Exact Enumeration of Alternating Sign Matrices

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Plan

1. Consider two bulk statistics (# of inversions & # of -1's) & four boundary statistics (positions of 1's in first & last row & column) for ASMs

2. Discuss exact enumerative results for various cases involving some or all of these six statistics

3. Sketch proofs of some of these results using methods involving the six-vertex model with domain-wall boundary conditions

Alternating Sign Matrices (ASMs)

$$\mathrm{ASM}(n) := \left\{ n \times n \text{ matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry 0, 1 or } -1 \\ \bullet \text{ along each row \& column, nonzero entries} \\ \quad \text{alternate in sign, starting \& ending with a 1} \end{array} \right\}$$

- Any permutation matrix is an ASM
- Any ASM contains a single 1 & no -1's in first & last row & column

- e.g. $\mathrm{ASM}(3) =$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

- e.g. $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{ASM}(6)$

ASM Statistics

For $A \in \text{ASM}(n)$

Bulk statistics:

- $\nu(A) := \sum_{\substack{1\le i<i'\le n \\ 1\le j'\le j\le n}} A_{ij}\, A_{i'j'} \;=\; \sum_{i,j=1}^{n} (\sum_{j'=1}^{j} A_{ij'})(\sum_{i'=1}^{i-1} A_{i'j})$

  $= \#$ of 'inversions' in A
- $\mu(A) := \#$ of -1's in A

Boundary statistics:

- $\rho_{\mathsf{T}}(A) := \#$ of 0's left of the 1 in top row of A
- $\rho_{\mathsf{R}}(A) := \#$ of 0's below the 1 in right-most column of A
- $\rho_{\mathsf{B}}(A) := \#$ of 0's right of the 1 in bottom row of A
- $\rho_{\mathsf{L}}(A) := \#$ of 0's above the 1 in left-most column of A

- e.g. $A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

$\Rightarrow\;\; \nu(A) = 5,\;\; \mu(A) = 3,\;\; \rho_{\mathsf{T}}(A) = 3,\;\; \rho_{\mathsf{R}}(A) = 1,\;\; \rho_{\mathsf{B}}(A) = 2,\;\; \rho_{\mathsf{L}}(A) = 2$

Multiply-Refined ASM Generating Functions

Refinement order $=$ # of boundary parameters

- Quadruply-refined generating function
  $Z_n^{\text{quad}}(x,y;z_1,z_2,z_3,z_4) := \sum_{A\in\text{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\text{T}}(A)}\, z_2^{\rho_{\text{R}}(A)}\, z_3^{\rho_{\text{B}}(A)}\, z_4^{\rho_{\text{L}}(A)}$
- Triply-refined generating function
  $Z_n^{\text{tri}}(x,y;z_1,z_2,z_3) := Z_n^{\text{quad}}(x,y;z_1,1,z_2,z_3) = \sum_{A\in\text{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\text{T}}(A)}\, z_2^{\rho_{\text{B}}(A)}\, z_3^{\rho_{\text{L}}(A)}$
- Adjacent-boundary doubly-refined generating function
  $Z_n^{\text{adj}}(x,y;z_1,z_2) := Z_n^{\text{quad}}(x,y;z_1,1,1,z_2) = \sum_{A\in\text{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\text{T}}(A)}\, z_2^{\rho_{\text{L}}(A)}$
- Opposite-boundary doubly-refined generating function
  $Z_n^{\text{opp}}(x,y;z_1,z_2) := Z_n^{\text{quad}}(x,y;z_1,1,z_2,1) = \sum_{A\in\text{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z_1^{\rho_{\text{T}}(A)}\, z_2^{\rho_{\text{B}}(A)}$
- Singly-refined generating function
  $Z_n(x,y;z) := Z_n^{\text{quad}}(x,y;z,1,1,1) = \sum_{A\in\text{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z^{\rho_{\text{T}}(A)}$
- Unrefined generating function
  $Z_n(x,y) := Z_n^{\text{quad}}(x,y;1,1,1,1) = \sum_{A\in\text{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}$

- e.g. ASM(3) =

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$\Rightarrow$ $Z_3^{\text{quad}}(x,y;z_1,z_2,z_3,z_4) =$

$$1 + x\,z_1\,z_4 + x\,z_2\,z_3 + x^2\,z_1\,z_2\,z_3^2\,z_4^2 + x^2\,z_1^2\,z_2^2\,z_3\,z_4 + x^3\,z_1^2\,z_2^2\,z_3^2\,z_4^2 + x\,y\,z_1\,z_2\,z_3\,z_4$$

- Behaviour of ASMs under reflection and rotation

  $\Rightarrow$ simple symmetry properties of generating functions

  e.g. $Z_n^{\text{quad}}(x,y;z_1,z_2,z_3,z_4) = Z_n^{\text{quad}}(x,y;z_4,z_3,z_2,z_1)$

  $$= x^{n(n-1)/2}\,(z_1z_2z_3z_4)^{n-1}\,Z_n^{\text{quad}}(\tfrac{1}{x},\tfrac{y}{x};\tfrac{1}{z_2},\tfrac{1}{z_3},\tfrac{1}{z_4},\tfrac{1}{z_1})$$

