Non-linear integral equation approach to $sl(2|1)$ integrable network models

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Outline

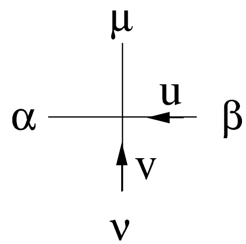
- *Quantum Hall systems, electrons in random potentials; black hole CFTs*
- R-matrices for fundamental representations of $sl(2|1)$
- transfer matrices and Hamiltonians
- Bethe ansatz, short review of work by Gade and Essler, Frahm, Saleur
- derivation of non-linear integral equations
  - tJ-model thermodynamics
  - network model

Work in collaboration with M. Brockmann

Integrable network models: R-matrices, Yang-Baxter equation

Consider R-matrix acting on tensor products of "standard" fundamental representation of $sl(2|1)$

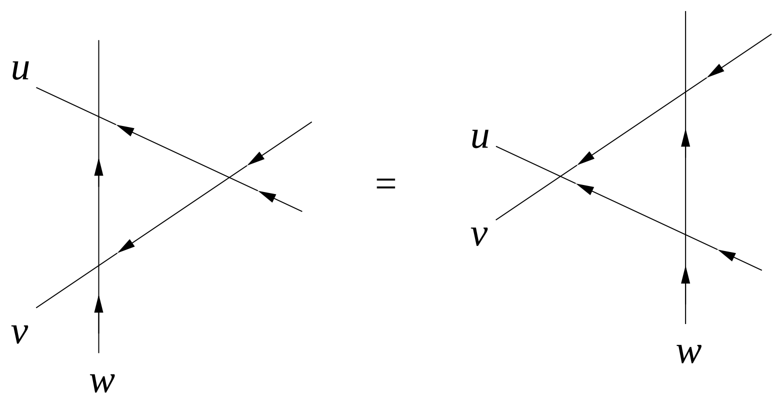
$R^{\alpha\mu}_{\beta\nu}(u,v) =$



$$R(u,v) = \mathcal{P} - \frac{1}{2}(u-v)I$$

$\mathcal{P}$: graded permutation operator, u and v are complex variables, and indices α, β, μ, ν take three values.

R-matrix satisfies Yang-Baxter equation

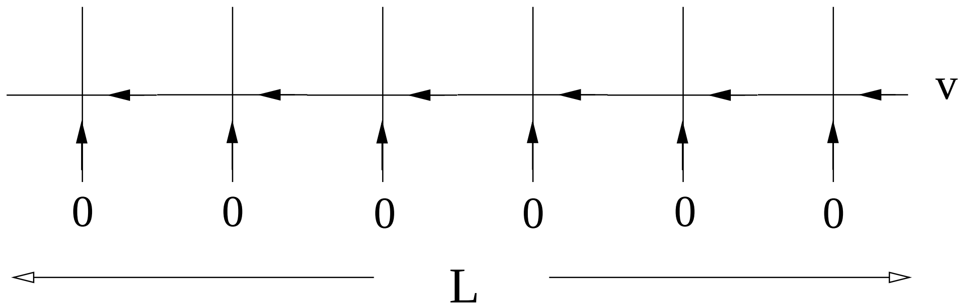


Generalization to mixed representations (standard fundamental and its conjugate visualized by left and right or up and down pointing arrows) possible!

In fact, the three new R-matrices are essentially obtained from rotations of above R-matrix by 90, 180, and 270 degrees. Yang-Baxter equation still holds where only arrow directions differ from above pictorial visualization (Gade 1998; Links, Foerster 1999; Abad, Rios 1999; Derkachov, Karakhanyan, Kirschner 2000).

Transfer matrices, Hamiltonians

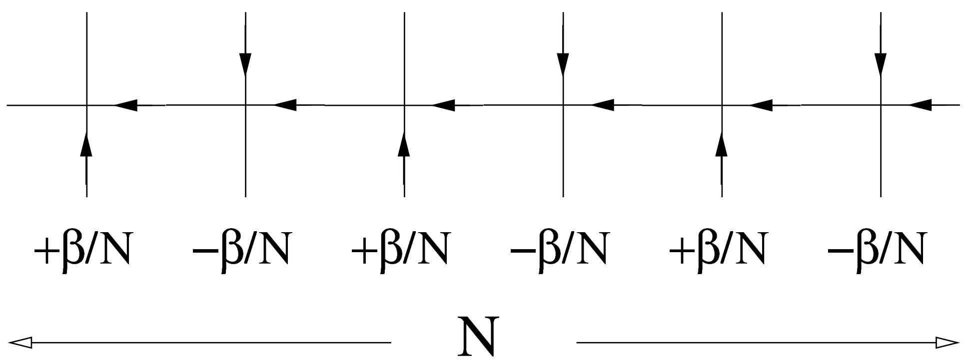
1) Product of R-matrices with same representations



defines transfer matrix whose logarithmic derivative yields Hamiltonian of supersymmetric tJ-model ($2t = J$)

$$\mathcal{H} = -t\sum_{j,\sigma}\mathcal{P}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\mathcal{P} + J\sum_{j}(\vec{S}_j\vec{S}_{j+1} - n_j n_{j+1}/4),$$

