Littlewood–Richardson coefficients and integrable tilings

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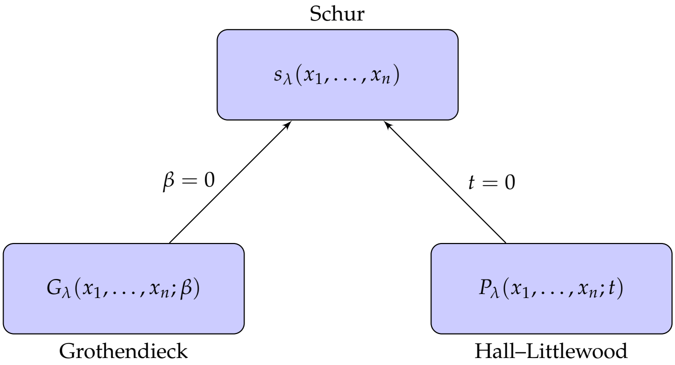
"Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. I was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there. The first part of this story might be an exaggeration."

– Gordon James



Schur polynomials and SSYT

- The Schur polynomials $s_\lambda(x_1,\dots,x_n)$ are the characters of irreducible representations of $GL(n)$. They are given by the Weyl formula:

$$s_\lambda(x_1,\dots,x_n) = \frac{\det_{1\leqslant i,j\leqslant n}\left[x_i^{\lambda_j-j+n}\right]}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)} = \sum_{\sigma\in S_n}\prod_{i=1}^{n} x_{\sigma(i)}^{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\left(\frac{x_{\sigma(i)}}{x_{\sigma(i)}-x_{\sigma(j)}}\right)$$

- A semi-standard Young tableau of shape λ is an assignment of one symbol $\{1,\dots,n\}$ to each box of the Young diagram λ, such that

  1. The symbols have the ordering $1<\cdots<n$.
  2. The entries in λ increase weakly along each row.
  3. The entries in λ increase strictly down each column.

- The Schur polynomial $s_\lambda(x_1,\dots,x_n)$ is also given by a weighted sum over semi-standard Young tableaux T of shape λ:

$$s_\lambda(x_1,\dots,x_n) = \sum_T \prod_{k=1}^{n} x_k^{\#(k)} = \sum_T \prod_{k=1}^{n} x_k^{|\lambda^{(k)}|-|\lambda^{(k-1)}|}$$

SSYT and sequences of interlacing partitions

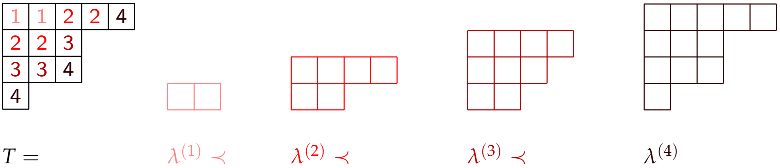
- Two partitions λ and μ interlace, written $\lambda \succ \mu$, if

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$$

  across all parts of the partitions. It is the same as saying $\lambda - \mu$ is a horizontal strip.
- One can interpret a SSYT as a sequence of interlacing partitions:

$$T = \{0 \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} \equiv \lambda\}$$

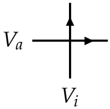
- The correspondence works by "peeling away" partition $\lambda^{(k)}$ from T, for all k:



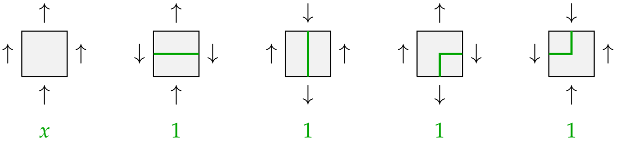
Schur polynomials from five-vertex models (I)

- Define the following L matrix, which is a limit of the rational six-vertex model:

$$L_{ai}(x) = \left(\begin{array}{cc|cc} x & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)_{ai} =$$

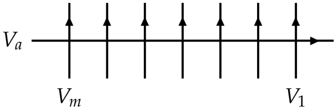


- The entries of the L matrix can be represented graphically as tiles:



- We are interested in the monodromy matrix, which is formed by rows of tiles:

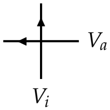
$$T_a(x) =$$



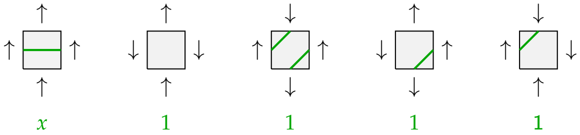
Schur polynomials from five-vertex models (II)

- We can use the same L matrix, but with the auxiliary and quantum spaces switched:

$$L_{ia}(x) = \left(\begin{array}{cc|cc} x & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)_{ia} =$$

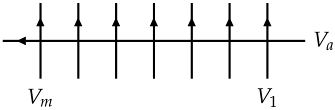


- Again, we represent the entries graphically:



- The monodromy matrix is now:

$$T^*_a(x) =$$



Two matrix product expressions for the Schur polynomial

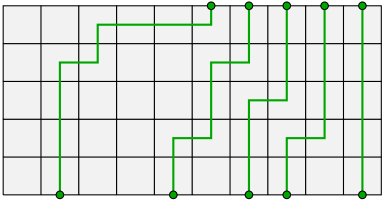
Theorem

The Schur polynomial $s_\lambda(x_1,\dots,x_n)$ *can be expressed in two different ways:*

$$s_\lambda(x_1,\dots,x_n) = \langle \lambda | T^*_{\circ\bullet}(x_n)\dots T^*_{\circ\bullet}(x_1)|0\rangle$$

$$s_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n} x_i^{m-n} \langle \lambda | T_{\circ\circ}(\bar{x}_n)\dots T_{\circ\circ}(\bar{x}_1)|0\rangle$$

- We give an example of the second expression. For the partition $\lambda = (4,2,1,1)$ and $n = 5$, a typical lattice configuration:



Littlewood–Richardson coefficients

- The Littlewood–Richardson coefficients are the structure constants in a product of two Schur polynomials:

$$s_\mu(x_1,\dots,x_n)s_\nu(x_1,\dots,x_n) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x_1,\dots,x_n)$$

- They satisfy some rather obvious properties:

$$c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}, \qquad c^\lambda_{\mu,\nu} = 0, \text{ unless } |\mu| + |\nu| = |\lambda|$$

- And some less obvious properties:

$$c^\lambda_{\mu,\nu} = c^{\bar{\mu}}_{\nu,\bar{\lambda}} = c^{\bar{\nu}}_{\bar{\lambda},\mu}$$

  where a barred partition is the complement of the Young diagram in a rectangular box.