- Properties of ASMs with a 1 in corner

  $\Rightarrow$ setting boundary parameters to 0 in generating functions reduces n to $n-1$

  e.g. $Z_n^{\text{quad}}(x,y;0,z_2,z_3,z_4) = Z_n^{\text{quad}}(x,y;z_1,z_2,z_3,0) = Z_{n-1}^{\text{adj}}(x,y;z_2,z_3)$

- ASMs with a boundary 1 separated from a corner by a single 0 also have relatively simple properties, giving some further generating function identities

Results with Bulk Parameter $y = 0$

- $y = 0$ corresponds to enumeration of permutation matrices with prescribed number of inversions & prescribed positions of 1's on boundaries

- Standard combinatorial arguments for permutations give

$$Z_n^{\text{quad}}(x,0;z_1,z_2,z_3,z_4) = x^2 z_1 z_2 z_3 z_4 \sum_{0\le i<j\le n-3}(x^{n+i-j-3} z_1^i z_3^{n-j-3} + x^{n-i+j-4} z_1^{n-i-3} z_3^j) \times \sum_{0\le i<j\le n-3}(x^{n+i-j-3} z_2^{n-j-3} z_4^i + x^{n-i+j-4} z_2^j z_4^{n-i-3})\,[n-4]_x! + (xz_4z_1\,[n-2]_{xz_4}\,[n-2]_{xz_1} + z_1z_2(xz_3z_4)^{n-1}\,[n-2]_{xz_1}\,[n-2]_{xz_2} + xz_2z_3\,[n-2]_{xz_2}\,[n-2]_{xz_3} + z_3z_4(xz_1z_2)^{n-1}\,[n-2]_{xz_3}\,[n-2]_{xz_4})\,[n-3]_x! + (1 + x^{2n-3}(z_1z_2z_3z_4)^{n-1})\,[n-2]_x!$$

where, as usual, $[n]_x = 1 + x + \ldots + x^{n-1}$, $[n]_x! = [n]_x[n-1]_x \ldots [1]_x$

- Setting boundary parameters to 1 gives other generating functions

e.g. $Z_n(x,0;z) = [n]_{xz}\,[n-1]_x!$

$Z_n(x,0) = [n]_x!$

Methods of Proof for General Results

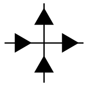
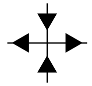
For most further, general results in this talk, all known methods of proof involve:

1. Izergin–Korepin determinant formula for partition function of six-vertex model with domain-wall boundary conditions, and possibly also Okada–Stroganov formula for this partition function at "combinatorial point", or
2. Fischer operator formula for monotone triangles or trapezoids with certain prescribed boundary entries, or
3. Zeilberger constant-term identities

- Will only discuss Method 1 in this talk

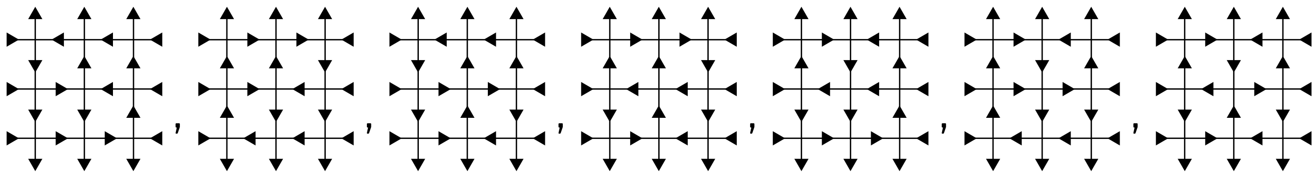
- No bijective/combinatorial proofs of such results currently known

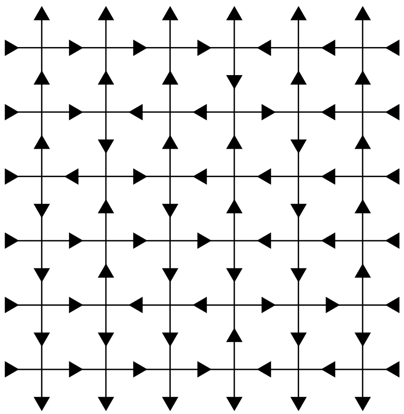
Summary of Six-Vertex Model Method

- Apply bijection between ASM(n) & set of configurations of six-vertex model on $n \times n$ grid with domain-wall boundary conditions (DWBC)
- Identify ASM statistics with certain six-vertex DWBC model statistics
  e.g. $(\#$ inversions$) = \Big(\#$  vertex configs.$\Big)$, $(\# -1$'s$) = \Big(\#$  vertex configs.$\Big)$
- Consider partition function for six-vertex DWBC model with crossing parameter q, row spectral parameters $t_1, r, \ldots, r, t_3$ & column spectral parameters $t_4, s, \ldots, s, t_2$
- Relate partition function to quadruply-refined ASM generating function in which x, y, z_1, z_2, z_3 & z_4 are parameterised in terms of q, r, s, t_1, t_2, t_3 & t_4
- Use Izergin–Korepin formula to write partition function as certain factor multiplied by determinant of $n \times n$ matrix
- Manipulate determinant in Izergin–Korepin formula
- Possibly set $q = e^{2\pi i/3}$, for which determinant gives single Schur polynomial
- Take into account ASM corners