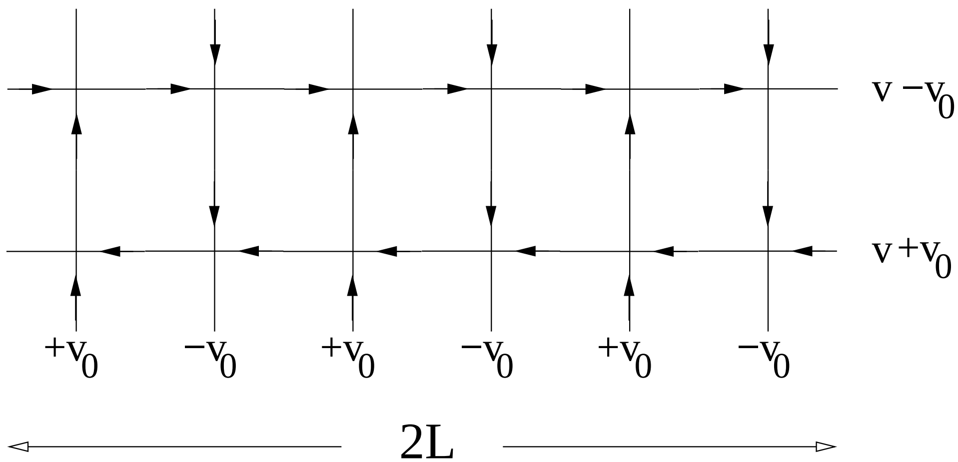
2) Product of R-matrices with alternating representations



yields "quantum transfer matrix" whose largest eigenvalue yields free energy of supersymmetric tJ-model

Transfer matrices, Hamiltonians

3) Transfer matrix with two rows and alternation of representations from column to column (and row to row)



defines transfer matrix whose logarithmic derivative yields a local Hamiltonian.

Alternatively:
lattice constructed from repeated application of double row yields realization of an integrable Chalker-Coddington network with or without relevance for spin-quantum Hall effect; black hole CFTs, emerging non-compact degrees of freedom, continuous spectrum (Saleur, Jacobsen, Ikhlef; Frahm, Seel).

Derivation and proof of integrability by R. Gade (1998); extensive investigations of spectrum by Essler, Frahm, Saleur (2005)

Our goal: Analytical calculation of largest eigenvalues of $T_1(v+v_0)T_2(v-v_0)$ where T_1 and T_2 are transfer matrices with "standard" and conjugated fundamental representations of $sl(2|1)$ in auxiliary space.

Eigenvalues of transfer matrices $T_1(v)$ and $T_2(v)$ (...Links, Foerster 1999; Göhmann, Seel 2004)

$$\Lambda_1(v) = \lambda_1^{(-)}(v) + \lambda_1^{(0)}(v) + \lambda_1^{(+)}(v), \qquad \Lambda_2(v) = \lambda_2^{(-)}(v) + \lambda_2^{(0)}(v) + \lambda_2^{(+)}(v),$$

where

$$\lambda_1^{(-)}(v) = \mathrm{e}^{-\mathrm{i}\varphi}\Phi_+(v+\mathrm{i}/2)\Phi_-(v+3\mathrm{i}/2)\,\frac{q_u(v-\frac{3\mathrm{i}}{2})}{q_u(v+\frac{\mathrm{i}}{2})}$$

$$\lambda_1^{(0)}(v) = 1\cdot\Phi_+(v+\mathrm{i}/2)\Phi_-(v-\mathrm{i}/2)\,\frac{q_u(v-\frac{3\mathrm{i}}{2})}{q_u(v+\frac{\mathrm{i}}{2})}\,\frac{q_\gamma(v+\frac{3\mathrm{i}}{2})}{q_\gamma(v-\frac{\mathrm{i}}{2})}, \qquad (\varphi\to\pi)$$

$$\lambda_1^{(+)}(v) = \mathrm{e}^{+\mathrm{i}\varphi}\Phi_+(v-3\mathrm{i}/2)\Phi_-(v-\mathrm{i}/2)\,\frac{q_\gamma(v+\frac{3\mathrm{i}}{2})}{q_\gamma(v-\frac{\mathrm{i}}{2})}$$

and formulas for $\lambda_2^{(\pm,0)}$ are obtained from those above by simultaneous exchange $\Phi_+ \leftrightarrow \Phi_-$ and $q_u \leftrightarrow q_\gamma$

"Vacuum functions" $\Phi_\pm$ and q-functions in terms of Bethe ansatz rapidities u_j and γ_α

$$\Phi_\pm(v) := (v \pm v_0)^L, \qquad q_u(v) := \prod_{k=1}^{N}(v-u_k), \qquad q_\gamma(v) := \prod_{\beta=1}^{M}(v-\gamma_\beta),$$

Bethe Ansatz equations

Eigenvalue functions have to be analytic → cancellation of poles by zeros yielding Bethe ansatz equations

$$\frac{\Phi_-(u_j+\mathrm{i})}{\Phi_-(u_j-\mathrm{i})} = -e^{\mathrm{i}\varphi}\frac{q_\gamma(u_j+\mathrm{i})}{q_\gamma(u_j-\mathrm{i})}, \quad j=1,\dots,N$$

