Matrix Product Ansatz for nonequilibrium steady states of driven quantum systems: XXZ, Hubbard and others

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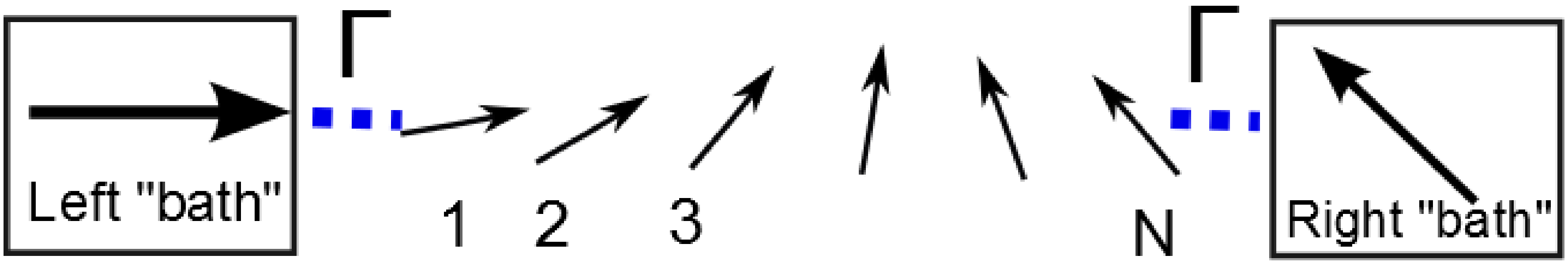
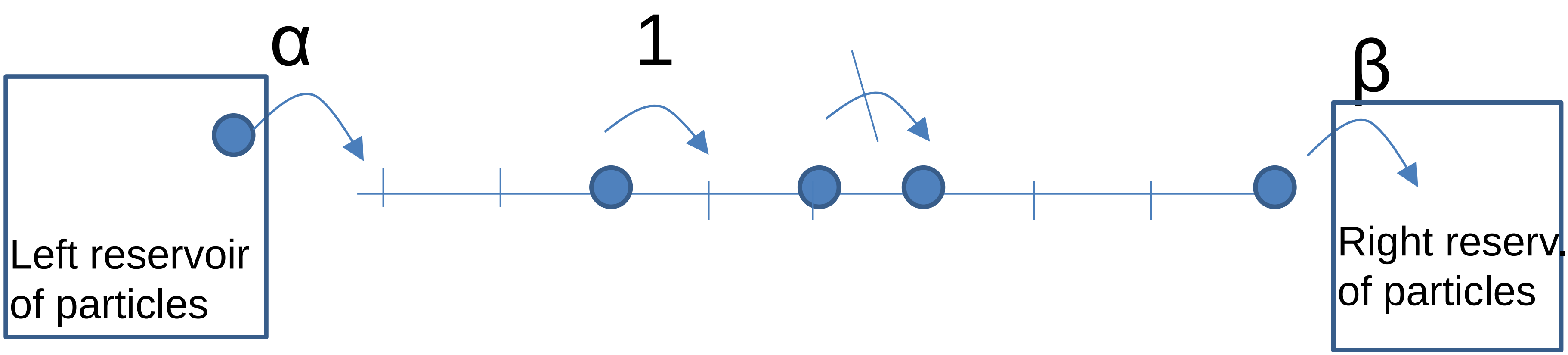
Galileo Galilei Institute, Firenze, 28 May 2015

DRIVEN QUANTUM SYSTEM OF SPINS



DRIVEN SYSTEM OF CLASSICAL PARTICLES (TASEP)