Configurations of Six-Vertex Model with DWBC

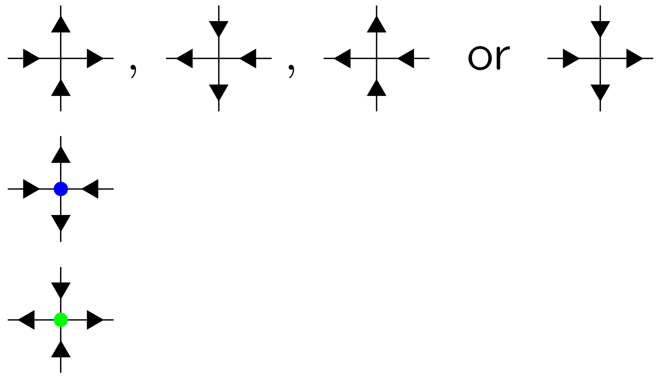
6VDW(n) := { edge orientations of $n \times n$ grid | • 2 inward & 2 outward arrows at each internal vertex ($\Rightarrow$ 6 possible vertex configurations) • upper & lower boundary arrows all outward, left & right boundary arrows all inward }

- e.g. 6VDW(3) = {  }

- e.g.  $\in$ 6VDW(6)

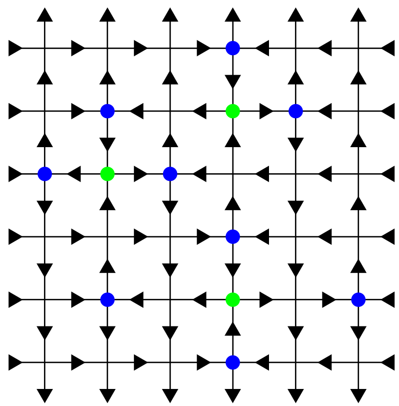
ASM(n) − 6VDW(n) **Bijection**

ASM		six-vertex model with DWBC
0	$\longleftrightarrow$	, , or
1	$\longleftrightarrow$	
-1	$\longleftrightarrow$	

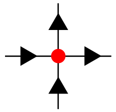
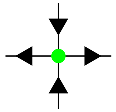
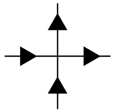
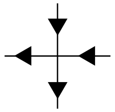
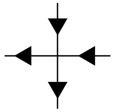
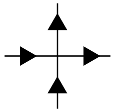


- e.g.

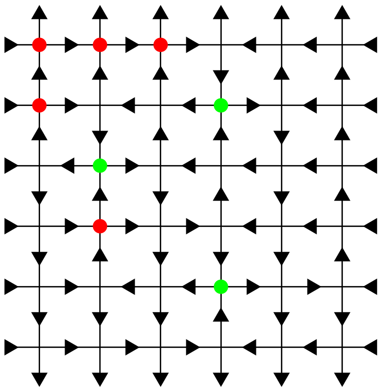
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow$$



For corresponding $A \in \mathsf{ASM}(n)$ & $C \in \mathsf{6VDW}(n)$

- $\nu(A)$ = # of  in C
- $\mu(A)$ = # of  in C
- $\rho_{\mathsf{T}}(A)$ = # of  in top row of C
- $\rho_{\mathsf{R}}(A)$ = # of  in right-most column of C
- $\rho_{\mathsf{B}}(A)$ = # of  in bottom row of C
- $\rho_{\mathsf{L}}(A)$ = # of  in left-most column of C

- e.g.

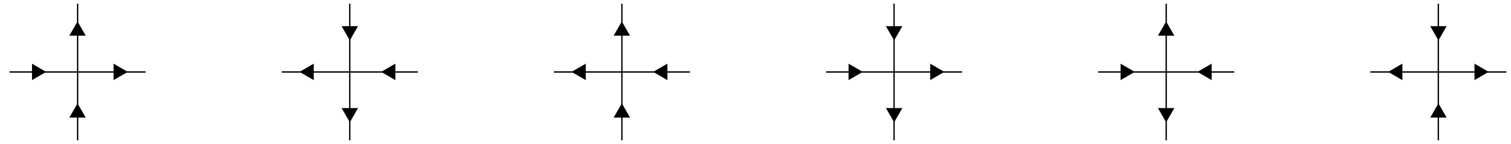


$\nu(C) = 5$, $\mu(C) = 3$,

$\rho_{\mathsf{T}}(A) = 3$, $\rho_{\mathsf{R}}(A) = 1$, $\rho_{\mathsf{B}}(A) = 2$, $\rho_{\mathsf{L}}(A) = 2$

Vertex Weights & Partition Function

- Integrable vertex weights:



$a(u,v)$ $\quad a(u,v)$ $\quad b(u,v)$ $\quad b(u,v)$ $\quad c(u,v)$ $\quad c(u,v)$

$$a(u,v) = u\,q^{1/2} - v\,q^{-1/2} \qquad b(u,v) = v\,q^{1/2} - u\,q^{-1/2} \qquad c(u,v) = (q - q^{-1})\,u^{1/2}\,v^{1/2}$$

u: row spectral parameter, v: column spectral parameter, q: crossing parameter

- Yang–Baxter equation satisfied

- Partition function

$$Z_n^{\mathrm{6V}}(u_1,\ldots,u_n;v_1,\ldots,v_n) \;:=\; \sum_{C\in\mathrm{6VDW}(n)} \prod_{i,j=1}^{n} \begin{pmatrix}\text{weight at vertex } (i,j) \text{ with} \\ \text{parameters } u_i,\, v_j \text{ for config'n } C\end{pmatrix}$$