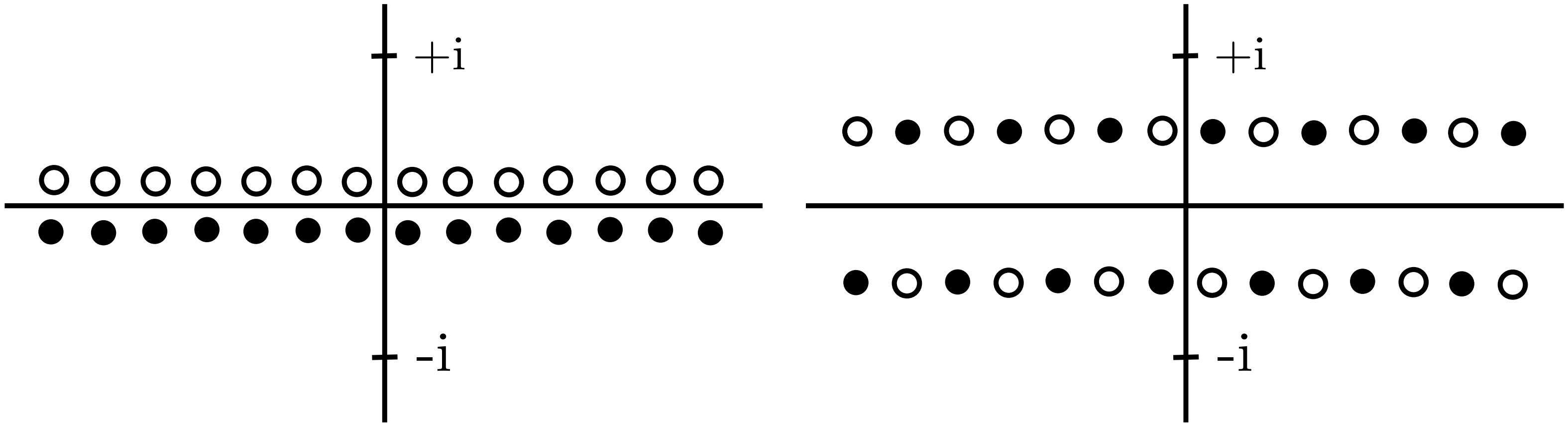
$$\frac{\Phi_+(\gamma_\alpha+\mathrm{i})}{\Phi_+(\gamma_\alpha-\mathrm{i})} = -e^{\mathrm{i}\varphi}\frac{q_u(\gamma_\alpha+\mathrm{i})}{q_u(\gamma_\alpha-\mathrm{i})}, \quad \alpha=1,\dots,M$$

These equations are the same for the QTM of the tJ model and for the supersymmetric network model.

Characterization of largest eigenvalue differs:

tJ: maximum value of Λ_1

network model: maximum value(s) of $\Lambda_1 \cdot \Lambda_2$



"strange strings" (Essler, Frahm, Saleur 2005)

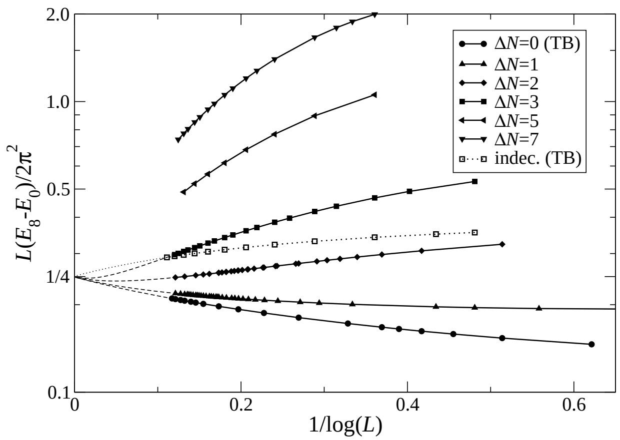
Bethe Ansatz: root distributions

Some results from Essler, Frahm, Saleur (2005) (numerical work for L up to approx. 5000):

- groundstate for $\varphi = \pi$ given by "degenerate solution" $u_j = -v_0$, $\gamma_\alpha = +v_0$ for all $j, \alpha = 1, \ldots, L$. groundstate energy is $E_0 = -4L$ and hence central charge $c = 0$.
- excited states are given by seas of "strange strings", i.e. one u and one γ rapidity with condition

  $\mathrm{Re}\, u = \mathrm{Re}\, \gamma$ and $\mathrm{Im}\, u = +\frac{1}{2} + \varepsilon$, $\mathrm{Im}\, \gamma = -\frac{1}{2} - \varepsilon$; or

  $\mathrm{Re}\, u = \mathrm{Re}\, \gamma$ and $\mathrm{Im}\, u = -\frac{1}{2} + \varepsilon$, $\mathrm{Im}\, \gamma = +\frac{1}{2} - \varepsilon$
- infinite number of excited states with same scaling dimension, differing by logarithmic corrections



- For special case $v_0 = 0$: simplification for states with identical sets of u rapidities and γ rapidities, $u_j = \gamma_j$ ($j = 1, \ldots, N$)

  two sets of BA equations coincide as $\Phi_+ = \Phi_-$ and $q_u = q_\gamma$

  remaining set of BA equations equivalent to Takhtajan-Babujian solution of spin-1 $su(2)$ chain

Functional equations: Definition of auxiliary functions

tJ model motivated ansatz of suitable auxiliary functions

$$b := \frac{\lambda_1^{(0)} + \lambda_1^{(+)}}{\lambda_1^{(-)}}, \qquad B := 1 + b = \frac{\lambda_1^{(-)} + \lambda_1^{(0)} + \lambda_1^{(+)}}{\lambda_1^{(-)}},$$

$$\bar{b} := \frac{\lambda_1^{(-)} + \lambda_1^{(0)}}{\lambda_1^{(+)}}, \qquad \bar{B} := 1 + \bar{b} = \frac{\lambda_1^{(-)} + \lambda_1^{(0)} + \lambda_1^{(+)}}{\lambda_1^{(+)}},$$

$$c := \frac{\lambda_1^{(0)} \left[\lambda_1^{(-)} + \lambda_1^{(0)} + \lambda_1^{(+)}\right]}{\lambda_1^{(-)}\lambda_1^{(+)}}, \qquad C := 1 + c = \frac{\left[\lambda_1^{(-)} + \lambda_1^{(0)}\right]\left[\lambda_1^{(0)} + \lambda_1^{(+)}\right]}{\lambda_1^{(-)}\lambda_1^{(+)}},$$

Factorization into “elementary factors” ...