- We will often write $c^\lambda_{\mu,\nu} = c_{\mu,\nu,\bar{\lambda}}$ and permute the indices freely.
- From the point of view of combinatorics, they stand to be interesting, since they are non-negative integers.

The Littlewood–Richardson rule

- Fix three Young diagrams λ, μ, ν such that $|\mu| + |\nu| = |\lambda|$.
- A Littlewood–Richardson tableau is a filling of the boxes of $\lambda - \mu$ such that $\#(k) = \nu_k$, and
  1. The rows are weakly increasing.
  2. The columns are strictly increasing.
  3. Reading the filling from right to left, top to bottom, any initial subword has at least as many symbols k as $k+1$.

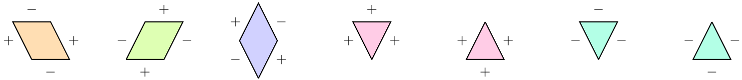
Theorem (Littlewood, Richardson, Schützenberger)

$c^{\lambda}_{\mu,\nu}$ *is the number of such tableaux.*

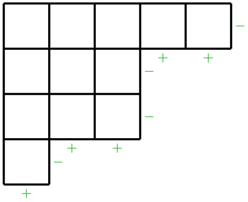
- As alluded to at the start of this talk, it took many years to prove this statement after it was first conjectured.

Knutson–Tao puzzles

- The subject of this talk are Knutson–Tao puzzles, an alternative way of calculating the Littlewood–Richardson coefficients.
- Consider the following set of puzzle pieces:



- Each edge of a piece is labeled with either $+$ or $-$, and when joining pieces these labels must match.
- A Knutson–Tao puzzle is a tiling of a triangle by these pieces, where the three sides of the triangle are fixed strings of $+$ and $-$. Every binary string corresponds with a unique partition:



Knutson–Tao puzzles

Theorem (Knutson, Tao)

$c_{\mu,\nu}^{\lambda} = c_{\mu,\nu,\bar{\lambda}}$ *is the number of Knutson–Tao puzzles with boundaries* $\mu, \nu, \bar{\lambda}$.

- The fact that these two combinatorial rules are equivalent is not at all obvious, but a direct correspondence was found by Zinn-Justin.
- We will describe an "integrable" proof of the coproduct identity:

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{\nu} c_{\mu,\nu}^{\lambda} s_{\nu}(x_1, \ldots, x_n)$$

Note that, because of the self-duality of Schur polynomials, this is an equivalent way of defining the Littlewood–Richardson coefficient $c_{\mu,\nu}^{\lambda}$.

Proof of coproduct identity

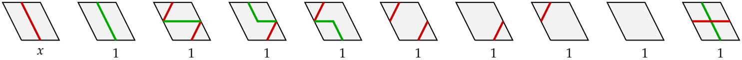
- The most important aspect of the proof is to embed the $SU(2)$ model describing the Schur polynomials into $SU(3)$.
- We consider the following L and R matrices:

$$L_{ia}(x) = \left(\begin{array}{ccc|ccc|ccc} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)_{ia} \qquad R_{ab}(x-y) = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x-y & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x-y & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x-y & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)_{ab}$$

which satisfy the intertwining equation

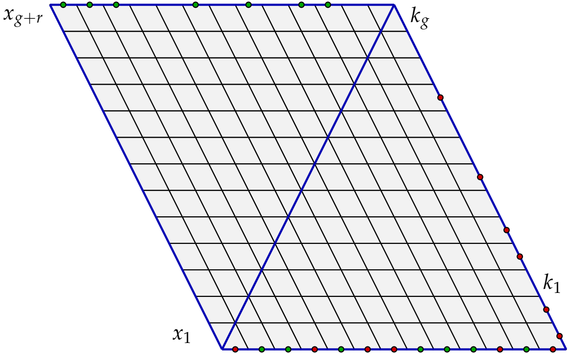
$$L_{ia}(x)L_{ib}(y)R_{ab}(x-y) = R_{ab}(x-y)L_{ib}(y)L_{ia}(x)$$

- We can represent the entries of the L matrix graphically, in many different ways. For example:



Proof of coproduct identity

- Consider the following partition function in the lattice model just defined:



- We can write this algebraically as

$$\langle \lambda_{\color{red}{\bullet}\color{green}{\bullet}} | \mathcal{O}_1(x_1) \dots \mathcal{O}_{g+r}(x_{g+r}) | \mu_{\circ\color{green}{\bullet}} \rangle$$

where $\mathcal{O}_i(x_i) = T_{\circ\circ}(x_i)$ if $i \in \{k_1, \dots, k_g\}$, and $\mathcal{O}_i(x_i) = T_{\circ\color{red}{\bullet}}(x_i)$ otherwise.

Proof of coproduct identity

- We can calculate this partition function explicitly, by using the commutation relations between the elements of the monodromy matrix:

$$T_{\circ{\color{red}\bullet}}(y)T_{\circ\circ}(x) = \frac{1}{x-y}T_{\circ\circ}(x)T_{\circ{\color{red}\bullet}}(y) + \frac{1}{y-x}T_{\circ\circ}(y)T_{\circ{\color{red}\bullet}}(x)$$

- We start off by calculating the transfer of a single $T_{\circ\circ}(x_i)$ to the left:

$$\langle\lambda_{{\color{red}\bullet}{\color{green}\bullet}}|T_{\circ{\color{red}\bullet}}(x_1)\dots T_{\circ{\color{red}\bullet}}(x_{k-1})T_{\circ\circ}(x_k) = \sum_{i=1}^{k}\langle\lambda_{{\color{red}\bullet}{\color{green}\bullet}}|T_{\circ\circ}(x_i)\prod_{\substack{j=1\\ j\neq i}}^{k}\frac{T_{\circ{\color{red}\bullet}}(x_j)}{(x_i-x_j)}$$