- Parameterize $\quad x = \left(\frac{a(r,s)}{b(r,s)}\right)^2, \quad y = \left(\frac{c(r,s)}{b(r,s)}\right)^2, \quad z_1 = \frac{a(t_1,s)\,b(r,s)}{a(r,s)\,b(t_1,s)}, \quad z_2 = \frac{a(r,t_2)\,b(r,s)}{a(r,s)\,b(r,t_2)},$

$$z_3 = \frac{a(t_3,s)\,b(r,s)}{a(r,s)\,b(t_3,s)}, \quad z_4 = \frac{a(r,t_4)\,b(r,s)}{a(r,s)\,b(r,t_4)}$$

- $r = s$ gives $x = 1$, $y = (q^{1/2} + q^{-1/2})^2$,

  $r = s$ & $q = e^{2\pi i/3}$ gives $x = y = 1$

- Using ASM(n) – 6VDW(n) bijection,

  $Z_n^{\mathrm{quad}}(x,y;z_1,z_2,z_3,z_4) \approx Z_n^{\mathrm{6V}}(t_1,r,\ldots,r,t_3;t_4,s,\ldots,s,t_2)$

- Izergin–Korepin formula:

$$Z_n^{6\mathsf{V}}(u_1,\ldots,u_n;v_1,\ldots,v_n) = \frac{\prod_{i,j=1}^n a(u_i,v_j)\,b(u_i,v_j)}{\prod_{1\le i<j\le n}(u_i-u_j)(v_j-v_i)} \det_{1\le i,j\le n}\left(\frac{c(u_i,v_i)}{a(u_i,v_j)\,b(u_i,v_j)}\right)$$

(Izergin 1987)

Can be proved by showing that each side satisfies & is uniquely determined by certain properties, e.g., symmetry in $u_1,\ldots,u_n$ & in $v_1,\ldots,v_n$

Alternative proof: *Bogoliubov, Pronko, Zvonarev, 2002*

- At $q=e^{2\pi i/3}$

$$Z_n^{6\mathsf{V}}(u_1,\ldots,u_n;u_{n+1},\ldots,u_{2n})|_{q=e^{2\pi i/3}} = i^{n^2}\,3^{n/2}\,u_1^{1/2}\ldots u_{2n}^{1/2}\;s_{(n-1,n-1,\ldots,2,2,1,1)}(u_1,\ldots,u_{2n})$$

(Okada 2006, Stroganov 2006)

where $s_{(n-1,n-1,\ldots,2,2,1,1)}(\,.\,)$ = Schur polynomial indexed by double-staircase partition $(n-1,n-1,\ldots,2,2,1,1)$

Therefore, at $q=e^{2\pi i/3}$, $Z_n^{6\mathsf{V}}(u_1,\ldots,u_n;u_{n+1},\ldots,u_{2n})$ is symmetric in all $u_1,\ldots,u_{2n}$

Results with Bulk Parameters $x = y = 1$

- Define:

  Unrefined ASM numbers $\mathcal{A}_n := |\mathrm{ASM}(n)|$

  Singly-refined ASM numbers $\mathcal{A}_{n,k} := |\{A \in \mathrm{ASM}(n) \mid A_{1,k+1} = 1\}|$

  Opposite-boundary doubly-refined ASM numbers

  $$\mathcal{A}^{\mathrm{opp}}_{n,k_1,k_2} := |\{A \in \mathrm{ASM}(n) \mid A_{1,k_1+1} = A_{n,n-k_2} = 1\}|$$

  Adjacent-boundary doubly-refined ASM numbers

  $$\mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2} := |\{A \in \mathrm{ASM}(n) \mid A_{1,k_1+1} = A_{k_2+1,1} = 1\}|$$

  i.e.

  $$Z_n(1,1) = \mathcal{A}_n$$
  $$Z_n(1,1;z) = \textstyle\sum_{k=0}^{n-1} \mathcal{A}_{n,k}\, z^k$$
  $$Z^{\mathrm{opp}}_n(1,1;z_1,z_2) = \textstyle\sum_{k_1,k_2=0}^{n-1} \mathcal{A}^{\mathrm{opp}}_{n,k_1,k_2}\, z_1^{k_1} z_2^{k_2}$$
  $$Z^{\mathrm{adj}}_n(1,1;z_1,z_2) = \textstyle\sum_{k_1,k_2=0}^{n-1} \mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2}\, z_1^{k_1} z_2^{k_2}$$

- Elementary identities follow from definitions, ASM symmetry properties & properties of ASMs with a boundary 1 in a corner or separated from a corner by a single 0

  e.g. $\sum_{k=0}^{n-1} \mathcal{A}_{n,k} = \mathcal{A}_n, \quad \mathcal{A}_{n,0} = \mathcal{A}_{n-1}, \quad \mathcal{A}_{n,1} = \frac{n}{2}\mathcal{A}_{n-1},$