Lindblad Master equation

$$\frac{\mathbf{d}}{\mathbf{d}t}\rho = -i\left[H,\rho\right] + D[\rho]$$

$$D[\rho] = \sum_{\alpha} L_{\alpha}\rho L^{\dagger}{}_{\alpha} - \frac{1}{2}\left\{\rho, L^{\dagger}{}_{\alpha}L_{\alpha}\right\}$$

$$Tr\,\rho = 1$$

Most general time evolution preserving positivity and trace of a reduced density matrix and having a semigroup property

$$\frac{d}{dt}\left(Tr\,\rho\right) = 0$$

→ Trace is conserved

$$\frac{d}{dt}\left(Tr\,\rho^{2}\right) \neq 0$$

→ Non-unitary evolution

$$\lim_{t\to\infty}\rho(t) = \rho_{NESS}$$

Our goal: to investigate a nonequilibrium steady state ρ_{NESS}

Lindblad Master equation

$$\frac{\mathbf{d}}{\mathbf{d}t}\rho = -i\left[H,\rho\right] + \Gamma\left(\mathbf{D_1}(\theta_{LEFT},\varphi_{LEFT})[\rho] + \mathbf{D_N}(\theta_{RIGHT},\varphi_{RIGHT})[\rho]\right)$$

$$\mathbf{D_k}(\theta,\varphi)[\rho] = L_k\rho L^{\dagger}{}_k - \frac{1}{2}\left\{\rho, L^{\dagger}{}_k L_k\right\}$$

$$L_k(\theta,\varphi) = (\cos\theta\cos\varphi)\sigma_k^x + (\cos\theta\sin\varphi)\sigma_k^y - (\sin\theta)\sigma_k^z$$

$$-i(\sin\varphi)\sigma_k^x + i(\cos\varphi)\sigma_k^y$$

targets spin polarization at site k

$$\left\langle \overrightarrow{\sigma_k} \right\rangle = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$H = \sum_{k=1}^{N-1} \sigma_k{}^x\sigma_{k+1}{}^x + \sigma_k{}^y\sigma_{k+1}{}^y + \sigma_k{}^z\sigma_{k+1}{}^z$$

$$\lim_{t\to\infty}\rho(t) = \rho_{NESS}$$

Our goal: to investigate a nonequilibrium steady state ρ_{NESS}

Matrix product Ansatz for NESS

$$\rho_{NESS} = \frac{S_N S_N^{\dagger}}{Tr(S_N S_N^{\dagger})}$$

Tomaz Prosen, 2011

$$S_N = \langle \phi | \Omega^{\otimes N} | \psi \rangle$$

D. Karevski, V. Popkov and G. Schütz, *Phys. Rev. Lett.* **110**, 047201 (2013)

where Ω satisfies local divergence condition

$$[h, \Omega \otimes \Omega] = \Xi \otimes \Omega - \Omega \otimes \Xi$$

$$\Omega = \begin{pmatrix} S_Z & S_+ \\ S_- & -S_Z \end{pmatrix}, \quad \Xi = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