- Iterating this equation, we obtain the multiple integral expression

$$\langle\lambda_{{\color{red}\bullet}{\color{green}\bullet}}|\mathcal{O}_1(x_1)\dots\mathcal{O}_{g+r}(x_{g+r})|\mu_{\circ{\color{green}\bullet}}\rangle =$$

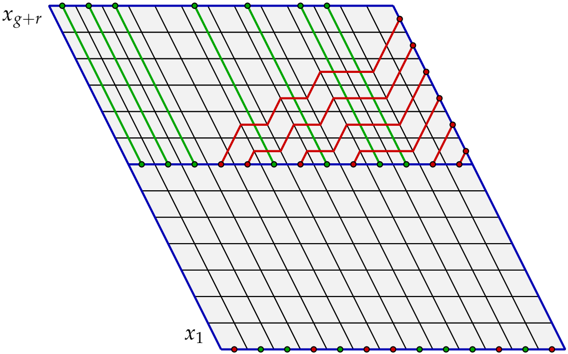
$$\oint_{w_g}\frac{dw_g}{2\pi\imath}\cdots\oint_{w_1}\frac{dw_1}{2\pi\imath}\frac{\prod_{1\leqslant i<j\leqslant g}(w_j-w_i)}{\prod_{i=1}^{g}\prod_{j=1}^{k_i}(w_i-x_j)}\langle\lambda_{{\color{red}\bullet}{\color{green}\bullet}}|T_{\circ\circ}(w_1)\dots T_{\circ\circ}(w_g)T_{\circ{\color{red}\bullet}}\dots T_{\circ{\color{red}\bullet}}|\mu_{\circ{\color{green}\bullet}}\rangle$$

Proof of coproduct identity

- We examine the completely "ordered" matrix product

$$\langle \lambda_{{\color{red}\bullet}{\color{green}\bullet}} | T_{\circ\circ}(x_1) \dots T_{\circ\circ}(x_g) T_{\circ{\color{red}\bullet}}(x_{g+1}) \dots T_{\circ{\color{red}\bullet}}(x_{g+r}) | \mu_{\circ{\color{green}\bullet}} \rangle$$

as a partition function:



- This partition function factorizes into a skew Schur polynomial, and a trivial region. We are thus able to write

$$\langle \lambda_{{\color{red}\bullet}{\color{green}\bullet}} | T_{\circ\circ}(x_1) \dots T_{\circ\circ}(x_g) T_{\circ{\color{red}\bullet}}(x_{g+1}) \dots T_{\circ{\color{red}\bullet}}(x_{g+r}) | \mu_{\circ{\color{green}\bullet}} \rangle = s_{\lambda/\mu}(\bar{x}_1, \dots, \bar{x}_g) \prod_{i=1}^{g} x_i^r$$

Proof of coproduct identity

- Returning to the multiple integral, we have

$$\langle \lambda_{\bullet\bullet} | \mathcal{O}_1(x_1) \dots \mathcal{O}_{g+r}(x_{g+r}) | \mu_{\circ\bullet} \rangle = \oint_{w_g} \frac{dw_g}{2\pi\imath} \cdots \oint_{w_1} \frac{dw_1}{2\pi\imath} \frac{\prod_{1 \leqslant i < j \leqslant g} (w_j - w_i)}{\prod_{i=1}^{g} \prod_{j=1}^{k_i} (w_i - x_j)} s_{\lambda/\mu}(\bar{w}_1, \dots, \bar{w}_g) \prod_{i=1}^{g} w_i^r$$

- Let us examine what happens when we set all $x_i = 0$. From the multiple integral expression, it is clear that

$$\begin{aligned} &\langle \lambda_{\bullet\bullet} | \mathcal{O}_1(0) \dots \mathcal{O}_{g+r}(0) | \mu_{\circ\bullet} \rangle \\ &= \mathrm{Coeff}\left(\prod_{i=1}^{g} w_i^r \prod_{1 \leqslant i < j \leqslant g} (w_j - w_i) s_{\lambda/\mu}(\bar{w}_1, \dots, \bar{w}_g), w_1^{k_1 - 1} \dots w_g^{k_g - 1} \right) \\ &= \mathrm{Coeff}\left(\prod_{1 \leqslant i < j \leqslant g} (z_i - z_j) s_{\lambda/\mu}(z_1, \dots, z_g), z_1^{\nu_1 - 1 + g} \dots z_g^{\nu_g} \right) \end{aligned}$$

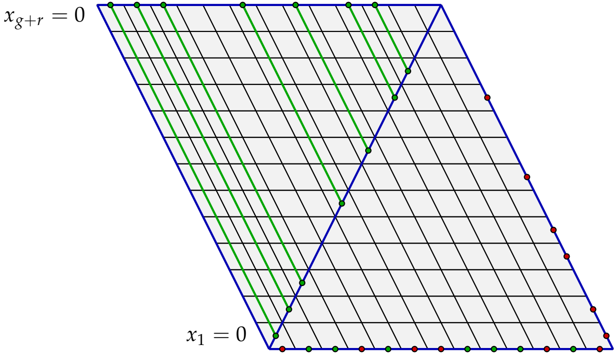
where we have defined $\nu_i = r - k_i + i$.

- We have thus shown that

$$\boxed{\langle \lambda_{\bullet\bullet} | \mathcal{O}_1(0) \dots \mathcal{O}_{g+r}(0) | \mu_{\circ\bullet} \rangle = c^{\lambda}_{\mu,\nu}}$$

Proof of coproduct identity

- By studying the original partition function with all parameters set to zero, we get a combinatorial rule for $c_{\mu,\nu}^{\lambda}$.
- At this special value of the parameters, the upper region is trivialized:



- The remaining region is precisely a Knutson–Tao puzzle.

Grothendieck polynomials

- Grothendieck polynomials were introduced by Lascoux and Schützenberger. They represent K-theory classes of Schubert varieties in the Grassmannian/flag manifold.

