The Fibonacci family of non-equilibrium universality classes

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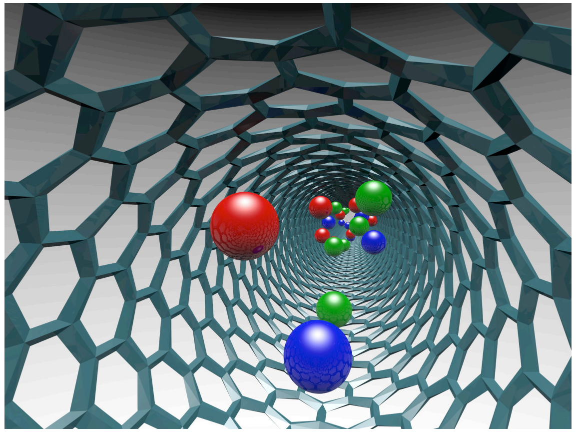
- Introduction
- Nonlinear fluctuating hydrodynamics
- Mode coupling theory
- Simulation results
- Conclusions

1. Introduction

Bulk-driven Particle systems with several conservation laws:

- Interacting stochastic particle systems on lattice with biased hopping
  ==> Non-reversible Markovian dynamics

- A few (!) examples:
  (1) Multi-species exclusion processes
  (2) Bricklayer model
  (3) Multilane exclusion processes



- Many Applications:
  (1) Diffusion in carbon nano tubes
  (2) Molecular Motors
  (3) Automobile traffic flow

- Rich behaviour, e.g.,
  - Phase transitions (phase separation, spontaneous symmetry breaking)
  - Hydrodynamic equations sensitive to regularization
  - Intricate interplay of shocks and rarefaction waves
  - Universal fluctuations: z=2 (Diffusive), z=3/2 (KPZ) ➔ Is that all?

Invariant measures for driven diffusive systems:

Assume translation invariance and ergodicity for fixed values of conserved particle numbers N_α

==> “canonical” invariant measure μ is unique and translation invariant

Construct “grandcanonical” invariant measures with chemical potentials ϕ^α

- Stationary densities of particles of tye α: $\rho_\alpha(\{\phi\})$
- Stationary current of particle of type α: $j_\alpha(\{\phi\})$

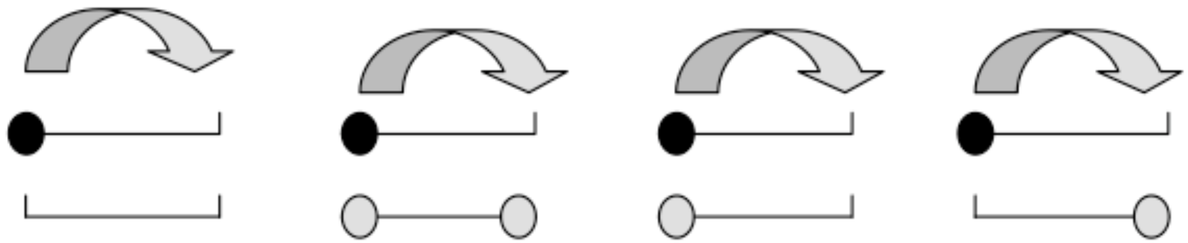
Onsager-type current symmetry $$\frac{\partial j^\alpha}{\partial \phi^\beta} = \frac{\partial j^\beta}{\partial \phi^\alpha}$$ [Grisi, GMS, 2011]

- Compressibility matrix K: $(K)_{\alpha\beta} = \partial\rho_\alpha/\partial\phi_\beta = 1/L \langle(N_\alpha - L\rho_\alpha)(N_\beta - L\rho_\beta)\rangle$

- Current Jacobian J: $(J)_{\alpha\beta} = \partial j_\alpha/\partial\rho_\beta$

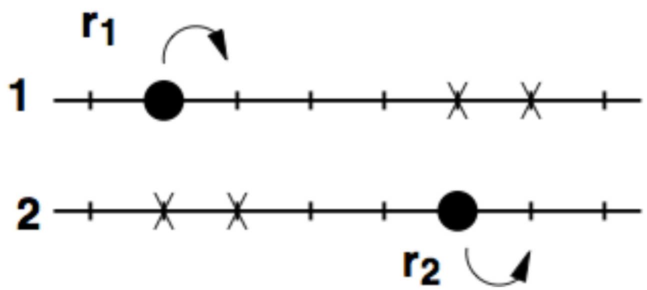
Current symmetry ➔ $JK = (JK)^T$

- Model: Interacting multi-lane TASEPs with densities ρ_i



For simplicity focus on two-lane model:

- $r_1 = 1 + \gamma\, n^{(2)}/2, \;\; r_2 = b + \gamma\, n^{(1)}/2$ [Popkov, Salerno (2004)]



- Invariant measure: Canonical: Uniform ➔ Grandcanonical: Product

$$j_1(\rho_1, \rho_2) = \rho_1(1-\rho_1)(1+\gamma\rho_2)$$
$$j_2(\rho_1, \rho_2) = \rho_2(1-\rho_2)(b+\gamma\rho_1)$$

2. Nonlinear fluctuating hydrodynamics

Study large-scale dynamics under Eulerian scaling $x = ka$, $t = \tau a$, $a \to 0$:

==> Law large numbers: Occupation numbers on lattice $n_{\alpha k}(t) \to \rho_\alpha(x,t)$ (Coarse-grained particle densities)

==> Local stationarity: Microscopic current $j^\alpha_k(t) \to j^\alpha(\{\rho\})$: Associated locally stationary currents

==> Lattice continuity equation → System of hyperbolic conservation laws

$$\frac{\partial}{\partial t}\vec{\rho} + A\frac{\partial}{\partial x}\vec{\rho} = 0$$

- Origin of hyperbolicity: Onsager-type symmetry

- General validity: Driven Diffusive Systems, Anharmonic chains, Hamiltonian dynamics, …

Introduce fluctuation fields $u_i(x,t) = \rho_i(x,t) - \rho_i$ and expand in u_i

A) <u>Linear theory:</u>

- Diagonalize A: $RAR^{-1} = \mathrm{diag}(v_i)$, Normalization $RCR^T = 1$

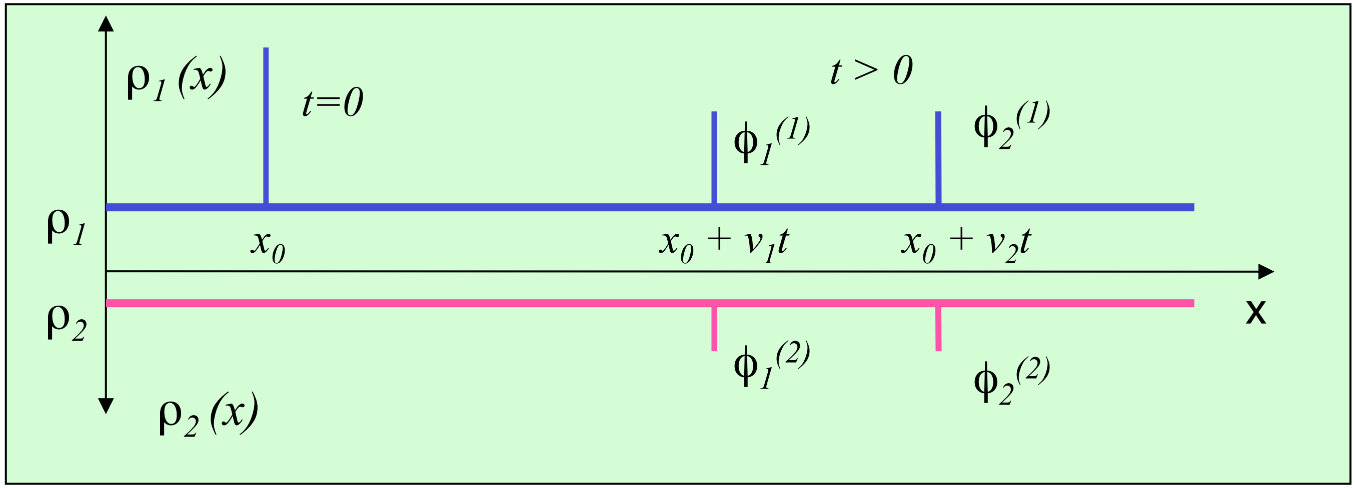
==> Eigenmode equation for normal modes $\phi = R\,\mathbf{u}$: $\partial_t \phi_i = - v_i \partial_x \phi_i$