where S_Z, S_+, S_- are operators in an auxiliary space

Solution of the Matrix Product Ansatz

S_+, S_-, S_z satisfy $SU(2)$

$$[S_+, S_-] = 2S_z$$

$$[S_z, S_\pm] = \pm S_\pm$$

$$A \equiv I$$

Representation

$$S_z = \sum_{k=0}^{\infty} (p-k) | k \rangle\langle k |$$

$$S_+ = \sum_{k=0}^{\infty} (k+1) | k \rangle\langle k+1 |$$

$$S_- = \sum_{k=0}^{\infty} (2p-k) | k+1 \rangle\langle k |$$

$$p = \frac{i}{\Gamma}$$

Boundary vectors

$$\langle \phi | = \langle 0 |$$

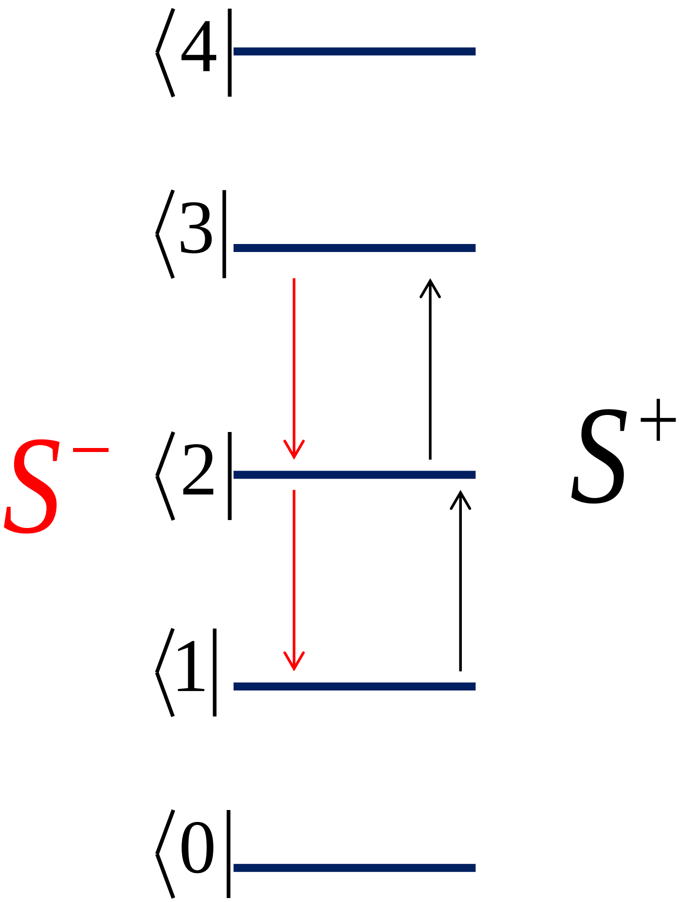
$$| \psi \rangle = \sum_{k=0}^{\infty} \frac{(S_-)^k \psi^k}{k!} | 0 \rangle = \sum_{k=0}^{\infty} \psi^k \binom{2p}{k} | 0 \rangle$$

$$\psi = -\tan\frac{\theta_R}{2}$$

$$S_N = \sum_{\substack{\alpha_1\alpha_2..\alpha_N \\ \alpha_i = z,+,-}} \langle\ 0\ | S^{\alpha_1} S^{\alpha_1} ... S^{\alpha_N} \,|\, \mu(\theta,\varphi)\rangle \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes ... \otimes \sigma^{\alpha_N}$$

$$\rho_{NESS} \sim S_N S_N{}^{\dagger}$$

XXX model



$$\langle\ k\ | S_z = (p-k)\langle\ k\ |$$

$$\langle\ k\, | S_+ = (k+1)\langle\ k+1\ |$$

$$\langle\ k+1 | S_- = (2p-k)\langle\ k\ |$$

$$p = \frac{i}{\Gamma}$$

MPA solution for XXZ model:

for XXZ Heisenberg model and $\theta = \pi$

$$q + q^{-1} = 2\Delta$$

$$| \psi \rangle = | 0 \rangle, \langle \phi | = \langle 0 |$$

$$2\Gamma = i(q^{p} + q^{-p}) / [p]_q$$

for XXX Heisenberg model and arbitrary twisting θ

$$q = 1, \quad SU_q(2) \to SU(2)$$

1D Hubbard model

T. Prosen, *Phys. Rev. Lett.* **112** (2014)
V. P. and T. Prosen , *Phys. Rev. Lett.* **114**, (2015)

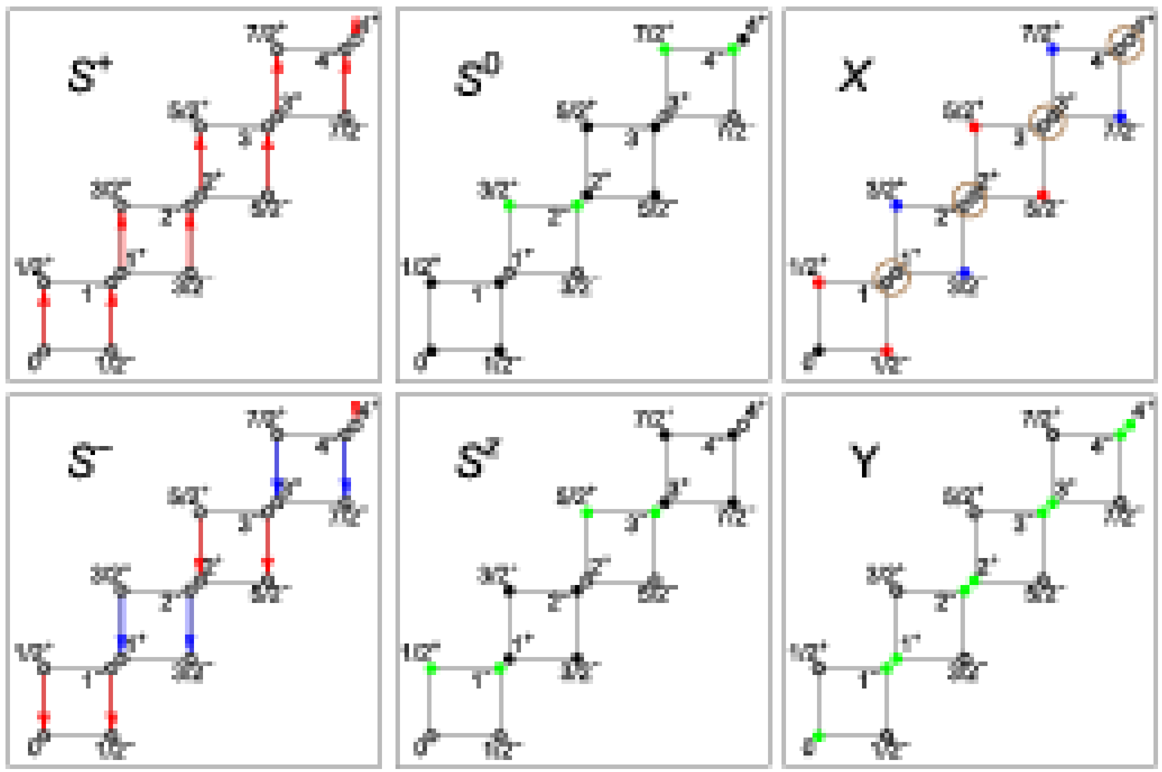
$$S_N = \sum_{\substack{\alpha_1\alpha_2..\alpha_N \\ \alpha_i=0,z,+,-}} \langle 0,0|S^{\alpha_1}T^{\beta_1}X...S^{\alpha_N}T^{\beta_N}X|0,0\rangle \sigma^{\alpha_1}\otimes\tau^{\beta_1}\otimes...\otimes\sigma^{\alpha_N}\otimes\tau^{\beta_N}$$

$$\rho^{Hubbard}{}_{NESS} \sim S_N S_N{}^{\dagger}$$

$$\left(S^+\right)^2 = \left(S^-\right)^2 = 0$$

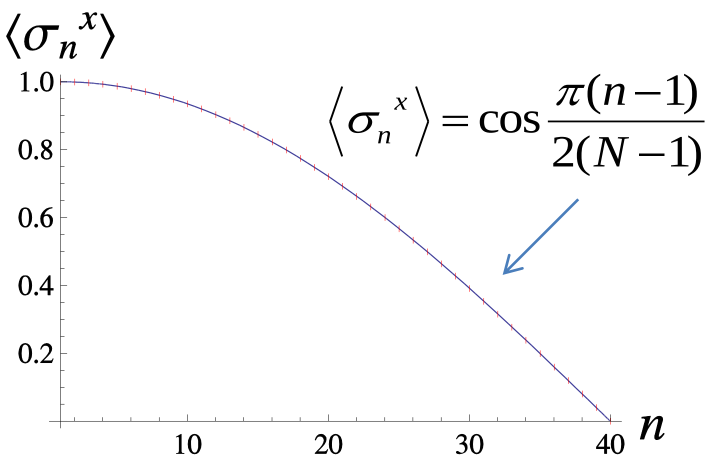
$$\left\{S^+, S^-\right\} = S^z$$

+ many other comm. relations



Exact observables for steady state and twisting angle π/2 in XY plane

$\Gamma = 4, \; N = 40$



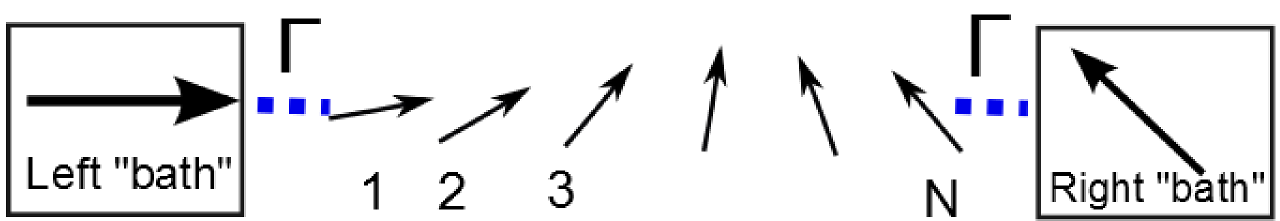
X- magnetization profile, from MPA, along the XXX spin chain, for chain of 40 sites

$$M_{k,N}{}^{\alpha} = Tr\left(\sigma_k{}^{\alpha}\rho\right) \rightarrow M^{\alpha}(x)$$