- They are inhomogeneous symmetric polynomials, parametrized by an additional parameter β, which continue to admit a determinant form:

$$G_\lambda(x_1, \ldots, x_n; \beta) = \frac{\det_{1 \leqslant i,j \leqslant n} \left[x_i^{\lambda_j - j + n} (1 + \beta x_i)^{j-1} \right]}{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)}$$

- The Grothendieck polynomials admit a description in terms of SS set-valued tableaux. These are fillings of a Young diagram by sets of distinct natural numbers, such that
  1. The largest entry in a box is weakly less than the smallest entry in the box to the right.
  2. The largest entry in a box is strictly less than the smallest entry in the box below.

Grothendieck polynomials

- The formula, in terms of semi-standard set-valued tableaux, is

$$G_\lambda(x_1,\dots,x_n;\beta) = \sum_{\mathbb{T}} \beta^{|\mathbb{T}|-|\lambda|} \prod_{k=1}^{n} x_k^{\#(k)}$$
$$= \sum_{\mathbb{T}} \prod_{k=1}^{n} x_k^{|\lambda^{(k)}|-|\lambda^{(k-1)}|} g_{\lambda^{(k)}/\lambda^{(k-1)}}(x_k;\beta)$$

where

$$g_{\lambda/\mu}(x;\beta) = \prod_{i=1}^{\ell(\mu)} (1 + \beta x - \beta x \delta_{\lambda_{i+1},\mu_i})$$

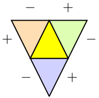
- This way of defining the Grothendieck polynomials is due to Buch.

K-theoretic Littlewood–Richardson rules

- We will focus on the structure constants for the product operation:

$$G_\mu(x_1,\ldots,x_n;\beta)G_\nu(x_1,\ldots,x_n;\beta) = \sum_\lambda c^\lambda_{\mu,\nu}(\beta)G_\lambda(x_1,\ldots,x_n;\beta)$$

- The first rule for calculating $c^\lambda_{\mu,\nu}(\beta)$ was obtained by Buch. A subsequent formula, in the spirit of Knutson–Tao puzzles, was published by Vakil. The puzzles now acquire an extra piece:



- Here we would like to use quantum integrability as a framework for recovering these earlier results, and new ones.

- Remark. From the point of view of integrability (also in K-theory), the x_i variables are not the most convenient. We re-parametrize as follows:

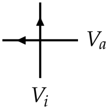
$$x_i = (u_i - 1)/\beta, \quad \forall\ 1 \leqslant i \leqslant n,$$

and write $G_\lambda(x_1,\ldots,x_n;\beta) \equiv G_\lambda(u_1,\ldots,u_n)$.

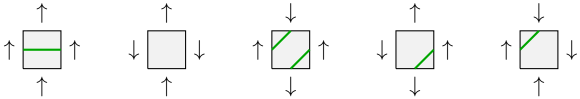
Grothendieck polynomials from a five-vertex model

- We define the following L matrix, which is limit of the trigonometric six-vertex model:

$$L_{ia}(u;\beta) = \left(\begin{array}{cc|cc} (u-1)/\beta & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & u & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)_{ia} =$$



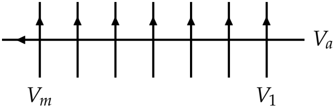
- The entries of the L matrix can be represented graphically:



$(u-1)/\beta$	1	1	u	1
x	1	1	$1+\beta x$	1

- Define as before the monodromy matrix:

$$T_a^*(u) =$$



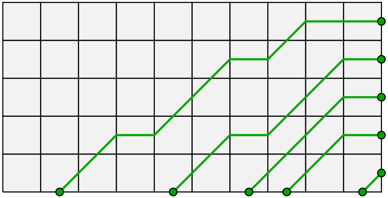
Grothendieck polynomials from a five-vertex model

Theorem

The Grothendieck polynomial $G_\lambda(u_1, \ldots, u_n)$ is given by

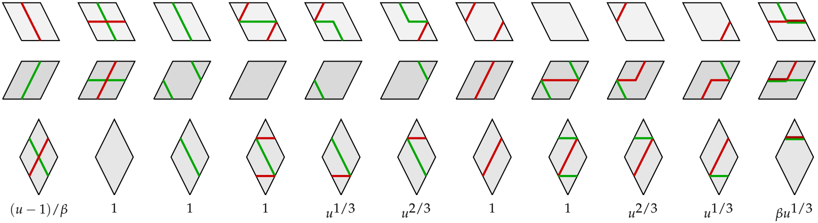
$$\prod_{i=1}^{n} u_i G_\lambda(u_1, \ldots, u_n) = \langle \lambda | T^*_{\circ\circ\bullet}(u_n) \ldots T^*_{\circ\circ\bullet}(u_1) | 0 \rangle$$

- For the partition $\lambda = (4, 2, 1, 1)$ and $n = 5$, a typical lattice configuration:

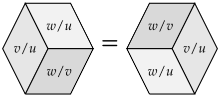


Three types of rhombi

- We consider rhombi in three different orientations:

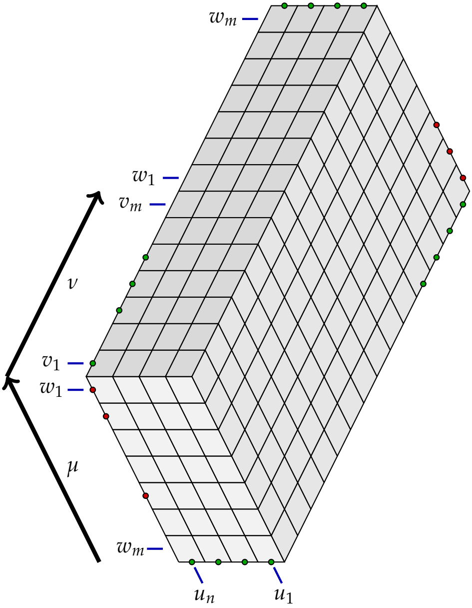


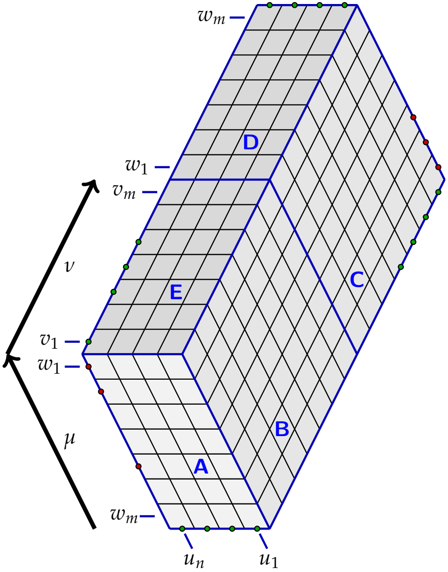
- The Yang–Baxter equation is satisfied:



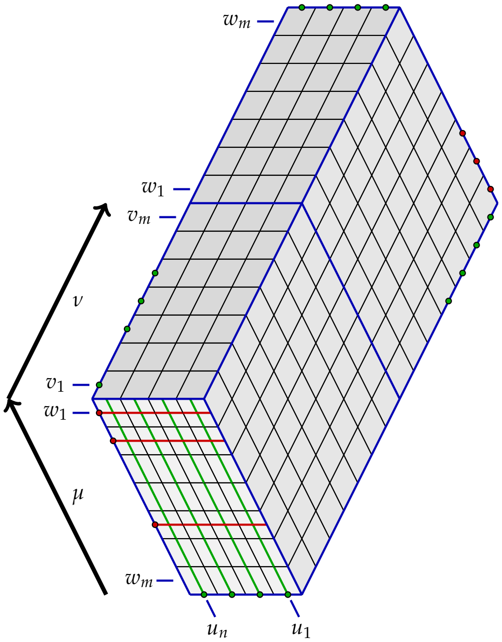
- This relation is a rather intricate limit of the $U_q(\widehat{sl_3})$ Yang–Baxter equation.

The left hand side





- Taking into account the boundary conditions and the tiles at our disposal, we can conclude that each of these regions is (a) trivial, (b) a Grothendieck polynomial, or (c) a new object.



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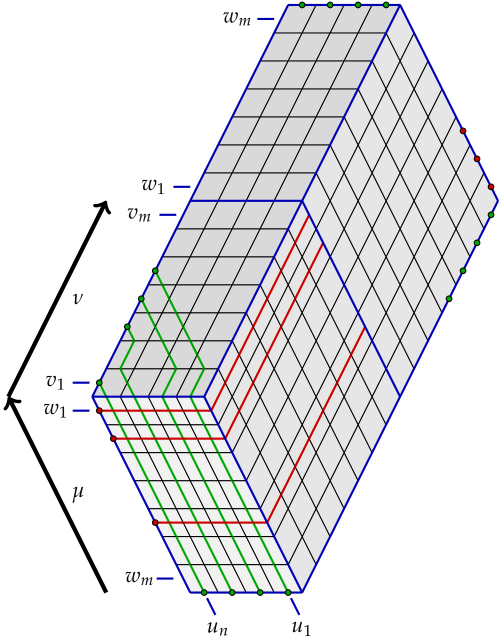

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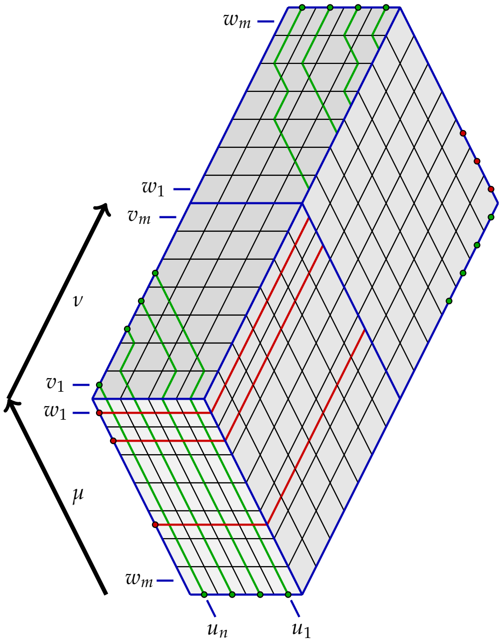

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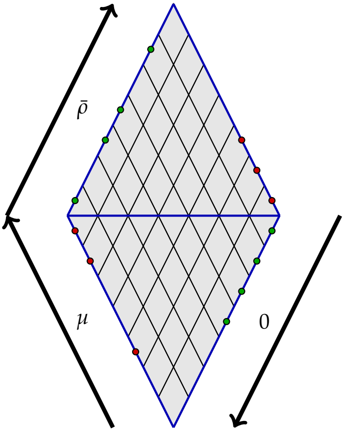
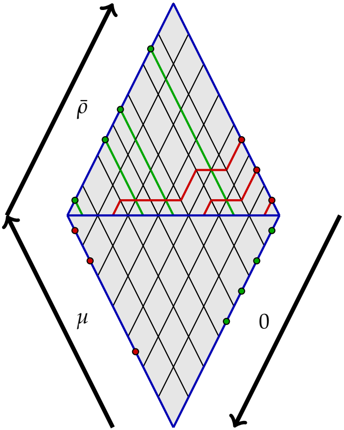
u_n
u_1

- We are left with the following diamond-shaped region:



- Along the central blue line, the spectral parameters coincide. From the Boltzmann weights, we see that the tile  vanishes.

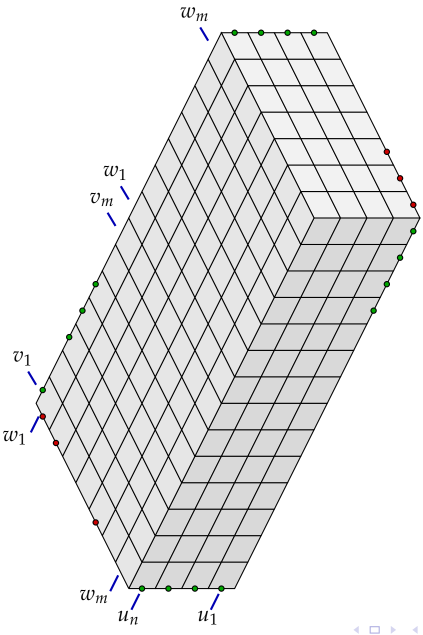
- This is sufficient to freeze the entire top half of the diamond.