... yields integral equations for logs: $\log b =: -L\varepsilon, \quad \log(1+b) = \log(1+\mathrm{e}^{-L\varepsilon})$ etc.

Functional equations: factorization

Factorization into "elementary factors" q_u, q_γ, D_u, D_γ, Λ_1

$$b(v) = e^{i\varphi} \frac{\Phi_-(v-i/2) q_\gamma(v+3i/2) D_\gamma(v-i/2)}{\Phi_+(v+i/2)\Phi_-(v+3i/2) q_u(v-3i/2)}, \qquad B(v) = e^{i\varphi} \frac{q_u(v+i/2)\Lambda_1(v)}{\Phi_+(v+i/2)\Phi_-(v+3i/2) q_u(v-3i/2)}$$

$$\bar{b}(v) = e^{-i\varphi} \frac{\Phi_+(v+i/2) q_u(v-3i/2) D_u(v+i/2)}{\Phi_-(v-i/2)\Phi_+(v-3i/2) q_\gamma(v+3i/2)}, \qquad \bar{B}(v) = e^{-i\varphi} \frac{q_\gamma(v-i/2)\Lambda_1(v)}{\Phi_-(v-i/2)\Phi_+(v-3i/2) q_\gamma(v+3i/2)}$$

$$c(v) = \frac{\Lambda_1(v)}{\Phi_+(v-3i/2)\Phi_-(v+3i/2)}, \qquad C(v) = \frac{D_u(v+i/2) D_\gamma(v-i/2)}{\Phi_+(v-3i/2)\Phi_-(v+3i/2)},$$

where

$$D_u(v) := \frac{1}{q_u(v)} \left[\Phi_-(v-i) q_\gamma(v+i) + e^{-i\varphi} \Phi_-(v+i) q_\gamma(v-i) \right]$$

$$D_\gamma(v) := \frac{1}{q_\gamma(v)} \left[\Phi_+(v+i) q_u(v-i) + e^{i\varphi} \Phi_+(v-i) q_u(v+i) \right]$$

are polynomials due to the Bethe ansatz equations.

Usual treatment: taking logarithm and then Fourier transform. However, from the three expressions for B, $\bar{B}$, and C the functions q_u, q_γ, D_u, D_γ and Λ_1 can not be resolved!

Apparent reason: too many unknowns (5) in comparison to number of equations (3)

Solution of functional equations: tJ-Model

Interesting case: thermodynamics of tJ-model (Jüttner, AK, J. Suzuki 1997)

- q_u and D_u are free of zeros above the real axis, q_γ and D_γ are free of zeros below the real axis,
- "effective number" of unknowns: 3

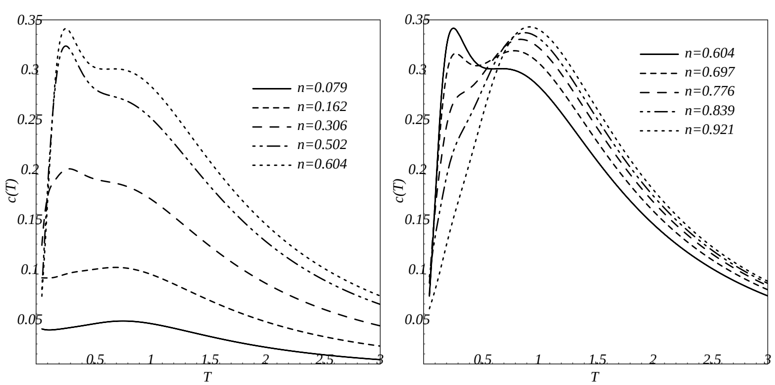
Concrete calculations are done for Fourier transforms of logarithms of all involved functions. Final equations are integral equations of convolution type with kernels $\kappa(x) = \frac{1}{2\pi}\frac{1}{x^2+1/4}$, $\kappa_\pm(x) = \kappa(x \pm \mathrm{i}/2)$,

$$\log b(x) = -\frac{\beta}{x^2+1/4} + \beta(\mu + h/2) - \kappa_+ * \log \overline{B} - \kappa * \log C,$$

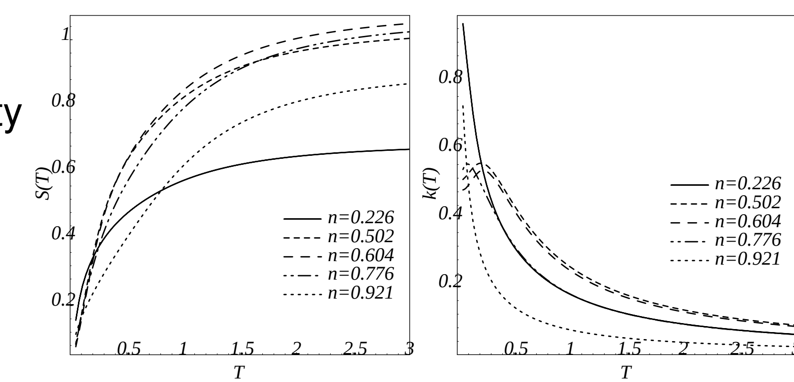
$$\log \overline{b}(x) = -\frac{\beta}{x^2+1/4} + \beta(\mu - h/2) - \kappa_- * \log B - \kappa * \log C,$$

$$\log c(x) = -\frac{2\beta}{x^2+1} + 2\beta\mu - \kappa * \log \overline{B} - \kappa * \log B - (\kappa_+ + \kappa_-) * \log C$$