- Travelling waves (eigenmodes) $\phi_i(x,t) = \phi_i(x-v_i t)$

- Characteristic speeds $v_{1,2}(\rho_1, \rho_2)$ = eigenvalues of current Jacobian A

- Strict hyperbolicity for two-lane model: $v_1 \neq v_2 \;\; \forall\, (\rho_1, \rho_2) \in (0,1) \times (0,1)$

- Microscopic: Stationary center of mass motion of localized perturbation
  [Popkov, GMS (2003)]



B) Nonlinear fluctuating theory

• Expand to second order, add phenomenological diffusion term and noise [Spohn]

$$\partial_t \phi_i = -\partial_x [c_i \phi_i + \langle \vec{\phi}, G^{(i)} \vec{\phi} \rangle - \partial_x (D\vec{\phi})_i + (B\vec{\xi})_i]$$

diffusion = regularization, noise B and diffusion matrix D related by FDT

- Mode coupling coefficients for eigenmodes

$$G^{(i)} = (1/2) \sum_j R_{ij} (R^{-1})^T H^{(j)} R^{-1}$$

- Hessian $H^{(\gamma)}$ with matrix elements $\partial^2 j_\gamma / (\partial\rho_\alpha \partial\rho_\beta)$

One component: $\partial_t \phi = - \partial_x [c \phi + g \phi^2 - D \partial_x \phi + B \xi]$ (KPZ equation, $g = j''/2$)

Two components ==> Two coupled KPZ equations

Remarks:

1) Higher order terms irrelevant in RG sense (if second order non-zero)

2) Offdiagonal terms neglible for strictly hyperbolic systems (no overlap between modes)

3) Self-coupling terms $G^{(\alpha)}_{\alpha\alpha}$ leading, other diagonal terms $G^{(\alpha)}_{\beta\beta}$ subleading

$$==> \begin{cases} \partial_t \phi_1 = - \partial_x [c_1 \phi_1 + G^{(1)}_{11} (\phi_1)^2 + G^{(1)}_{22} (\phi_2)^2 + \text{diff.} + \text{noise}] \\ \partial_t \phi_2 = - \partial_x [c_2 \phi_2 + G^{(2)}_{11} (\phi_1)^2 + G^{(2)}_{22} (\phi_2)^2 + \text{diff.} + \text{noise}] \end{cases}$$

3. Mode coupling theory

Go beyond LLN and study fluctuations:

- Dynamical structure function (lattice)

$$S_{\alpha\beta}(p,t) = \sum_k e^{ikp} <(\xi^{\alpha}_k (t) - \rho_\alpha)(\xi^{\beta}_0(0) - \rho_\beta) > = <u_\alpha(p,t)\ u_\beta(-p,t)>$$

where $u_\alpha(p,t)$ = Fourier transform of locally conservd quantity $\xi^{\alpha}_k (t) - \rho_\alpha$

- <u>One conservation law:</u> Scaling form $S(p,t) = F(p^z t)$

- KPZ universality class z=3/2, universal scaling function F [Praehofer, Spohn (2002)]

- <u>Several conservation laws:</u> Different universality classes in the same DDS

- Known cases for two-component DDS: (a) Both KPZ (generic)
  (b) KPZ and Diffusive (z=2)
  [Das et al (2001), Rakos, GMS (2005)]

==> Is that all there is?

Mode coupling scenarios [van Beijeren (2012), Spohn (2013), Popkov, Schmidt, GMS (2014)]

Some scenarios:

A) Both self-coupling coefficients nonzero: $G^{(1)}_{11} \neq 0$, $G^{(2)}_{22} \neq 0$

==> two KPZ modes ($z_1 = 3/2$, $z_2=3/2$)

B) One self-coupling coefficient nonzero, all other diagonal terms of mode-coupling matrices 0, e.g., $G^{(1)}_{11} \neq 0$, $G^{(1)}_{22} = G^{(2)}_{22} = G^{(2)}_{11} = 0$

==> one KPZ mode, one diffusive mode ($z_1 = 3/2$, $z_2=2$)