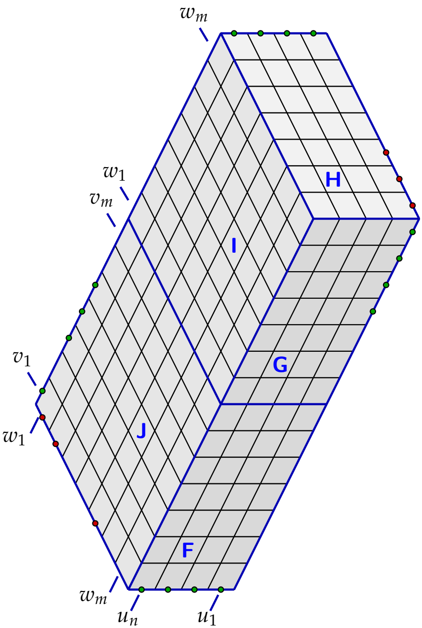


- The lower half of the diamond is not frozen, however. We denote this remaining region by $c^{\nabla}_{\mu,\bar{\rho},0}(\beta)$.
- The entire left hand side is equal to

$$\prod_{i=1}^{n} u_i G_\nu(u_1,\ldots,u_n) \sum_{\rho} c^{\nabla}_{\mu,\bar{\rho},0}(\beta) G_\rho(u_1,\ldots,u_n)$$

The right hand side




w_m
w_1
v_m
v_1
w_1
w_m
u_n
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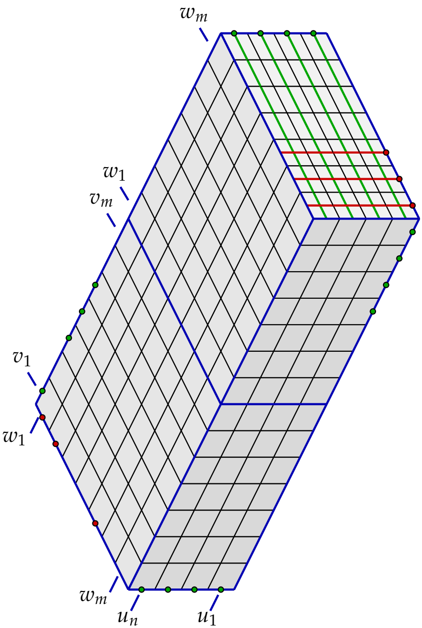
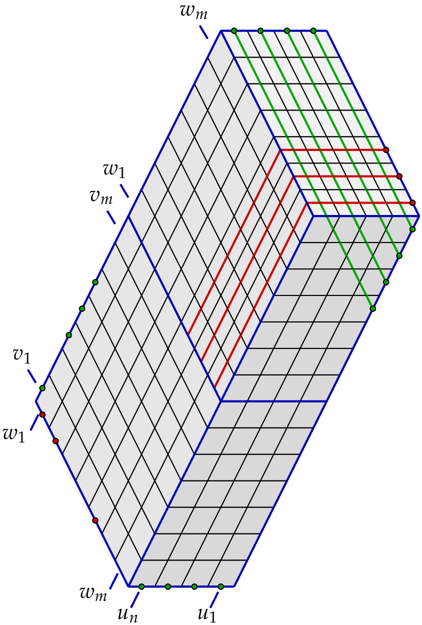
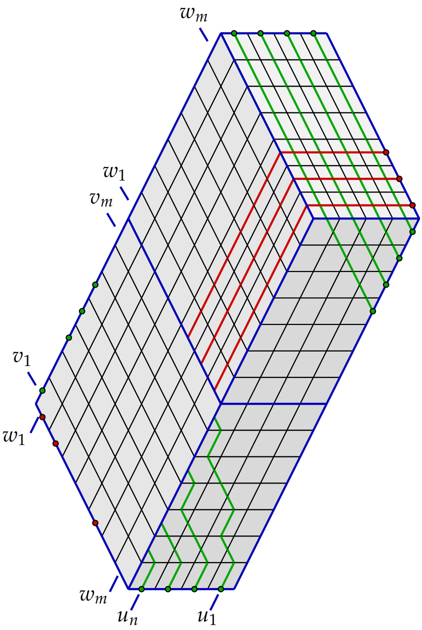
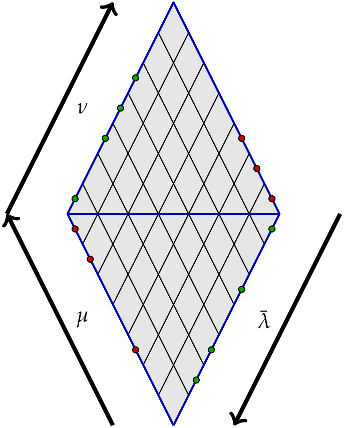

w_m
w_1
v_m
v_1
w_1
w_m
u_n
u_1



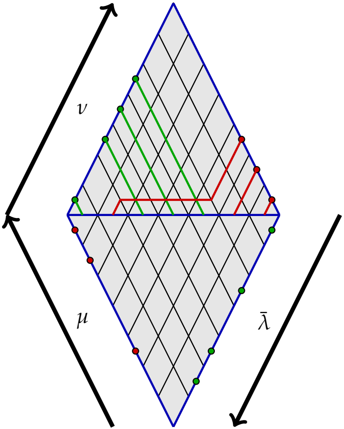
w_m
w_1
v_m
v_1
w_1
w_m
u_n
u_1


w_m
w_1
v_m
v_1
w_1
w_m
u_n
u_1

- We are left, once again, with a diamond-shaped region:



- In the non-equivariant case, the spectral parameters agree at every vertex. This means, in particular, no tile  can occur on the horizontal blue line.
- Hence we conclude that the top half of the diamond is frozen, by previous arguments.