Specific heat



Compressibility



Compact notation for non-linear integral equations: tJ

tJ model

3 non-linear integral equations take the compact form

$$y = d + K * Y$$

where the abbreviations have been used

$$y := \begin{pmatrix} \log b \\ \log \bar{b} \\ \log c \end{pmatrix}, \; Y := \begin{pmatrix} \log(1+b) \\ \log(1+\bar{b}) \\ \log(1+c) \end{pmatrix}, \; d := \beta \begin{pmatrix} -\frac{1}{x^2+1/4} + \mu + h/2 \\ -\frac{1}{x^2+1/4} + \mu - h/2 \\ -\frac{2}{x^2+1} + 2\mu \end{pmatrix}, \; K = -\begin{pmatrix} 0 & \kappa_+ & \kappa \\ \kappa_- & 0 & \kappa \\ \kappa & \kappa & \kappa_+ + \kappa_- \end{pmatrix}$$

and κ's as above: $\kappa(x) = \frac{1}{2\pi}\frac{1}{x^2+1/4}$, $\kappa_\pm(x) = \kappa(x \pm \mathrm{i}/2)$.

Solution of functional equations: network model

Successful strategy for network model: (Brockmann, AK 200*)

define two sets of auxiliary functions $b_i, \bar{b}_i, c_i \ldots \quad (i = 1, 2)$

- the above introduced auxiliary functions b, $\bar{b}$, c ... are denoted by $b_1, \bar{b}_1, c_1 \ldots$,
- $b_2, \bar{b}_2, c_2 \ldots$ are obtained by simply replacing all subscripts 1 by 2 and exchanging $\Phi_+ \leftrightarrow \Phi_-$, $q_u \leftrightarrow q_\gamma$, $D_u \leftrightarrow D_\gamma$ in the definition

Now there are

- 6 equations for $B_1, \bar{B}_1, C_1, B_2, \bar{B}_2, C_2$ and
- 6 unknowns q_u, q_γ, D_u, D_γ, Λ_1, and Λ_2

which can be solved. In the last step $b_1, \bar{b}_1, c_1, b_2, \bar{b}_2, c_2$ can be expressed in terms of $B_1, \bar{B}_1, C_1, B_2, \bar{B}_2, C_2$

Concrete calculations are done for Fourier transforms of logarithms of all involved functions. Final equations are integral equations of convolution type.

Compact notation for NLIEs: network model (version I)

Supersymmetric network model: 6 non-linear integral equations, version I

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} d \\ d \end{pmatrix} + \begin{pmatrix} A-B & B \\ B & A-B \end{pmatrix} * \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

where y_1 and y_2 are two copies of the 3d vector y, and Y_1 and Y_2 are two copies of the 3d vector Y.
Driving terms

$$d := \begin{pmatrix} L \log \mathrm{th} \frac{\pi}{2} x - \mathrm{i}\varphi/2 \\ L \log \mathrm{th} \frac{\pi}{2} x + \mathrm{i}\varphi/2 \\ 0 \end{pmatrix},$$

and kernel matrices (in Fourier representation)

$$A(k) = \frac{1}{2\cosh k/2} \begin{pmatrix} \mathrm{e}^{-|k|/2} & -\mathrm{e}^{-|k|/2-k} & 1 \\ -\mathrm{e}^{-|k|/2+k} & \mathrm{e}^{-|k|/2} & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad B(k) = \begin{pmatrix} \frac{1}{2\sinh|k|} & -\frac{\mathrm{e}^{-k}}{2\sinh|k|} & -\frac{\mathrm{e}^{-k/2}}{2\sinh(k)} \\ -\frac{\mathrm{e}^{k}}{2\sinh|k|} & \frac{1}{2\sinh|k|} & \frac{\mathrm{e}^{k/2}}{2\sinh(k)} \\ \frac{\mathrm{e}^{k/2}}{2\sinh(k)} & -\frac{\mathrm{e}^{-k/2}}{2\sinh(k)} & 0 \end{pmatrix}$$

Good properties: symmetry $A(-k)^T = A(k)$, $B(-k)^T = B(k)$ may allow for analytic calculations of CFT
bad properties: B is very singular! Kernel of integral equations not integrable!

Compact notation for NLIEs: network model (versions II & III)

NLIE version II

Technical trick: particle-hole transformation

$$\log B = \log(1+b) = \log(1+1/b) + \log b = \log \tilde{B} - \log \tilde{b} \quad \text{where} \quad \tilde{b} = 1/b$$

Then rewrite equations for $\log \tilde{b}$ etc. in terms of $\log \tilde{B}$ etc.

$$y = d + K * Y \quad \Leftrightarrow \quad -\tilde{y} = d + K * (\tilde{Y} - \tilde{y}) \quad \Leftrightarrow \quad \tilde{y} = -(1-K)^{-1} * (d + K * \tilde{Y})$$

The new kernel is regular(!) but now $\log \tilde{B}$ and $\log \tilde{\tilde{B}}$ are singular at $x \to \pm\infty$ and 0!

NLIE version III

New idea: write y in terms of Y as well as $\tilde{Y} (= Y - y)$, difficult to find as redundant and not unique:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} d \\ d \end{pmatrix} + \frac{1}{2} \begin{pmatrix} K & K \\ K & K \end{pmatrix} * \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \tilde{K} & -\tilde{K} \\ -\tilde{K} & \tilde{K} \end{pmatrix} * \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix}$$

with regular $K = A$ (as above) and regular $\tilde{K}$!

Note: some singular behaviour of the $\tilde{Y}$ cancels in the difference!