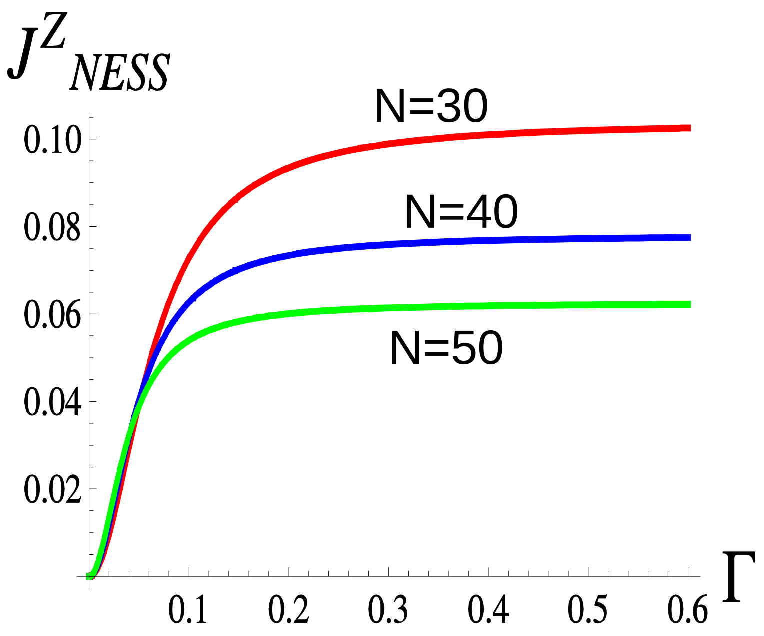
$$\frac{\partial^2 M^{\alpha}(x)}{\partial x^2} + \theta^2 M^{\alpha}(x) = 0$$

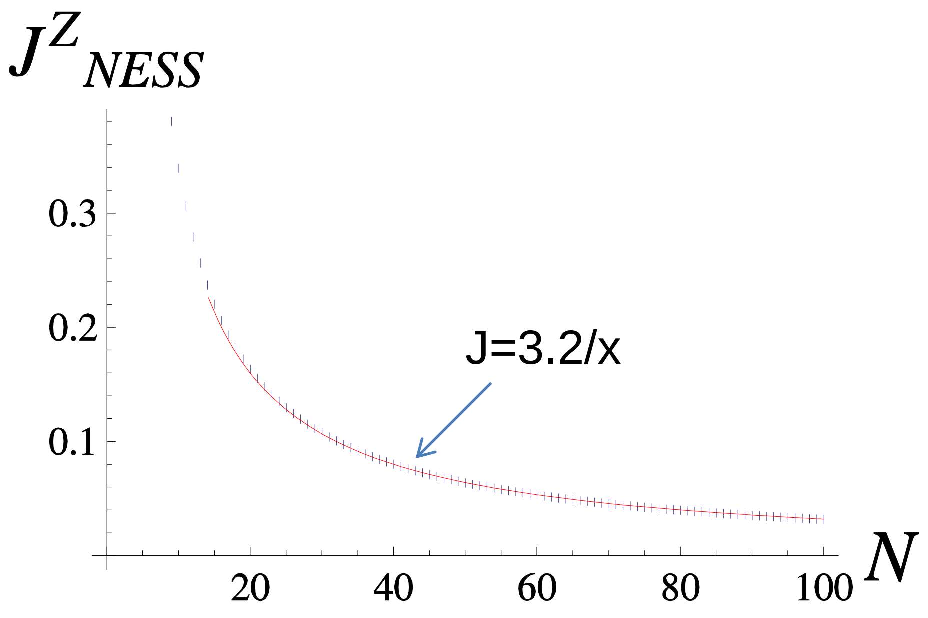
$$\text{for } \Gamma > \Gamma^* \approx 1/N$$