- The lower half of the diamond is, by our previous conventions, called $c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta)$.
- These coefficients are 120° rotationally invariant. This is only obvious using the tile conventions of Knutson and Tao.
- The entire right hand side is thus

$$\prod_{i=1}^{n} u_i \sum_{\lambda} c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta) G_{\lambda}(u_1, \ldots, u_n)$$

Equating the two sides

- Putting everything together and cancelling the common factor $\prod_{i=1}^{n} u_i$, we find that

$$G_\nu(u_1,\dots,u_n) \sum_\rho c^{\nabla}_{\mu,\bar{\rho},0}(\beta) G_\rho(u_1,\dots,u_n) = \sum_\lambda c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta) G_\lambda(u_1,\dots,u_n) \qquad (\star)$$

- This is not yet satisfactory, because we wish to obtain a left hand side which is a pure product.
- We specialize $(\star)$ to the case $\mu = 0$, which gives

$$G_\nu(u_1,\dots,u_n) \sum_\rho c^{\nabla}_{0,\bar{\rho},0}(\beta) G_\rho(u_1,\dots,u_n) = \sum_\lambda c^{\nabla}_{0,\nu,\bar{\lambda}}(\beta) G_\lambda(u_1,\dots,u_n)$$

- It is easy to show that

$$c^{\nabla}_{0,\bar{0},0}(\beta) = 1, \qquad c^{\nabla}_{0,\overline{\square},0}(\beta) = \beta, \qquad c^{\nabla}_{0,\bar{\rho},0}(\beta) = 0, \ \forall \rho \neq 0, \square$$

$$G_0 = 1, \qquad G_\square = \Big(\prod_{i=1}^{n} u_i - 1\Big)/\beta$$

Equating the two sides

- Hence we obtain the identity

$$\prod_{i=1}^{n} u_i G_\nu(u_1,\dots,u_n) = \sum_\lambda c^{\nabla}_{0,\nu,\bar{\lambda}}(\beta) G_\lambda(u_1,\dots,u_n) = \sum_\lambda c^{\nabla}_{\nu,\bar{\lambda},0}(\beta) G_\lambda(u_1,\dots,u_n)$$

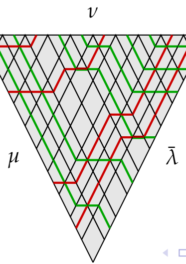
with the final equality coming from the cyclic invariance of the coefficients.

- Substituting this result back into our starting equation ($\star$), we obtain

Theorem (W, Zinn-Justin)

$$\prod_{i=1}^{n} u_i G_\mu(u_1,\dots,u_n) G_\nu(u_1,\dots,u_n) = \sum_\lambda c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta) G_\lambda(u_1,\dots,u_n)$$

$c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta) =$



180° rotation

- The tiles we are using are not invariant under 180° rotations. This is precisely due to the new K-tile.
- If we rotate our previous partition functions by 180°, preserving the orientation of the tiles themselves, we expect to obtain something new.
- The final result of the calculation is

$$G_\nu(u_1,\ldots,u_n)\sum_\rho c^{\nabla}_{0,\mu,\bar{\rho}}(\beta)G_\rho(u_1,\ldots,u_n)=\sum_\rho\sum_\lambda c^{\Delta}_{\bar{\lambda},\rho,\mu}(\beta)c^{\nabla}_{0,\bar{\rho},\nu}(\beta)G_\lambda(u_1,\ldots,u_n)$$

- The left hand side is something we have seen already. It is the left hand side of $(\star)$. Equating the two right hand sides, we thus obtain

$$\sum_\lambda c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta)G_\lambda(u_1,\ldots,u_n)=\sum_\rho\sum_\lambda c^{\Delta}_{\bar{\lambda},\rho,\mu}(\beta)c^{\nabla}_{0,\bar{\rho},\nu}(\beta)G_\lambda(u_1,\ldots,u_n)$$

180° rotation

- By the linear independence of the Grothendieck polynomials, we find the following relation between the coefficients:

$$c^{\nabla}_{\mu,\nu,\bar{\lambda}}(\beta) = \sum_{\rho} c^{\Delta}_{\bar{\lambda},\rho,\mu}(\beta) c^{\nabla}_{0,\bar{\rho},\nu}(\beta)$$

- Multiplying by $G_{\bar{\nu}}$ and summing over $\bar{\nu}$, after a lot of simplification we find that

$$G_{\mu}(u_1,\ldots,u_n) G_{\bar{\lambda}}(u_1,\ldots,u_n) = \sum_{\rho} c^{\Delta}_{\bar{\lambda},\rho,\mu} G_{\bar{\rho}}(u_1,\ldots,u_n)$$

- Using the cyclic invariance of the coefficients and relabeling the partitions, this looks more normal:

Theorem (Vakil)

$$G_{\mu}(u_1,\ldots,u_n) G_{\nu}(u_1,\ldots,u_n) = \sum_{\lambda} c^{\Delta}_{\mu,\nu,\bar{\lambda}} G_{\lambda}(u_1,\ldots,u_n)$$

Hall–Littlewood polynomials

- Hall–Littlewood polynomials are t-generalizations of Schur polynomials. They can be defined as a sum over the symmetric group:

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{\sigma(i)}^{\lambda_i} \prod_{1 \leqslant i < j \leqslant n} \left(\frac{x_{\sigma(i)} - t x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} \right)$$

- Alternatively, the Hall–Littlewood polynomial $P_\lambda(x_1, \ldots, x_n; t)$ is given by a weighted sum over semi-standard Young tableaux T of shape λ:

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_T \prod_{k=1}^{n} \left(x_k^{\#(k)} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(t) \right)$$

where the function $\psi_{\lambda/\mu}(t)$ is given by

$$\psi_{\lambda/\mu}(t) = \prod_{\substack{i \geqslant 1 \\ m_i(\mu) = m_i(\lambda)+1}} \left(1 - t^{m_i(\mu)} \right)$$

