q^{4.78}+\cdots$
$n=15$	$1+q+q^{2.04}+q^{2.97}+q^{3.11}+q^{3.89}+q^{4.11}+q^{4.24}+q^{4.68}+q^{4.83}+\cdots$
$n=17$	$1+q+q^{2.03}+q^{2.98}+q^{3.09}+q^{3.91}+q^{4.09}+q^{4.20}+q^{4.76}+q^{4.87}+\cdots$
$n=19$	$1+q+q^{2.02}+q^{2.98}+q^{3.07}+q^{3.93}+q^{4.07}+q^{4.16}+q^{4.81}+q^{4.90}+\cdots$
$\widehat{\mathrm{ch}}(\mathcal{K}_{1,2})$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6+\cdots$

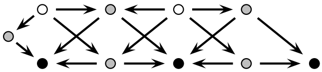
- The character only provides **partial information:**

Corner entries :

Boundary entries :



Interior entries :



Evidence from TL_n representation theory

- Applies for the case where there is **no seam** ($k = 0$).
- **Lattice deformations of Virasoro modes:** (Koo, Saleur 1994)

$$L_m^{(n)} = \frac{n}{\pi}\left[-\frac{1}{v_s}\sum_{j=1}^{n-1}(e_j - f_{bulk})\cos\left(\frac{\pi m j}{n}\right) + \frac{1}{v_s^2}\sum_{j=1}^{n-2}[e_j, e_{j+1}]\sin\left(\frac{\pi m j}{n}\right)\right] + \frac{c}{24}\delta_{m,0}.$$

- The structure of the limiting Virasoro module can be deduced from:
  - the character
  - the computation of the first eigenstates of $\mathcal{H}$ for small system size
  - the known structure of $\mathsf{K}_{n,0}^d$

Example 1:

$$(\mathsf{K}_{n,0}^d) \qquad\qquad (\mathcal{K}_{1,s})$$

$$\mathsf{I}_{n,0}^d \downarrow \mathsf{I}_{n,0}^{d'} \quad \xrightarrow{n\to\infty} \quad \bullet \longrightarrow \bullet$$

The other cases $\bullet \longleftarrow \bullet$ and $\bullet \oplus \bullet$ can be ruled out.

Evidence from TL_n representation theory

- Applies for the case where there is **no seam** ($k = 0$).
- **Lattice deformations of Virasoro modes:** (Koo, Saleur 1994)

$$L_m^{(n)} = \frac{n}{\pi}\left[-\frac{1}{v_s}\sum_{j=1}^{n-1}(e_j - f_{bulk})\cos\left(\frac{\pi m j}{n}\right) + \frac{1}{v_s^2}\sum_{j=1}^{n-2}\left[e_j, e_{j+1}\right]\sin\left(\frac{\pi m j}{n}\right)\right] + \frac{c}{24}\delta_{m,0}.$$

- The structure of the limiting Virasoro module can be deduced from:
  - the character
  - the computation of the first eigenstates of $\mathcal{H}$ for small system size
  - the known structure of $\mathsf{K}_{n,0}^d$

$$(\mathsf{K}_{n,0}^d) \qquad\qquad (\mathcal{K}_{1,s})$$

Example 2: $\quad \mathsf{I}_{n,0}^d \quad \xrightarrow{n\to\infty} \quad \bullet$

Here, the character already determines the structure.

Evidence from $\mathsf{B}_{n,k}$ representation theory

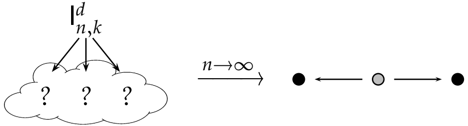
- Recall: Lattice Kac modules $\mathsf{K}^d_{n,k}$ are really modules over $\mathsf{B}_{n,k}$.
- The representation theory of $\mathsf{B}_{n,k}$ is **not known.**

Partial analysis of the module structure of $\mathsf{K}^d_{n,k}$	$\rightarrow$	Partial understanding of the structure of $\mathcal{K}_{r,s}$

- Same strategy as for $k = 0$ and TL_n:

$(\mathsf{K}^d_{n,k})$ $\qquad\qquad$ $(\mathcal{K}_{r,s})$

Example 1:



- **This analysis is consistent with the conjecture** in every case we looked at.

Evidence from $\mathsf{B}_{n,k}$ representation theory

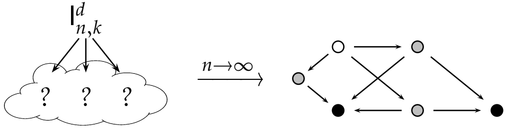
- Recall: Lattice Kac modules $\mathsf{K}^d_{n,k}$ are really modules over $\mathsf{B}_{n,k}$.
- The representation theory of $\mathsf{B}_{n,k}$ is **not known.**

Partial analysis of the module structure of $\mathsf{K}^d_{n,k}$	$\longrightarrow$	Partial understanding of the structure of $\mathcal{K}_{r,s}$

- Same strategy as for $k = 0$ and TL_n:

$(\mathsf{K}^d_{n,k})$ $\qquad\qquad$ $(\mathcal{K}_{r,s})$

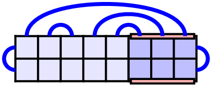
Example 2:



- **This analysis is consistent with the conjecture** in every case we looked at.