NLIEs version III: kernels

Fourier transforms

$$K(k) = \frac{1}{2\cosh k/2}\begin{pmatrix} e^{-|k|/2} & -e^{-|k|/2-k} & 1 \\ -e^{-|k|/2+k} & e^{-|k|/2} & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad K(k) = K^T(-k)$$

$$\tilde{K}(k>0) = \begin{pmatrix} -\frac{1}{e^k+1} & e^{-k} - e^{-2k} + \frac{e^{-k}}{e^k+1} & e^{-k/2} - e^{-3k/2} \\ \frac{e^k}{e^k+1} & -\frac{1}{e^k+1} & 0 \\ 0 & e^{-k/2} - e^{-3k/2} & -e^{-k} \end{pmatrix}, \qquad \tilde{K}(k<0) := \tilde{K}^T(-k)$$

Most compact notation of NLIE as two weakly coupled 3×3 systems

$$y_i = d \pm \tilde{d} + K * Y_i, \qquad i = 1,2 \quad \text{for which } +,- \text{ applies}$$

and additional driving term

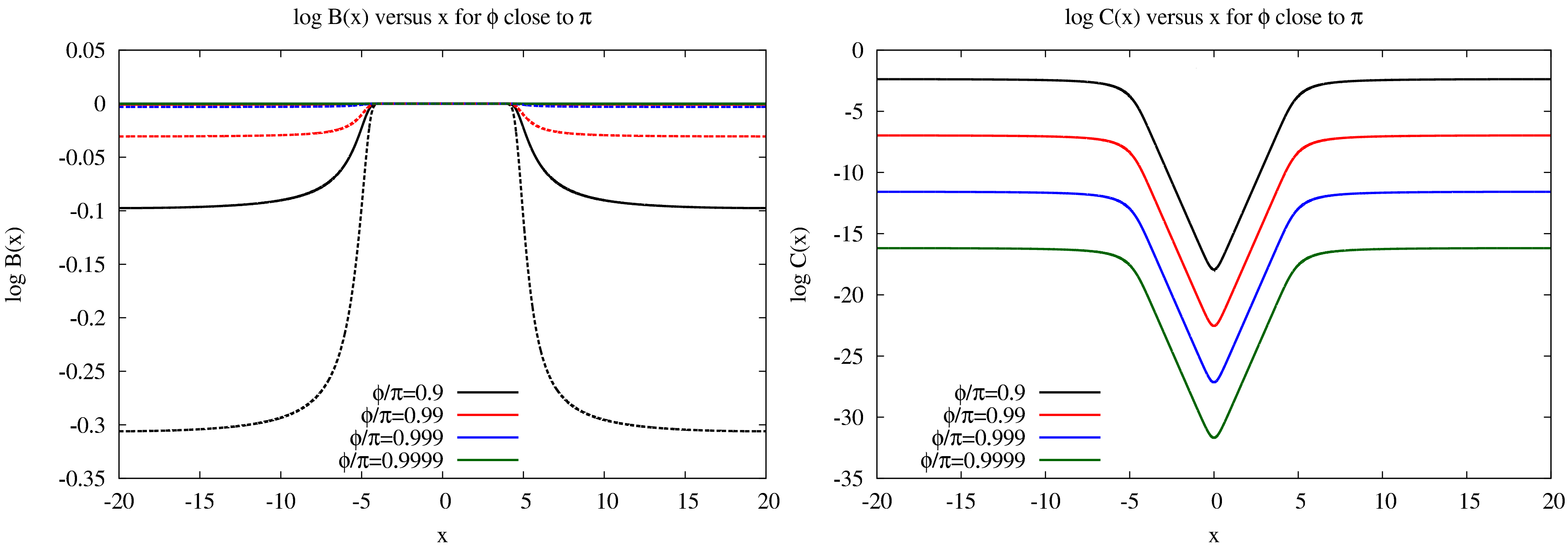
$$\tilde{d} := \frac{1}{2}(\tilde{K} - K) * (Y_1 - Y_2) - \frac{1}{2}\tilde{K} * (y_1 - y_2)$$

Numerical solution to NLIE: ground-state

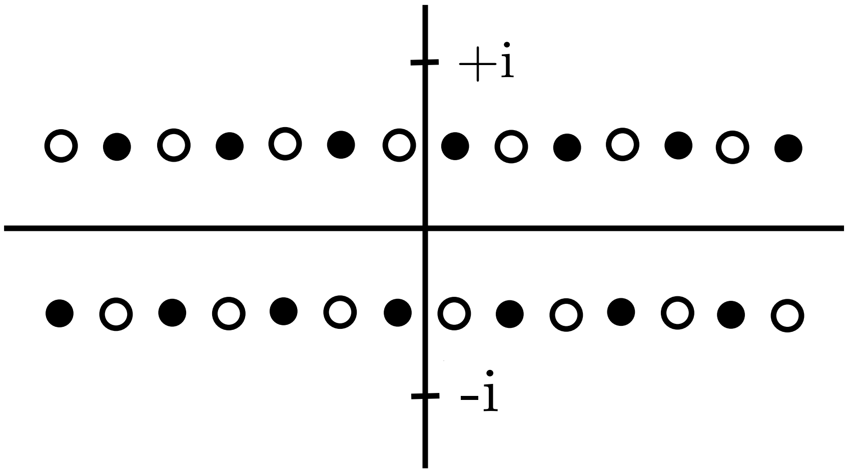
Ground state of model with $\varphi = \pi$ completely degenerate, but not for $\varphi \neq \pi$.
For $\varphi = \pi$ we know

$$b_j = \bar{b}_j = 0,\; B_j = \bar{B}_j = 1, \quad c_j = -1, C_j = 0$$

For $\varphi \neq \pi$ with $\tilde{d} = 0$ we find numerically ($L = 10^6$)



