XXZ spin chain with generic boundaries.

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References:

Complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms, S. Faldella, N.K. and G. Niccoli, J. Stat. Mech. (2014) P01011. **arXiv:1307.3960**

Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from SOV, N. K., J. M. Maillet, G. Niccoli, J. Stat. Mech. (2014) P05015, **arXiv:1401.4901**

On determinant representations of scalar products and form factors in the SoV approach: the XXX case, N. K., J. M. Maillet, G. Niccoli, V. Terras **arXiv:1506.???? (online tomorrow)**

The XXZ spin-1/2 Heisenberg chain

Open chain XXZ chain

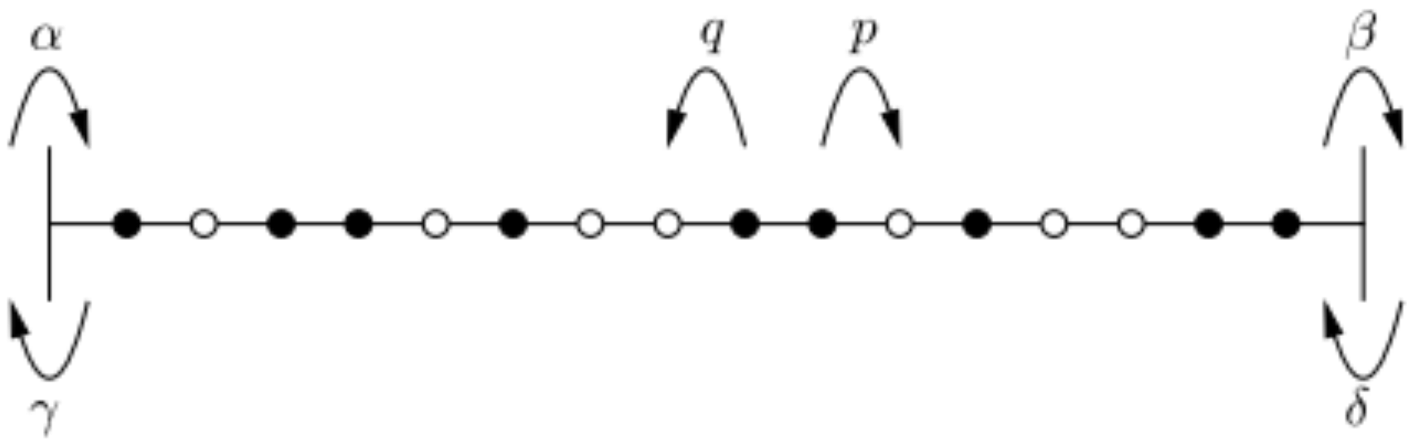
$$H = \sum_{m=1}^{N-1} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) \right)$$
$$- h_-^x \sigma_1^x - h_-^y \sigma_1^y - h_-^z \sigma_1^z - h_+^x \sigma_N^x - h_+^y \sigma_N^y - h_+^z \sigma_N^z$$

$h_\pm^a$, $a = x, y, z$ - boundary magnetic fields.

Due to the $U(1)$ symmetry of the bulk Hamiltonian **5 generic parameters**

Motivations

- All the attributes of the integrability but one: there was no exact solution for generic boundary terms.
- A simple model for the interaction with an environment
- Relation with open **Asymmetric Simple Exclusion Process** (ASEP)



- We need eigenstates, overlaps, form factors, correlation functions

Open chain. Diagonal boundaries

Diagonal boundary terms **(2 parameters)**:

- **coordinate** Bethe ansatz: F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel 1987
- **Algebraic** Bethe ansatz E.K. Sklyanin 1988.
- **Correlation functions**, vertex operators approach: M. Jimbo, R. Kedem, T. Kojima, H. Konno, T.Miwa, 1995
- **Partition function**, Izergin-type determinant formula O. Tsuchiya, 1998
- **Correlation functions**, Algebraic Bethe Ansatz approach: N. K., K.K. Kozlowski, J.M. Maillet, G. Niccoli, N. Slavnov, V. Terras, 2007.