Hall–Littlewood polynomials and t-bosons

- Hall–Littlewood polynomials are most naturally expressed in terms of bosons.
- Consider the L and R matrices

$$L_a(x) = \begin{pmatrix} 1 & \phi^\dagger \\ x\phi & x \end{pmatrix}_a \qquad R_{ab}(x/y) = \left(\begin{array}{cc|cc} x-ty & 0 & 0 & 0 \\ 0 & t(x-y) & (1-t)y & 0 \\ \hline 0 & (1-t)x & x-y & 0 \\ 0 & 0 & 0 & x-ty \end{array}\right)_{ab}$$

which satisfy the intertwining equation

$$R_{ab}(x/y)L_a(x)L_b(y) = L_b(y)L_a(x)R_{ab}(x/y),$$

where $\phi, \phi^\dagger$ satisfy the t-boson algebra:

$$\phi\phi^\dagger - t\phi^\dagger\phi = 1 - t.$$

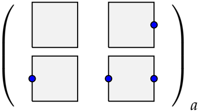
- We we will use the Fock representation of this algebra:

$$\phi^\dagger|m\rangle = |m+1\rangle, \qquad \phi|m\rangle = (1-t^m)|m-1\rangle, \ \forall m \geqslant 0.$$

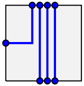
Hall–Littlewood polynomials and t-bosons

- It is then natural to represent the elements of the L matrix as follows:

$$L_a(x) = \begin{pmatrix} 1 & \phi^\dagger \\ x\phi & x \end{pmatrix}_a =$$



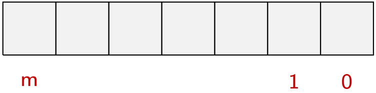
where the top and bottom edges of the tiles have no limitation on their occupation numbers. For example,



$$= x\langle 3|\phi|4\rangle = x(1-t^4)$$

- We construct a monodromy matrix in the usual way:

$$T_a(x) = L_a^{(m)}(x) \dots L_a^{(0)}(x) =$$



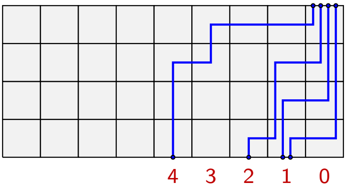
Hall–Littlewood polynomials and t-bosons

Theorem (Tsilevich)

The Hall–Littlewood polynomial can be expressed as

$$\prod_{i=1}^{n}(1-t^i)P_\lambda(x_1,\dots,x_n;t) = \langle\lambda|T_{\circ\circ}(x_n)\dots T_{\circ\circ}(x_1)|0\rangle$$

- In the case $\lambda=(4,2,1,1)$, $n=4$, a typical lattice configuration would be:



Structure constants and (inverse) Kostka–Foulkes polynomials

- We are interested in the t-analogues of the Littlewood–Richardson coefficients, which can be defined in two ways:

$$P_\mu(x_1,\dots,x_n;t)P_\nu(x_1,\dots,x_n;t) = \sum_\lambda f^\lambda_{\mu,\nu}(t) P_\lambda(x_1,\dots,x_n;t)$$

$$P_{\lambda/\mu}(x_1,\dots,x_n;t) = \sum_\nu f^\lambda_{\mu,\nu}(t) P_\nu(x_1,\dots,x_n;t)$$

which are equivalent due to the self-duality of Hall–Littlewood polynomials.

- One can also think about expressing Hall–Littlewood polynomials in the Schur basis:

$$P_\lambda(x_1,\dots,x_n;t) = \sum_\mu K^{-1}_{\lambda\mu}(t) s_\mu(x_1,\dots,x_n)$$

$$Q_\lambda(x_1,\dots,x_n;t) = \sum_\mu K_{\mu\lambda}(t) S_\mu(x_1,\dots,x_n;t)$$

The resulting coefficients are the (inverse) Kostka–Foulkes polynomials.

- How can we obtain new combinatorial expressions for these quantities, using quantum integrability?

Extending to $sl(3)$ in different ways (I)

- Consider the L and R matrices

$$L_a(x) = \begin{pmatrix} 1 & \phi^\dagger & \phi^\dagger \\ x\phi t^{\mathcal{N}} & xt^{\mathcal{N}} & 0 \\ x\phi & x\phi^\dagger\phi & x \end{pmatrix}_a$$

$$R_{ab}(x/y) = \left(\begin{array}{ccc|ccc|ccc} x-ty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t(x-y) & 0 & (1-t)y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t(x-y) & 0 & 0 & 0 & (1-t)y & 0 & 0 \\ \hline 0 & (1-t)x & 0 & x-y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x-ty & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t(x-y) & 0 & (1-t)y & 0 \\ \hline 0 & 0 & (1-t)x & 0 & 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-t)x & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x-ty \end{array}\right)_{ab}$$

which satisfy the usual intertwining equation.

- Since both families of bosons give Hall–Littlewood polynomials, this is a good candidate for studying $f^{\lambda}_{\mu,\nu}(t)$.

Extending to $sl(3)$ in different ways (II)

- Consider the L and R matrices

$$L_a(x) = \begin{pmatrix} 1 & x\phi^\dagger & x\psi^\dagger \\ \phi t^{\mathcal{N}} & xt^{\mathcal{N}} & 0 \\ \psi & x\phi^\dagger\psi & x(-t)^{\mathcal{N}} \end{pmatrix}_a$$

$$R_{ab}(x/y) = \left(\begin{array}{ccc|ccc|ccc} x-ty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t(x-y) & 0 & (1-t)x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t(x-y) & 0 & 0 & 0 & (1-t)x & 0 & 0 \\ \hline 0 & (1-t)y & 0 & x-y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x-ty & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t(x-y) & 0 & (1-t)y & 0 \\ \hline 0 & 0 & (1-t)y & 0 & 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-t)x & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y-tx \end{array}\right)_{ab}$$

where the green particles are fermions:

$$\psi\psi = \psi^\dagger\psi^\dagger = 0, \qquad \psi\psi^\dagger + \psi^\dagger\psi = 1-t.$$

These matrices satisfy the intertwining equation.

- The fermions give rise to the "capital S" polynomial $S_\lambda(x_1,\ldots,x_n;t)$. Hence this model is natural for the study of $K_{\lambda\mu}(t)$.