Numerical solution to NLIE: excited states, $\varphi = \pi$



$L = 10$
9 rapidities of each type
"strange strings" (Essler, Frahm, Saleur 2005)

Here the functions $C_1(x) = 1 + c_1(x)$, $C_2(x) = 1 + c_2(x)$ have zeros at $\pm\theta_1$, $\pm\theta_2$ with

$$\theta_1 = 2.19559584..., \qquad \theta_2 = 1.39236116...$$

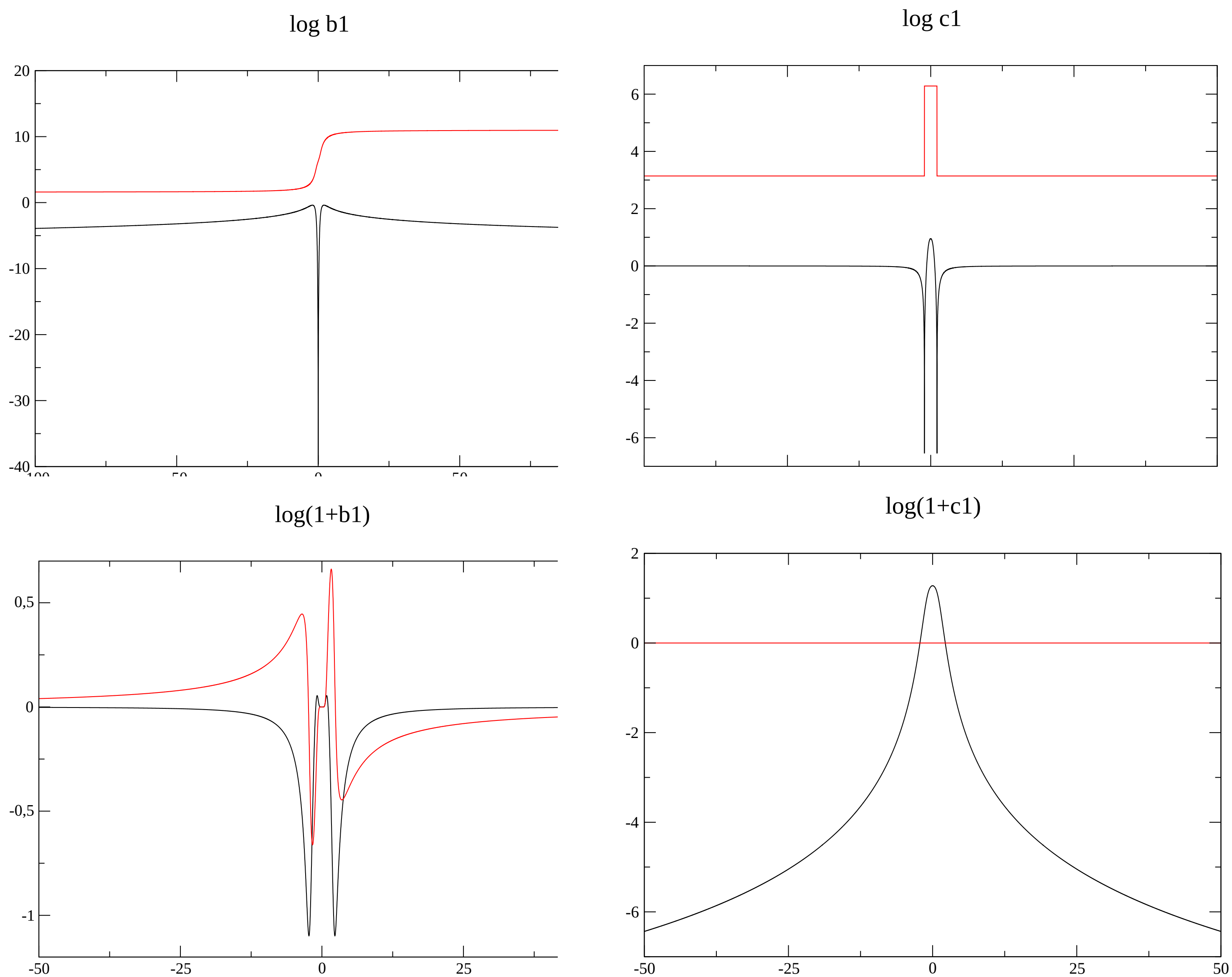
$\longrightarrow$ additional driving terms, additive in θ_1, θ_2

numerically: NLIE are satisfied
direct iteration does not converge, errors 'explode'
reason: consistency condition

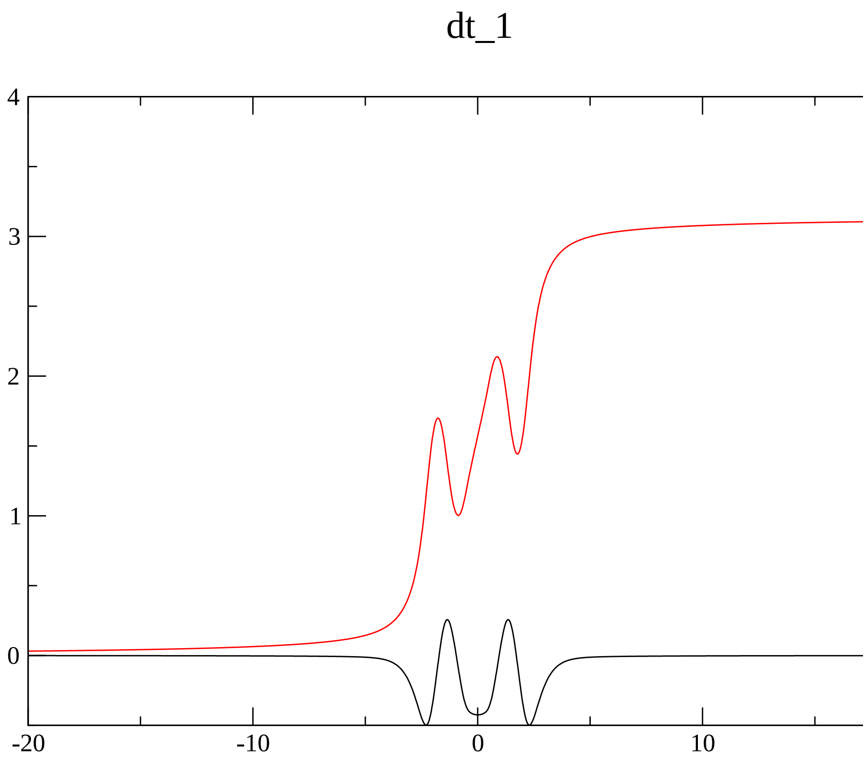
$$(1 + \tilde{K}) * (y_1 - y_2) = \tilde{K} * (Y_1 - Y_2)$$