Open chain. Non-diagonal boundaries with constraints

Non-diagonal boundary terms with boundary constraints **(one constraint, 4 parameters)**

- T-Q **equation**: R. Nepomechie 2002 (roots of unity), 2004 (general Δ);
- **Algebraic Bethe ansatz** with gauge transformation: J. Cao, H.-Q. Lin, K.-J. Shi, Y. Wang, 2003,
- Second reference state: W.-L. Yang, Y-.Z. Zhang 2007,
- Partition function, **determinant representations**: G. Filali, N. K. 2010
- **Separation of variables**, XXX chain: H. Frahm, A. Seel, T. Wirth 2008
- **Coordinate Bethe ansatz**: N. Crampé, E. Ragoucy, D. Simon 2010-2011

Bethe ansatz without boundary contraints

- **XXX chain, functional Bethe ansatz**: H. Frahm, J.H. Grelik, A. Seel, T. Wirth 2011: Spectrum of the XXX chain with generic boundary
- **Off-diagonal Bethe ansatz**: J.Cao, W.L. Yang, K. Shi, Y. Wang 2013: Inhomogeneous Baxter equation, Bethe-like equations.
  Problem: no description for the eigenstates, no possibility to distinguish admissible and inadmissible solutions. Can produce many different descriptions for the same state.
- **Separation of variable approach, construction of the eigenstates**. One triangular K-matrix (**4 parameters again**): Niccoli (2012), **generic** case: S. Faldella, N.K. Niccoli 2013 (construction of the eigenstates).
- **Modified Algebraic Bethe ansatz**: XXX chain, conjecture by S. Belliard, N. Crampé 2013, XXZ case new conjectures 2014
- Several other methods: W. Galleas, P. Baseilhac, V. Pasquier...

Quantum inverse scattering method

L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan (1979):

1. Yang-Baxter equation:

$$R_{12}(\lambda_{12})\ R_{13}(\lambda_{13})\ R_{23}(\lambda_{23}) = R_{23}(\lambda_{23})\ R_{13}(\lambda_{13})\ R_{12}(\lambda_{12}).$$

We consider the trigonometric solution with $\Delta = \cosh\eta$

2. Monodromy matrix.

$$M_a(\lambda) = R_{aN}(\lambda - \xi_N - \eta)\ldots R_{a1}(\lambda - \xi_1 - \eta) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[a]}$$

ξ_j are generic inhomogeneity parameters: $\xi_j \neq \xi_k + \epsilon\eta$, $\epsilon = 0, \pm 1$.

Reflection equation

Cherednik 1984

$$R_{12}(\lambda-\mu)\,K_1(\lambda)\,R_{12}(\lambda+\mu)\,K_2(\mu)=K_2(\mu)\,R_{12}(\lambda+\mu)\,K_1(\lambda)\,R_{12}(\lambda-\mu).$$

General 2×2 solution (Ghoshal Zamolodchikov 1994):

$$K(\lambda;\zeta,\kappa,\tau)=\frac{1}{\sinh\zeta}\begin{pmatrix}\sinh(\lambda-\eta/2+\zeta) & \kappa e^{\tau}\sinh(2\lambda-\eta)\\ \kappa e^{-\tau}\sinh(2\lambda-\eta) & \sinh(\zeta-\lambda+\eta/2)\end{pmatrix}$$

Right boundary: $K^{+}(\lambda)=K^{-}(\lambda+\eta)$,

Quantum inverse scattering method, Sklyanin 1988

$$\mathcal{U}_{-}(\lambda)=M(\lambda)K_{-}(\lambda)\widehat{M}(\lambda)=M(\lambda)K_{-}(\lambda)\sigma_0^y M^{t_0}(-\lambda)\sigma_0^y=\begin{pmatrix}\mathcal{A}_{-}(\lambda) & \mathcal{B}_{-}(\lambda)\\ \mathcal{C}_{-}(\lambda) & \mathcal{D}_{-}(\lambda)\end{pmatrix}$$