C) One self-coupling coefficient nonzero, subleading diagonal of other mode-coupling matrix 0, e.g., $G^{(1)}_{11} \neq 0$, $G^{(2)}_{11} \neq 0$, $G^{(2)}_{22} = 0$

==> one KPZ mode, second non-KPZ superdiffusive mode ($z_1 = 3/2$, $z_2=5/3$)

Remark: Heat mode with $z=5/3$, two KPZ sound modes in Hamiltonian dynamics with three conservation laws [van Beijeren (2012)]

General solution of mode coupling equations (cont')

Scaling ansatz, with $\tilde{\omega}_\alpha := \omega + iv_\alpha p \; \zeta_\alpha = \tilde{\omega}_\alpha |p|^{-z_\alpha}$

$$\tilde{S}_\alpha(p, \tilde{\omega}_\alpha) = p^{-z_\alpha} g_\alpha(\zeta_\alpha)$$

Subballistic scaling z>1 (short range interactions) and strict hyperbolicity:

$$g_\alpha(\zeta_\alpha) = \lim_{k\to 0} \left[\zeta_\alpha + D_\alpha |p|^{2-z_\alpha} + Q_{\alpha\alpha} \zeta_\alpha^{2-z_\alpha - \frac{1}{z_\alpha}} |p|^{3-2z_\alpha} + \sum_{\beta\neq\alpha} Q_{\alpha\beta} \left(-iv_p^{\alpha\beta}\right)^{\frac{1}{z_\beta}-1} |p|^{1+\frac{1}{z_\beta}-z_\alpha} \right]^{-1}$$

with $v_p^{\alpha\beta} := |v_\alpha - v_\beta| \mathrm{sgn}[p(v_\alpha - v_\beta)]$

$$Q_{\alpha\beta} = 2(G^\alpha_{\beta\beta})^2 \Gamma\left(1 - \frac{1}{z_\beta}\right) \Omega[\hat{S}_\beta] \geq 0$$

$\Omega[\hat{f}] = \int_{-\infty}^{\infty} \mathrm{d}p \, \hat{f}(p) \hat{f}(-p)$ (Integral over square of real-space scaling function)

General solution of mode coupling equations (cont')

Power counting:

$$z_\alpha = \begin{cases} 2 & \text{if} \quad \mathbb{I}_\alpha = \emptyset \\ 3/2 & \text{if} \quad \alpha \in \mathbb{I}_\alpha \\ \min_{\beta \in \mathbb{I}_\alpha} \left[\left(1 + \frac{1}{z_\beta} \right) \right] & \text{else} \end{cases}$$

where

$$\mathbb{I}_\alpha := \{\beta : G^\alpha_{\beta\beta} \neq 0\}$$

Structure of diagonal elements of mode coupling matrix ➔ universality classes

1) All diagonal elements of mode α vanish: Diffusive mode (z=2)

2) Self coupling $G^\alpha_{\alpha\alpha} \neq 0$: KPZ or modified KPZ (for non-zero coupling to diffusive mode [Stoltz, Spohn (2015)]

3) Otherwise: ?

General solution of mode coupling equations (Case 3)

A) Coupling to either Diffusive or KPZ-type mode:

(i) Consider sequential coupling with KPZ mode 1:

➔ Recursion for dynamical exponents $$z_\alpha = 1 + \frac{1}{z_{\alpha-1}}$$

Solution: Kepler ratios $$\boxed{z_\alpha = \frac{F_{\alpha+3}}{F_{\alpha+2}}}$$

of neighbouring Fibonacci numbers 1, 1, 2, 3, 5, 8,…

➔ z = 3/2, 5/3, 8/5, … → $\varphi = (1+\sqrt{5})/2 \approx 1.618$ (golden mean)

Scaling functions: z-stable Levy distributions

$$\hat{S}_\alpha(p,t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-iv_\alpha pt - E_\alpha |p|^{z_\alpha} t \left(1 - i\sigma_p^{\alpha\beta} \tan\left(\frac{\pi z_\alpha}{2}\right)\right)}$$

Time scales and asymmetries given recursively in terms of mode coupling coefficients and (numerically known) integral over square of KPZ scaling function

General solution of mode coupling equations (Case 3 cont')

(ii) Consider sequential coupling to *diffusive* mode 1:

➔ Same recursion for dynamical exponents, but shifted initial value

Solution: Shifted Kepler ratios $z = 2, 3/2, 5/3, 8/5, \ldots$

Scaling functions: Levy for $z<2$, time scales contain diffusion coefficient instead of KPZ-integral

(iii) Non-sequential coupling: Still valid for large general class of mode-coupling matrices (work in progress)

B) No coupling to either diffusive or KPZ-type mode

➔ Golden mean universality class

Universality classes for two-component systems

[Popkov, Schmidt, GMS (2015); Spohn, Stoltz (2015)]

$G^2 \diagdown G^1$	$\begin{pmatrix} \star & \\ & \bullet \end{pmatrix}$	$\begin{pmatrix} \star & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} & \\ & \bullet \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} & \\ & 0 \end{pmatrix}$
$\begin{pmatrix} \bullet & \\ & \star \end{pmatrix}$	(KPZ,KPZ)	(KPZ,KPZ)	($\frac{5}{3}$L,KPZ)	(D,KPZ’)
$\begin{pmatrix} 0 & \\ & \star \end{pmatrix}$	(KPZ,KPZ)	(KPZ,KPZ)	($\frac{5}{3}$L,KPZ)	(D,KPZ)
$\begin{pmatrix} \bullet & \\ & \mathbf{0} \end{pmatrix}$	(KPZ,$\frac{5}{3}$L)	(KPZ,$\frac{5}{3}$L)	(GM,GM)	(D,$\frac{3}{2}$L)
$\begin{pmatrix} 0 & \\ & \mathbf{0} \end{pmatrix}$	(KPZ’,D)	(KPZ,D)	($\frac{3}{2}$L,D)	(D,D)

4. Simulation results

Measure dynamical exponents z_i

1) Monte Carlo random sequential update

2) Excite modes at site k=L/2 at t=0 and measure dynamical structure function of each mode

3) Compute center of mass motion $<X_i(t)>$ of excitation ==> v_i ✔

4) Measure amplitudes $A_i(t)$ at maximum:
   Mass conservation $A_i(t) \sim 1/t^{1/z_i}$ ==> z_i

5) Fit predicted scaling functions

A) (D,(3/2L) universality class

Choose equal densities $\rho_1 = \rho_2 = \rho$, and symmetric lanes b=1

$$G^1 = -4A_0(1+\gamma\rho)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^2 = -4A_0\begin{pmatrix} 1+\gamma(1-\rho) & 0 \\ 0 & 1-\gamma(1-3\rho) \end{pmatrix}$$

$\gamma = 1/(1-3\rho)$: Diffusive and 3/2-Fibonacci mode

$$S_1(x,t) = \frac{1}{\sqrt{4\pi D_1 t}} \mathrm{e}^{-\frac{(x-v_1 t)^2}{4D_1 t}}$$

$$\hat{S}_2(p,t) = \frac{1}{\sqrt{2\pi}} \exp\left(-iv_2 pt - C_0|p|^{3/2} t\left[1 - i\,\mathrm{sgn}(p(v_1 - v_2))\right]\right)$$

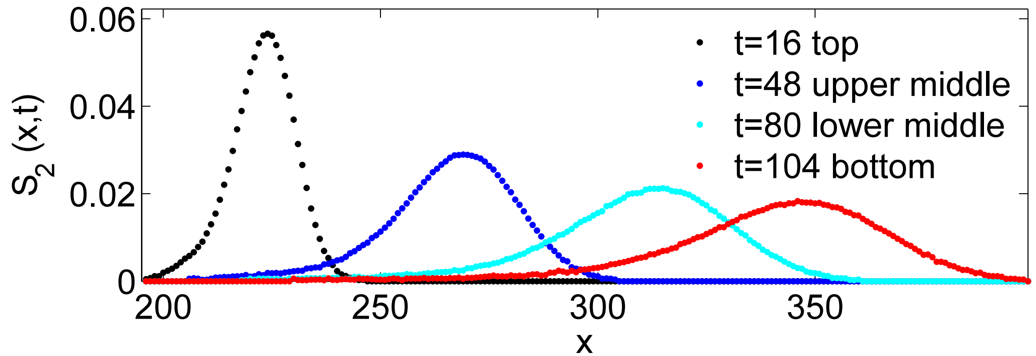
$$C_0 = \frac{(G^2_{11})^2}{2\sqrt{D_1|v_2 - v_1|}}$$