  $\mathcal{A}_{n,k} = \mathcal{A}_{n,n-1-k}, \quad \mathcal{A}^{\mathrm{opp}}_{n,k_1,k_2} = \mathcal{A}^{\mathrm{opp}}_{n,k_2,k_1}, \quad \mathcal{A}^{\mathrm{adj}}_{n,k_1,k_2} = \mathcal{A}^{\mathrm{adj}}_{n,k_2,k_1}$

- Explicit formulae for ASM numbers:

  Unrefined ASM numbers $\mathcal{A}_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ *(Zeilberger 1996, Kuperberg 1996)*

  Singly-refined ASM numbers $\mathcal{A}_{n,k} = \frac{(n+k-1)!\,(2n-k-2)!}{k!\,(n-k-1)!\,(2n-2)!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i-1)!}$

  (Zeilberger 1996, Colomo, Pronko 2005, Stroganov 2006, Fischer 2007)

  Opposite-boundary doubly-refined ASM numbers

  $$\mathcal{A}^{\text{opp}}_{n,k_1,k_2} = \frac{1}{\mathcal{A}_{n-1}} \sum_{i=0}^{\min(k_1,n-k_2-1)} (\mathcal{A}_{n,k_1-i}\,\mathcal{A}_{n-1,k_2+i} + \mathcal{A}_{n-1,k_1-i-1}\,\mathcal{A}_{n,k_2+i} - \mathcal{A}_{n,k_1-i-1}\,\mathcal{A}_{n-1,k_2+i} - \mathcal{A}_{n-1,k_1-i-1}\,\mathcal{A}_{n,k_2+i+1})$$

  (Stroganov 2006, Colomo, Pronko 2005)

  Adjacent-boundary doubly-refined ASM numbers

  $$\mathcal{A}^{\text{adj}}_{n,k_1,k_2} = \begin{cases} \mathcal{A}_{n-1}, & k_1 = k_2 = 0 \\ \binom{k_1+k_2-2}{k_1-1}\mathcal{A}_{n-1} - \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \binom{k_1+k_2-i-j}{k_1-i}\mathcal{A}^{\text{opp}}_{n,i-1,n-j}, & 1 \le k_1, k_2 \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

- Relation between opposite-boundary & adjacent-boundary doubly-refined cases:

  $$\mathcal{A}^{\text{opp}}_{n,k_1-1,n-k_2} = \mathcal{A}^{\text{adj}}_{n,k_1-1,k_2} + \mathcal{A}^{\text{adj}}_{n,k_1,k_2-1} - \mathcal{A}^{\text{adj}}_{n,k_1,k_2} + (\delta_{k_1,1} - \delta_{k_1,0})(\delta_{k_2,1} - \delta_{k_2,0})\,\mathcal{A}_{n-1}$$

  or $z_1\, z_2^n\, Z_n^{\text{opp}}(1,1;z_1,\frac{1}{z_2}) = (z_1+z_2-1)\,Z_n^{\text{adj}}(1,1;z_1,z_2) + (z_1-1)(z_2-1)\,\mathcal{A}_{n-1}$

  (Stroganov 2006, Fischer 2012)

- Further result for opposite-boundary doubly-refined ASM numbers:

  $$Z_n^{\mathsf{opp}}(1,1;z_1,z_2) = \frac{(q^2(z_1+q)(z_2+q))^{n-1}}{3^{n(n-1)/2}}\, s_{(n-1,n-1,\ldots,2,2,1,1)}\Big(\frac{qz_1+1}{z_1+q},\frac{qz_2+1}{z_2+q},\underbrace{1,\ldots,1}_{2n-2}\Big)\Big|_{q=e^{\frac{2\pi i}{3}}}$$

  Gives $\mathcal{A}_n = 3^{-n(n-1)/2} \times$ (number of semistandard Young tableaux of shape $(n-1,n-1,\ldots,2,2,1,1)$ with entries from $\{1,\ldots,2n\}$)

  (Okada 2006)

- Further relation between opposite-boundary doubly-refined & unrefined ASM numbers:

  $$\det_{0\le k_1,k_2\le n-1}\left(\mathcal{A}^{\mathsf{opp}}_{n,k_1,k_2}\right) = (-1)^{n(n+1)/2+1}\,(\mathcal{A}_{n-1})^{n-3}$$

  (Biane, Cantini, Sportiello 2011)

- Formulae for triply- and quadruply-refined generating functions at $x = y = 1$, e.g.