$$\frac{k}{N} = x$$

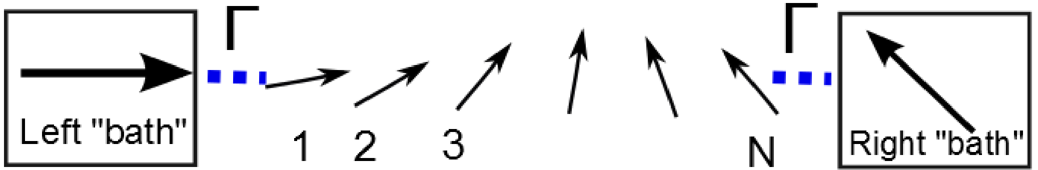


Exact observables for steady state and twisting angle π/2 in XY plane

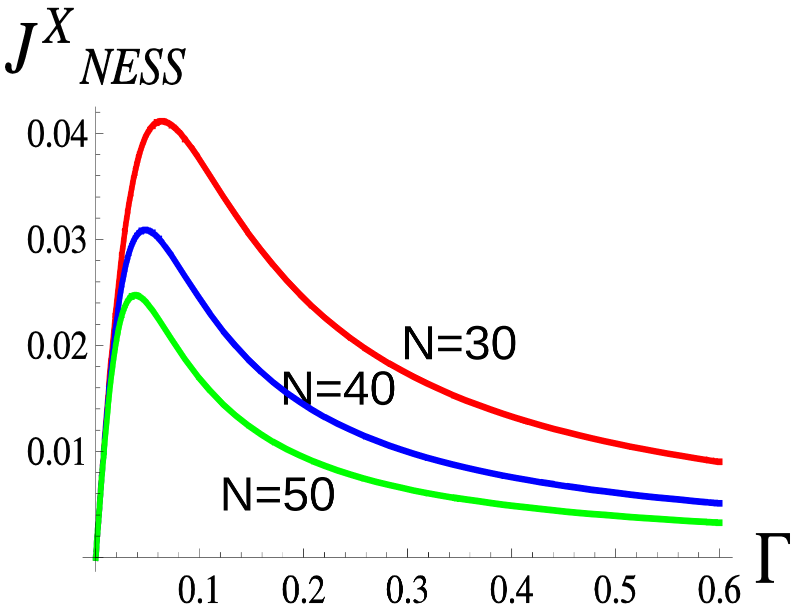


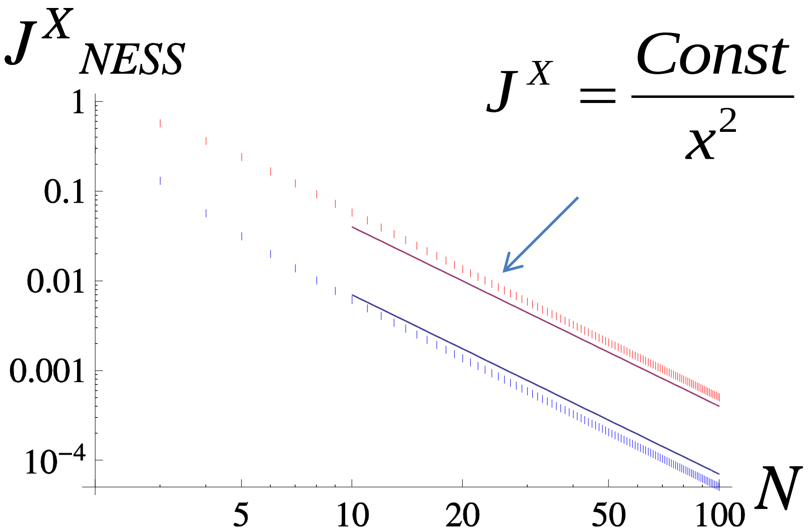


Z-magnetization current, from MPA ,as function of system size N and coupling



Exact observables for steady state and twisting angle π/2 in XY plane





X- and Y-magnetization current, from MPA ,as function of system size N and coupling Γ