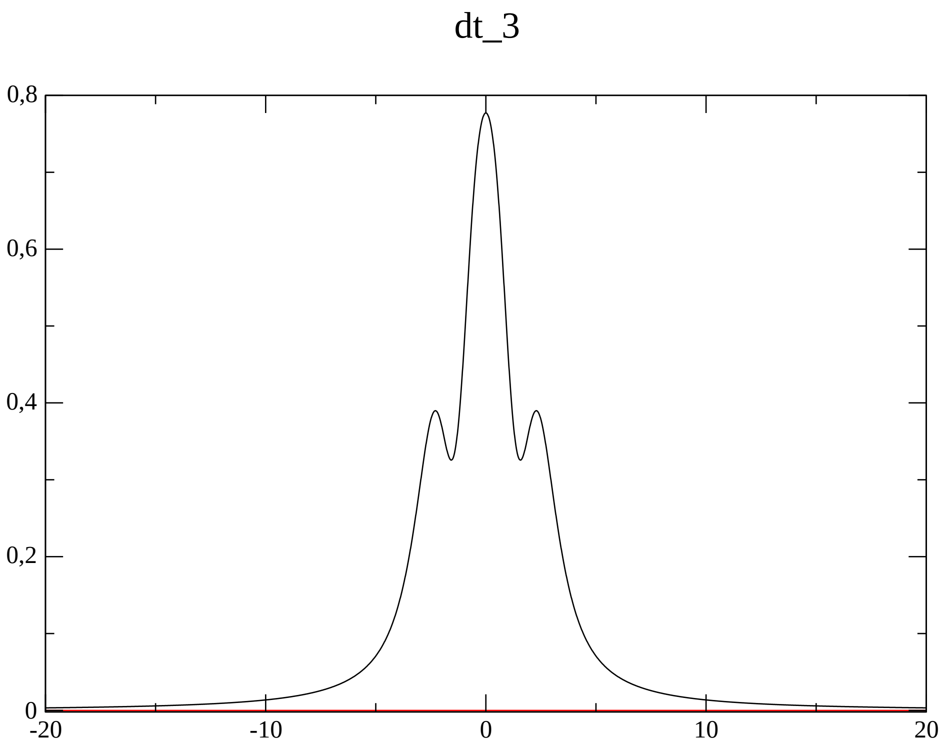
'solved' 1 time 'forward', 2 times 'backward'
result inserted into $\tilde{d} \longrightarrow$ convergence

Numerical solution to NLIE: excited states, $\varphi = \pi$



Numerical solution to NLIE: excited states, $\varphi = \pi$





Integral equations for network model

properties and merits of non-linear integral equations for 6 auxiliary functions

- equations are exact for any system size L, even for $L = 2$!
- kernel is regular $\rightarrow$ numerical and analytical solutions feasible
  goal: all scaling dimensions from $1/L$ excitations gaps; logarithmic corrections, e.g. $1/(L\log L)$
- physical rapidities, i.e. zeros of Λ_1 and Λ_2, enter the driving terms d via deformed contours approach
- Takhtajan-Babujian solutions (for $v_0 = 0$ and coinciding strings) lead to simplification
  $b_1 = b_2$, $\bar{b}_1 = \bar{b}_2$, $c_1 = c_2$ and set of non-linear integral equations reduce to the "truncated TBA" equations for spin-1 $su(2)$ (see J. Suzuki 99).
- general case can be understood as two 'weakly coupled' sets of Takhtajan-Babujian NLIE
- numerical solution by iteration: procedure not necessarily converging...

Some analytical result (Brockmann, AK) for:
$v_0 = 0$: $L/2 + L/2$ many strange strings of both types, pairwise "degenerate" corresponding to TB-state with $L/2$ many 2-strings

Excitation energy computable by use of "dilog-trick"

$$\Delta E = \frac{\pi^2}{2}\frac{1}{L}, \qquad \text{scaling dimension } x = \frac{1}{4}$$

of course: result is known, but now follows from completely analytical calculations

Summary

Results:

- presentation of non-linear integral equations for the staggered $sl(2|1)$ network model
- explicit numerical calculation for the ground state
- integration kernels are regular and symmetric
- solution functions $\log C_j(x)$ singular for $x \to \pm\infty$ if $\varphi = \pi$, unavoidable

To do:

- NLIEs also hold for the excited states, but need to be analysed in future work
- analytic and numerical calculations
- symmetry of integration kernel allows for “dilogarithmic-trick”

Advertisement: Textbook on Hubbard model and related systems