Monte-Carlo simulations

3/2L Fibonacci mode: $\rho_1 = \rho_2 = 0.2$, $\gamma = 2.5$, b = 1 (symmetry between lanes)

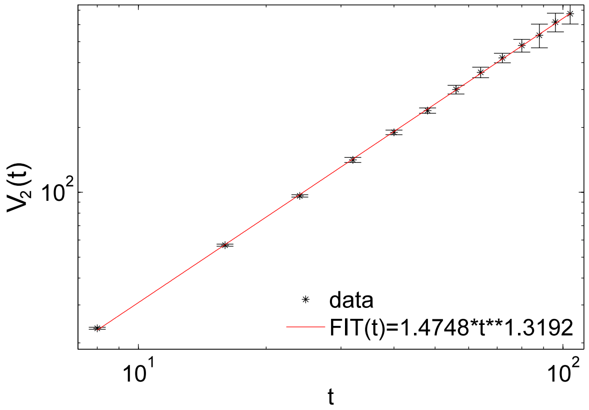
Theory: $v_2 = 1.3$, $2/z=1.333$, $\beta = -1$

$V_2 = 1.30(1)$ ✔

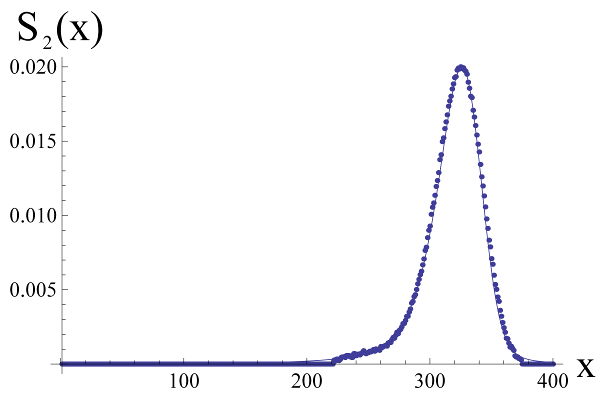


Variance: $t^{2/z}$
z = 1.52 ✔

$\beta = -0.692$ ✖



$1+\beta \sim t^{-1/6}$



t = 88

B) Golden mean universality class

$G^{(1)}_{11} = G^{(2)}_{22} = 0$, $G^{(1)}_{22} \neq 0$; $G^{(2)}_{11} \neq 0$;

Dynamical structure function:

$$\hat{S}_\pm(p,t) = \frac{1}{\sqrt{2\pi}} \exp\left(-iv_\pm pt - C_\pm |p|^\varphi t \left[1 \pm i \operatorname{sgn}(p(v_- - v_+)) \tan\left(\frac{\pi\varphi}{2}\right)\right]\right)$$

$\varphi = (1+\sqrt{5})/2 \approx 1.618$ (golden mean)

$$C_\pm = \frac{1}{2}|v_+ - v_-|^{1-\frac{2}{\varphi}} \left(\frac{2G^1_{22}G^2_{11}}{\varphi \sin\left(\frac{\pi\varphi}{2}\right)}\right)^{\varphi-1} \left(\frac{G^1_{22}}{G^2_{11}}\right)^{\pm(1+\varphi)}$$

➔ All parameters given by J and K!

(No free fitting parameters)