Evidence from fusion

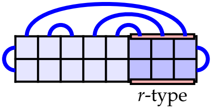
- **Lattice prescription for fusion:** (Cardy 1986; Pearce, Rasmussen, Zuber 2006)



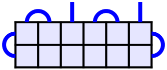
$$\mathsf{K}^0_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,1}$$

Evidence from fusion

- **Lattice prescription for fusion:** (Cardy 1986; Pearce, Rasmussen, Zuber 2006)



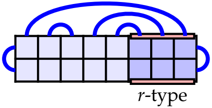
$$\mathsf{K}^0_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,1}$$



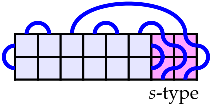
$$\mathsf{K}^d_{n,0} \xrightarrow{n\to\infty} \mathcal{K}_{1,s}$$

Evidence from fusion

- **Lattice prescription for fusion:** (Cardy 1986; Pearce, Rasmussen, Zuber 2006)



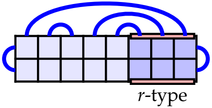
$$\mathsf{K}^0_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,1}$$



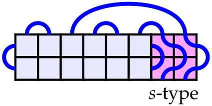
$$\mathsf{K}^d_{n,0} \xrightarrow{n\to\infty} \mathcal{K}_{1,s}$$

Evidence from fusion

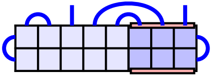
- **Lattice prescription for fusion:** (Cardy 1986; Pearce, Rasmussen, Zuber 2006)



$$\mathsf{K}^0_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,1}$$



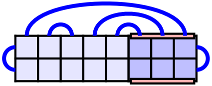
$$\mathsf{K}^d_{n,0} \xrightarrow{n\to\infty} \mathcal{K}_{1,s}$$



$$\mathsf{K}^d_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,s} \overset{?}{=} \mathcal{K}_{r,1} \times \mathcal{K}_{1,s}$$

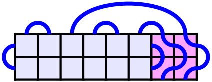
Evidence from fusion

- **Lattice prescription for fusion:** (Cardy 1986; Pearce, Rasmussen, Zuber 2006)



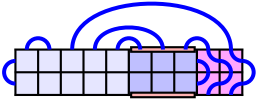
r-type

$$\mathsf{K}^0_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,1}$$



s-type

$$\mathsf{K}^d_{n,0} \xrightarrow{n\to\infty} \mathcal{K}_{1,s}$$

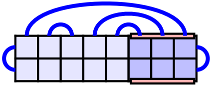


(r,s)-type

$$\mathsf{K}^d_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,s} \stackrel{?}{=} \mathcal{K}_{r,1} \times \mathcal{K}_{1,s}$$

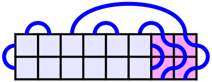
Evidence from fusion

- **Lattice prescription for fusion:** (Cardy 1986; Pearce, Rasmussen, Zuber 2006)



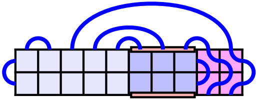
r-type

$$\mathsf{K}^0_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,1}$$



s-type

$$\mathsf{K}^d_{n,0} \xrightarrow{n\to\infty} \mathcal{K}_{1,s}$$



(r,s)-type

$$\mathsf{K}^d_{n,k} \xrightarrow{n\to\infty} \mathcal{K}_{r,s} \stackrel{?}{=} \mathcal{K}_{r,1} \times \mathcal{K}_{1,s}$$

- Evidence supporting that $\mathcal{K}_{r,s} = \mathcal{K}_{r,1} \times \mathcal{K}_{1,s}$ as Virasoro modules:
  - **Verlinde-like formula** for the characters:

$$\mathrm{ch}(\mathcal{K}_{r,1} \times \mathcal{K}_{1,s}) = \mathrm{ch}(\mathcal{K}_{r,s})$$

  - Construction of $\mathcal{K}_{r,1} \times \mathcal{K}_{1,s}$ at any desired grade using the **Nahm-Gaberdiel-Kausch algorithm**

Conclusion

Summary

- The **boundary seam algebras $\mathsf{B}_{n,k}$** are quotients of the one-boundary TL algebra.
- They describe dense loop models with a boundary seam.
- In the scaling limit, its modules become **Virasoro Kac modules**.

Outlook

- Work out the representation theory of $\mathsf{B}_{n,k}$.
- Understand what's happening in regime B.
- Study loop models with boundary seams on both sides.

Conclusion

Summary

- The **boundary seam algebras** $\mathsf{B}_{n,k}$ are quotients of the one-boundary TL algebra.
- They describe dense loop models with a boundary seam.
- In the scaling limit, its modules become **Virasoro Kac modules**.

Outlook

- Work out the representation theory of $\mathsf{B}_{n,k}$.
- Understand what's happening in regime B.
- Study loop models with boundary seams on both sides.

Thank you!