$$(z_1z_2 - z_1 + 1)(z_2z_3 - z_2 + 1)(z_3z_4 - z_3 + 1)(z_4z_1 - z_4 + 1)(z_2z_4)^{n-1}\, Z_n^{\text{quad}}(1, 1; z_1, \tfrac{1}{z_2}, z_3, \tfrac{1}{z_4}) =$$

$$\frac{z_1\, z_2\, z_3\, z_4}{\mathcal{A}_{n-1}\,\mathcal{A}_{n-2}\,\mathcal{A}_{n-3}\,(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)} \times$$

$$\det\begin{pmatrix} (z_1-1)^3 Z_n(1,1;z_1) & z_1(z_1-1)^2 Z_{n-1}(1,1,z_1) & z_1^2(z_1-1)Z_{n-2}(1,1;z_1) & z_1^3 Z_{n-3}(1,1;z_1) \\ (z_2-1)^3 Z_n(1,1;z_2) & z_2(z_2-1)^2 Z_{n-1}(1,1;z_2) & z_2^2(z_2-1)Z_{n-2}(1,1;z_2) & z_2^3 Z_{n-3}(1,1;z_2) \\ (z_3-1)^3 Z_n(1,1;z_3) & z_3(z_3-1)^2 Z_{n-1}(1,1;z_3) & z_3^2(z_3-1)Z_{n-2}(1,1;z_3) & z_3^3 Z_{n-3}(1,1;z_3) \\ (z_4-1)^3 Z_n(1,1;z_4) & z_4(z_4-1)^2 Z_{n-1}(1,1;z_4) & z_4^2(z_4-1)Z_{n-2}(1,1;z_4) & z_4^3 Z_{n-3}(1,1;z_4) \end{pmatrix}$$

$$+ (z_2-1)(z_3-1)(z_3z_4 - z_3 + 1)(z_1z_2 - z_1 + 1) z_4 z_1 z_2^{n-1} Z_{n-1}^{\text{opp}}(1,1;z_4,z_1) +$$

$$(z_3-1)(z_4-1)(z_4z_1 - z_4 + 1)(z_2z_3 - z_2 + 1) z_1 z_2 z_3^{n-1} Z_{n-1}^{\text{opp}}(1,1;z_1,z_2) +$$

$$(z_4-1)(z_1-1)(z_1z_2 - z_1 + 1)(z_3z_4 - z_3 + 1) z_2 z_3 z_4^{n-1} Z_{n-1}^{\text{opp}}(1,1;z_2,z_3) +$$

$$(z_1-1)(z_2-1)(z_2z_3 - z_2 + 1)(z_4z_1 - z_4 + 1) z_3 z_4 z_1^{n-1} Z_{n-1}^{\text{opp}}(1,1;z_3,z_4) +$$

$$(z_1-1)(z_2-1)(z_3-1)(z_4-1)\Big((z_1z_2 - z_1 + 1)(z_3z_4 - z_3 + 1)(z_2z_4)^{n-1} + (z_2z_3 - z_2 + 1)(z_4z_1 - z_4 + 1)(z_1z_3)^{n-1}\Big)\mathcal{A}_{n-2}$$

(Ayyer, Romik 2013)

Further Results

- **Objects in simple bijection with ASMs**

  e.g. – configurations of six-vertex model on square grid with DWBC
  - monotone (or Gog) triangles
  - osculating paths on square grid with certain boundary conditions
  - fully packed loop configurations on square grid with certain bound. conds.
  - 3-colorings of square grid with certain boundary conditions
  - integer fillings of square grid, weakly-decreasing along rows & columns, with certain boundary conditions

  All results could alternatively be expressed in terms of other such objects

- **Objects with nontrivial connections to ASMs**

  Certain ASM generating functions or numbers also appear (in some cases still conjecturally) in contexts of

  - totally symmetric self-complementary plane partitions *(e.g. Fonseca, Zinn-Justin 2008)*
  - descending plane partitions *(e.g. RB, Di Francesco, Zinn-Justin 2012, 2013)*
  - $O(1)$ dense loop models *(e.g. Batchelor, de Gier, Nienhuis 2001, Razumov, Stroganov 2004, Di Francesco, Zinn-Justin 2005, Cantini, Sportiello 2012, 2014)*

- **Case** $y = x + 1$
  - Sub-case $x = 1$ & $y = 2$ involves 2-enumeration of ASMs
  - Closely related to λ-determinants, domino tilings of Aztec diamond, free fermion case of six-vertex model
  - Many results known *(e.g. Robbins, Rumsey 1986, Elkies, Kuperberg, Larsen, Propp 1992, Ciucu 1996, Colomo, Pronko 2005, Okada 2006)*

- 3-**enumeration of ASMs**
  Various results known for $Z_n(1, 3)$ & $Z_n(1, 3; z)$
  (e.g. Kuperberg 2002, Stroganov 2003, Colomo, Pronko 2005, Okada 2006)

- **ASMs with fixed number of inversions**
  Certain expressions known for coefficients of x^k in $Z_n(x, 1)$ *(RB 2008)*

- **ASMs with fixed number of** -1**'s**
  Certain expressions known for coefficients of y^k in $Z_n(1, y)$
  (Lalonde 2002, Le Gac 2011)

- **ASMs invariant under symmetry operations**
  Many results known for enumeration of ASMs invariant under symmetry operations, in some cases also with prescribed values of bulk or boundary statistics
  (e.g. Kuperberg 2002, Okada 2006, Razumov, Stroganov 2006, Aval, Duchon 2010)

- **Generalisation of unrefined, singly-refined & opposite-boundary doubly-refined generating functions**