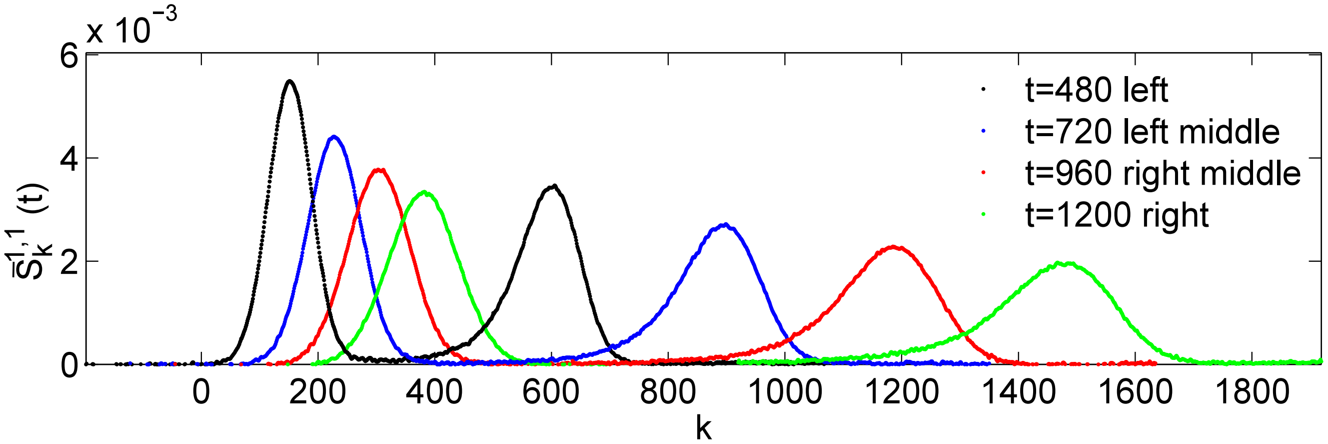
Simulation results:

Choose manifold: $\rho_1 = \frac{1-b}{3\gamma}, \quad \rho_2 = \frac{\gamma - 1}{3\gamma}$

$$v_{\pm} = (1+\gamma\rho_2)(1-2\rho_1) \pm \gamma\sqrt{\rho_1(1-\rho_1)\rho_2(1-\rho_2)}$$

$\rho_1 = 0.25$, $\rho_2 = 0.20$, $\gamma = 2.5$, $b = 0.625$ ➔ $v_- = 0.317$, $v_+ = 1.183$

$$G^1 = \begin{pmatrix} 0 & -0.406416 \\ -0.406416 & -0.105726 \end{pmatrix}, \quad G^2 = \begin{pmatrix} -0.812833 & -0.052863 \\ -0.052863 & 0 \end{pmatrix}$$

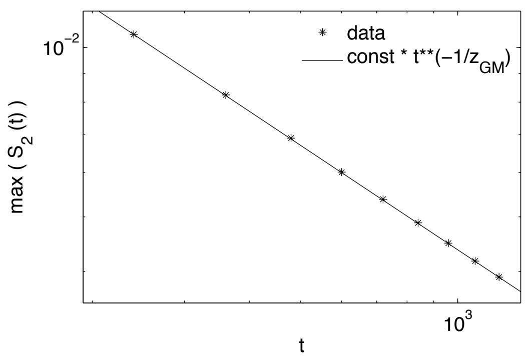


Simulation results (cont’)

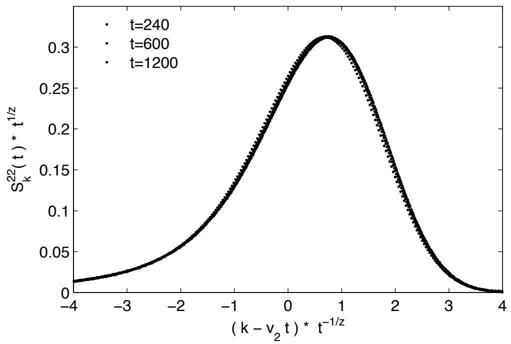
Measurement of center of mass: Error << 1%

Asymmetry: + mode: $\beta \approx -1$ for t=600;
- mode: $|\beta| < 1$ for all measured t (small coupling constant)

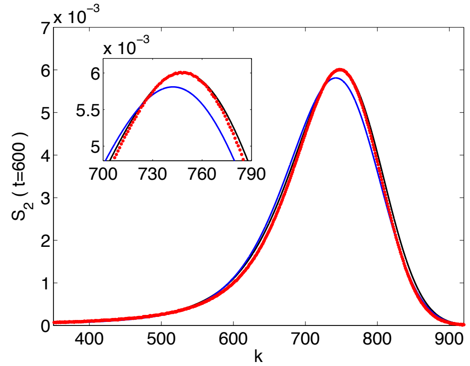
Amplitude at maximum:



➔ $z = 1.618(5)$



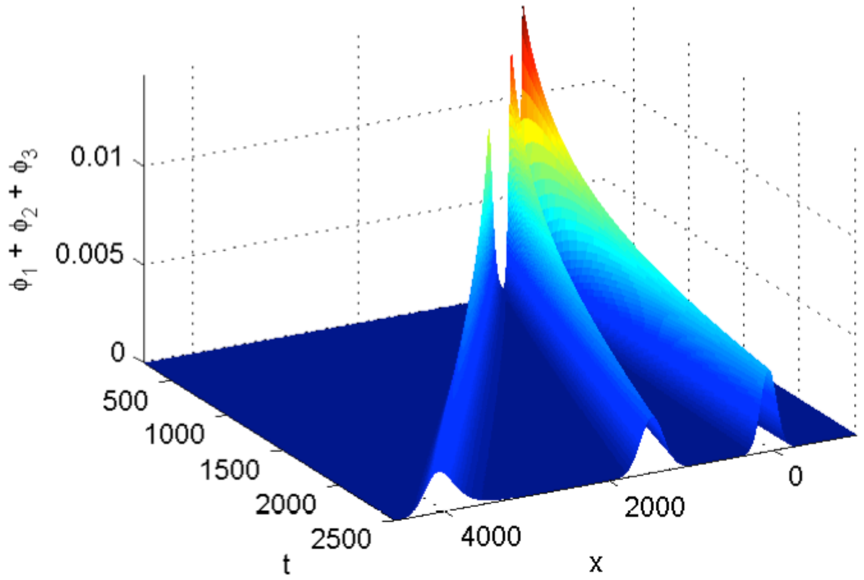
Scaling plot



Fit with max. asym. φ−Levy

Three lane model

New Fibonacci universality class: z= 2, 3/2, 5/3, 8/5



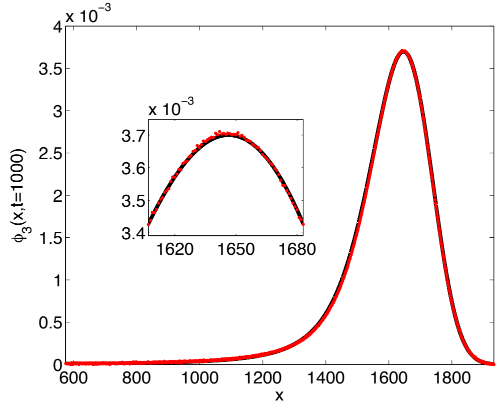
Mode 1: 8/5-Fibonacci, Mode 2: 5/3-Fibonacci, Mode 3: 3/2-KPZ.

Simulation results for 8/5 Fibonacci and GM

Choose large coupling constants ➔ maximal asymmetry at finite t

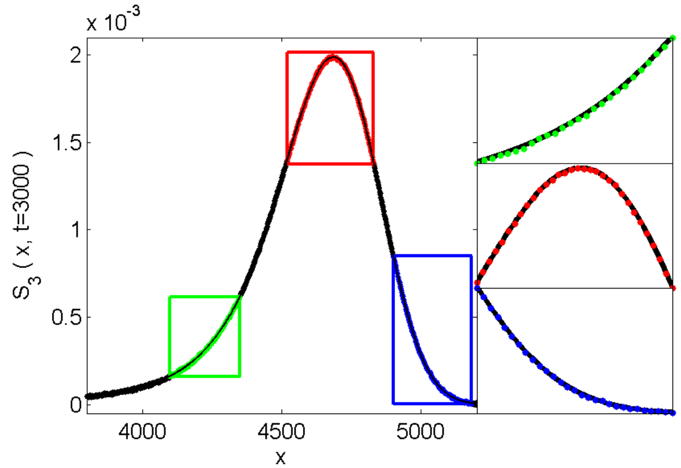
8/5 Fibonacci mode at t=1000:

Fit with max. asym. 8/5−Levy



Golden mean mode at t=3000:

Fit with max. asym. φ−Levy



5. Conclusions

- Mode coupling theory gives infinite discrete family of non-equilibrium universality classes for fluctuating hydrodynamics for hyperbolic systems
- Dynamical exponents are Kepler ratios of consecutive Fibonacci numbers or the golden mean limit
- Universality classes completely fixed by macroscopic stationary current-density relation
- Scaling functions completely fixed by current-density relation and macroscopic stationary compressibility matrix
- Stunning agreement of scaling functions with simulation data