Commutativity property

$$\left[S_N(p), S_N(p')\right] = 0$$

Note: $\left[\rho_N(p), \rho_N(p')\right] \neq 0$

Yang Baxter equation

$$R_{\beta\beta'}(u,v)\Omega_{\beta n}(u)\Omega_{\beta' n}(v) = \Omega_{\beta n}(v)\Omega_{\beta' n}(u)R_{\beta\beta'}(u,v)$$

plus "Reflection equations"

$$\langle 0| \otimes \langle 0| R = \langle 0| \otimes \langle 0|$$

R-matrix properties

$$R_{\beta\beta'}(u,v)\Omega_{\beta n}(u)\Omega_{\beta' n}(v)=\Omega_{\beta n}(v)\Omega_{\beta' n}(u)R_{\beta\beta'}(u,v)$$

Auxiliary spaces β,β' in which intertwining operator R acts nontrivially are infinite dimensional

$R_{ll'}^{kk'}=0,\ \text{if } k+k'\neq l+l'$ ice rule

$$R=\sum_{\alpha=0}^{\infty}\sum_{k=0}^{\alpha}\sum_{l=0}^{\alpha}R_{k,l}^{(\alpha)}\left|k,\alpha-k\right\rangle\left\langle l,\alpha-l\right|$$

$$R_{0,0}^{(0)}=1$$

All coefficients $R_{k,l}^{(\alpha)}$ are generically nonzero

YBE gives an infinite overdetermined set of recurrence relations for $R_{k,l}^{(\alpha)}$

T. Prosen, E.Ilievski and V.P., *New J. Phys.* **15** (2013)

Comparison with usual YBE for periodic isotropic Heisenberg model

$$\left[T_N(u), T_N(v)\right] = 0$$ Commutativity of transfer matrix $T_N(u)$

$$T_N = Tr_0\left(L_{01}(u)L_{02}(u)\ldots L_{0N}(u)\right)$$

$$R_{00'}(u,v)L_{0n}(u)L_{0'n}(v) = L_{0n}(v)L_{0'n}(v)R_{00'}(u,v)$$ Yang-Baxter Eq

$$R = uP + I = \begin{pmatrix} u+1 & 0 & 0 & 0 \\ 0 & 1 & u & 0 \\ 0 & u & 1 & 0 \\ 0 & 0 & & u+1 \end{pmatrix}, \quad R_{i'j'}{}^{ij} = 0 \text{ if } i+j \neq i'+j'$$

$$H_{XXX} = \sum_{n=1}^{N} \vec{\sigma}_n \vec{\sigma}_{n+1} = \frac{d}{du} \log T_N(u)\Big|_{u=0}$$

$$H^{(n)} = \frac{d^n}{d^n u} \log T_N(u)\Big|_{u=0}$$

$$\left[H^{(m)}, H^{(n)}\right] = 0$$

Comparison of two „transfer matrices

Equilibrium unitary problem

$[T_N(u), T_N(v)] = 0$

Auxiliary space is finite (dim=2)

T_N is a trace of monodromy matrix:

$T_N = Tr_0\left(L_{01}(u)L_{02}(u)...L_{0N}(u)\right)$

$T_N(u)$ is Hermitian

$T_N(u)$ is diagonalizable

Non-Equilibrium problem

$[S_N(p), S_N(q)] = 0$

Auxiliary space is infinite (dim=∞)

S_N is a matrix element of monodromy matrix:

$S_N(p) = \langle 0 | \Omega_{01}(p)\Omega_{02}(p)...\Omega_{0N}(p) | 0 \rangle$

$S_N(u)$ Non-Hermitian

$S_N(u)$ Non-diagonalizable (Jordan form)

Conclusions

- Fundamental integrable quantum statistical models (Heisenberg model, Hubbard, SU(N)) are integrable also in a nonequilibrium setting via Matrix Product Ansatz, at least for the Non-Equilibrium Steady State
- Respective L- matrices have infinite-dimensional auxiliary space,
- Monodromy matrix expectation w.r.t. vector form commuting family of operators in Hilbert space, depending on two continuous parameters

Very first review on the subject: T. Prosen, .
arXiv:1504.00783

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