  For any $1 \le k_1, \ldots, k_m \le n$ with $k_1, \ldots, k_m$ all distinct, define

  $$Z_n(x, y; z_1, \ldots, z_m) = \sum_{A \in \mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A) - \sum_{i=1}^m \mu_{k_i}(A)} \prod_{i=1}^m z_i^{\nu_{k_i}(A)} (x z_i^2 + (y-x-1) z_i + 1)^{\mu_{k_i}(A)}$$

  where $\nu_i(A) = \sum_{j=1}^n (\sum_{k=1}^j \sum_{l=1}^{i-1} A_{ik} A_{lj} + \sum_{k=j+1}^n \sum_{l=i}^n A_{ik} A_{lj})$

  $\mu_i(A) =$ number of -1's in row i of A

  Then $$Z_n(x, y; z_1, \ldots, z_m) = \frac{\det_{1 \le i,j \le m}(z_i^{\,j-1} (z_i - 1)^{m-j} Z_{n+1-j}(x, y; z_i))}{\prod_{1 \le i < j \le m}(z_i - z_j) \prod_{i=1}^{m-1} Z_{n-i}(x, y)}$$

  (Colomo, Pronko 2006, 2008)

  Case $m = 4$, & symmetry in all spectral parameters of partition function of six-vertex model with DWBC at $q = e^{2\pi i/3}$, leads to formula for $Z_n^{\text{quad}}(1, 1; z_1, z_2, z_3, z_4)$

- **ASMs in which several rows or columns closest to two opposite boundaries are prescribed**

  Certain formulae & relations known for ASM enumeration involving configurations of several rows or columns closest to two opposite boundaries

  (e.g. Fischer, Romik 2009, Karklinsky, Romik 2010, Fischer 2011, 2012)

General Results with Bulk Parameters x & y Arbitrary

- Relation satisfied by opposite-boundary doubly-refined & unrefined generating functions:

$$(z_1-z_2)\, Z_n^{\text{opp}}(x,y;z_1,z_2)\, Z_{n-1}(x,y) = (z_1-1)\, z_2\, Z_n(x,y;z_1)\, Z_{n-1}(x,y;z_2) \; - \\ z_1\,(z_2-1)\, Z_{n-1}(x,y;z_1)\, Z_n(x,y;z_2)$$

(Colomo, Pronko 2005)

(Corresponds to $m=2$ case of previous result)

- Determinant formula for opposite-boundary doubly-refined generating function:

$$Z_n^{\text{opp}}(x,y;z_1,z_2) = \det_{0\le i,j\le n-1}\left(-\delta_{i,j+1} + \begin{cases} \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}, & j \le n-3 \\ \sum_{k=0}^{i}\sum_{l=0}^{k} \binom{i-1}{i-k}\binom{n-l-2}{k-l} x^k y^{i-k} z_2^{l+1}, & j = n-2 \\ \sum_{k=0}^{i}\sum_{l=0}^{k}\sum_{m=0}^{l} \binom{i-1}{i-k}\binom{n-l-2}{k-l} x^k y^{i-k} z_1^m z_2^{l-m}, & j = n-1 \end{cases}\right)$$

(RB, Di Francesco, Zinn-Justin 2012)

Setting $z_1=1$ or $z_2=1$ gives determinant formulae for singly-refined & unrefined generating functions

e.g. $Z_n(x,y) = \det_{0\le i,j\le n-1}\left(-\delta_{i,j+1} + \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k y^{i-k}\right)$

- Relation satisfied by quadruply-refined, adjacent boundary doubly-refined & unrefined generating functions:

$$
\begin{aligned}
&y(z_1-z_3)(z_4-z_2)\,Z_{n-2}(x,y)\,Z_n^{\text{quad}}(x,y;z_1,z_2,z_3,z_4)=\\
&\quad((z_1-1)(z_2-1)-yz_1z_2)((z_3-1)(z_4-1)-yz_3z_4)\,Z_{n-1}^{\text{adj}}(x,y;z_1,z_4)\,Z_{n-1}^{\text{adj}}(x,y;z_2,z_3)\,-\\
&\quad(x(z_1-1)(z_4-1)-y)(x(z_2-1)(z_3-1)-y)\,z_1z_2z_3z_4\,\widetilde{Z}_{n-1}^{\text{adj}}(x,y;z_1,z_2)\,\widetilde{Z}_{n-1}^{\text{adj}}(x,y;z_3,z_4)\,-\\
&\quad(z_2-1)(z_3-1)((z_1-1)(z_4-1)-yz_1z_4)\,Z_{n-1}^{\text{adj}}(x,y;z_1,z_4)\,Z_{n-2}(x,y)\,-\\
&\quad(z_1-1)(z_4-1)((z_2-1)(z_3-1)-yz_2z_3)\,Z_{n-1}^{\text{adj}}(x,y;z_2,z_3)\,Z_{n-2}(x,y)\,+\\
&\quad(z_3-1)(z_4-1)(x(z_1-1)(z_2-1)-y)\,z_1z_2\,(xz_3z_4)^{n-1}\,\widetilde{Z}_{n-1}^{\text{adj}}(x,y;z_1,z_2)\,Z_{n-2}(x,y)\,+\\
&\quad(z_1-1)(z_2-1)(x(z_3-1)(z_4-1)-y)\,z_3z_4\,(xz_1z_2)^{n-1}\,\widetilde{Z}_{n-1}^{\text{adj}}(x,y;z_3,z_4)\,Z_{n-2}(x,y)\,+\\
&\quad(z_1-1)(z_2-1)(z_3-1)(z_4-1)(1-(x^2z_1z_2z_3z_4)^{n-1})\,Z_{n-2}(x,y)^2
\end{aligned}
$$

(RB 2013)

- $\widetilde{Z}_n^{\text{adj}}$ = certain simple transformation of Z_n^{adj}

- Proof involves applying Desnanot–Jacobi determinant identity $\det M\ \det M_{\text{C}} = \det M_{\text{TL}}\ \det M_{\text{BR}} - \det M_{\text{TR}}\ \det M_{\text{BL}}$ to matrix M in Izergin–Korepin formula (where M_{TL}, M_{TR}, M_{BR} & M_{BL} are top-left, top-right, bottom-right & bottom-left $(n-1)\times(n-1)$ submatrices & M_{C} is central $(n-2)\times(n-2)$ submatrix)

- Follows that quadruply-refined generating function can be obtained recursively using initial conditions at $n=1$ & 2, together with definitions $Z_n^{\text{adj}}(x,y;z_1,z_2)=Z_n^{\text{quad}}(x,y;z_1,1,1,z_2)$ & $Z_n(x,y)=Z_n^{\text{quad}}(x,y;1,1,1,1)$, taking care to avoid division by zero for $z_i=1$

Examples of consequences of quadruply-refined relation *(RB 2013)*

Setting relevant boundary parameters to 1 in the quadruply-refined relation gives formulae for other generating functions, e.g.:

- Triply-refined generating function satisfies

$$(z_2-z_1)(z_3-1)\,Z_n^{\mathrm{tri}}(x,y;z_1,z_2,z_3)\,Z_{n-2}(x,y)= \\ ((z_2-1)(z_3-1)-yz_2z_3)\,z_1\,Z_{n-1}^{\mathrm{adj}}(x,y;z_1,z_3)\,Z_{n-1}(x,y;z_2)\;- \\ (x(z_1-1)(z_3-1)-y)\,z_1z_2z_3\,\widetilde{Z}_{n-1}^{\mathrm{adj}}(x,y;z_2,z_3)\,Z_{n-1}(x,y;z_1)\;- \\ (z_1-1)(z_3-1)\,z_2\,Z_{n-1}(x,y;z_2)\,Z_{n-2}(x,y)\;+ \\ (z_2-1)(z_3-1)\,z_1\,(xz_2z_3)^{n-1}\,Z_{n-1}(x,y;z_1)\,Z_{n-2}(x,y)$$

- Opposite-boundary doubly-refined generating function satisfies

$$(z_1-z_2)\,Z_n^{\mathrm{opp}}(x,y;z_1,z_2)\,Z_{n-1}(x,y)=(z_1-1)\,z_2\,Z_n(x,y;z_1)\,Z_{n-1}(x,y;z_2)\;- \\ z_1\,(z_2-1)\,Z_{n-1}(x,y;z_1)\,Z_n(x,y;z_2)$$

  (as first obtained by *Colomo, Pronko 2005*)

- Adjacent-boundary doubly-refined generating function satisfies

$$Z_n^{\mathrm{adj}}(x,y;z_1,z_2)=Z_{n-1}(x,y)\left(1+\sum_{i=1}^{n-1}\left(\tfrac{y\,z_1\,z_2}{(z_1-1)(z_2-1)}\right)^{n-i}\left(1+\tfrac{(x(z_1-1)(z_2-1)-y)\,Z_i(x,y;z_1)\,Z_i(x,y;z_2)}{y\,Z_{i-1}(x,y)\,Z_i(x,y)}\right)\right)$$

- Coefficients $Z_n(x,y)_k$ of z^k in singly-refined generating function $Z_n(x,y;z)$ satisfy

$$Z_n(x,y)_k = Z_{n-1}(x,y)\,\delta_{k,0} + Z_{n-1}(x,y)\sum_{i=0}^{k-1}\Bigg(y^{i+1}\binom{k-1}{i}\binom{n-1}{i+1} + y^i\sum_{j_1=0}^{k-i-1}\sum_{j_2=0}^{n-i-2}\frac{Z_{n-i-1}(x,y)_{j_1}\,Z_{n-i-1}(x,y)_{j_2}}{Z_{n-i-1}(x,y)\,Z_{n-i-2}(x,y)}\left(x\binom{k-j_1-2}{i-1}\binom{n-j_2-2}{i} - y\binom{k-j_1-1}{i}\binom{n-j_2-1}{i+1}\right)\Bigg)$$

- Unrefined generating function satisfies

$$Z_n(x,y) = Z_{n-1}(x,y)\Bigg(1 + \sum_{i=0}^{n-2}\Bigg(y^{i+1}\binom{n-1}{i+1}^2 + \frac{xy^i\left(\sum_{j=0}^{n-i-2}\binom{n-j-2}{i}Z_{n-i-1}(x,y)_j\right)^2 - y^{i+1}\left(\sum_{j=0}^{n-i-2}\binom{n-j-1}{i+1}Z_{n-i-1}(x,y)_j\right)^2}{Z_{n-i-1}(x,y)\,Z_{n-i-2}(x,y)}\Bigg)\Bigg)$$

- Derivations of several of the earlier ASM enumeration results can also be obtained using the quadruply-refined relation