1. Reflection algebra

$$R_{12}(\lambda-\mu)\,(\mathcal{U}_-)_1(\lambda)\,R_{12}(\lambda+\mu-\eta)\,(\mathcal{U}_-)_2(\mu)$$
$$=(\mathcal{U}_-)_2(\mu)\,R_{12}(\lambda+\mu-\eta)\,(\mathcal{U}_-)_1(\lambda)\,R_{12}(\lambda-\mu)$$

2. Transfer matrix:

$$\mathcal{T}(\lambda)=\mathrm{tr}_0\{K_+(\lambda)\,\mathcal{U}_-(\lambda)\}.$$

$$[\mathcal{T}(\lambda),\mathcal{T}(\mu)]=0$$

3. Hamiltonian (homogeneous limit):

$$H=\frac{2(\sinh\eta)^{1-2N}}{\mathrm{tr}\{K_+(\eta/2)\}\,\mathrm{tr}\{K_-(\eta/2)\}}\frac{d}{d\lambda}\mathcal{T}(\lambda)_{|\lambda=\eta/2}+\text{constant.}$$

$$H = \sum_{i=1}^{N-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh\eta\, \sigma_i^z \sigma_{i+1}^z)$$
$$+ \frac{\sinh\eta}{\sinh\zeta_-} [\sigma_1^z \cosh\zeta_- + 2\kappa_-(\sigma_1^x \cosh\tau_- + i\sigma_1^y \sinh\tau_-)]$$
$$+ \frac{\sinh\eta}{\sinh\zeta_+} \left[(\sigma_N^z \cosh\zeta_+ + 2\kappa_+(\sigma_N^x \cosh\tau_+ + i\sigma_{\mathsf{N}}^y \sinh\tau_+)\right] .$$

Two obstacles: No reference state! $|0\rangle$, such that $\mathcal{C}_-(\lambda)|0\rangle = 0$, $\forall\lambda$ and K_+ mixes all the operators...

No usual Algebraic Bethe Ansatz $\longrightarrow$ **Separation of variables**

Model case: if K_+ is triangular separation of variables works (Niccoli 2012). Generic case can be reduced to triangular case but we need a gauge transformation

Quantum determinant

$$\frac{\det_q \mathcal{U}_-(\lambda)}{\sinh(2\lambda - 2\eta)} = \mathcal{A}_-(\epsilon\lambda + \eta/2)\mathcal{A}_-(\eta/2 - \epsilon\lambda) + \mathcal{B}_-(\epsilon\lambda + \eta/2)\mathcal{C}_-(\eta/2 - \epsilon\lambda)$$
$$= \mathcal{D}_-(\epsilon\lambda + \eta/2)\mathcal{D}_-(\eta/2 - \epsilon\lambda) + \mathcal{C}_-(\epsilon\lambda + \eta/2)\mathcal{B}_-(\eta/2 - \epsilon\lambda),$$

where $\epsilon = \pm 1$, Quantum determinant is a central element of the reflection algebra

$$[\det_q \mathcal{U}_-(\lambda), \mathcal{U}_-(\mu)] = 0.$$

Explicit expressions:

$$\begin{aligned} \det_q \mathcal{U}_-(\lambda) &= \det_q K_-(\lambda) \det_q M_0(\lambda) \det_q M_0(-\lambda) \\ &= \sinh(2\lambda - 2\eta)\mathsf{A}_-(\lambda + \eta/2)\mathsf{A}_-(-\lambda + \eta/2), \end{aligned}$$

Bulk quantum determinant:

$$\det_q M(\lambda) = a(\lambda + \eta/2)d(\lambda - \eta/2),$$

Quantum determinant for the boundary matrices

$$\det_q K_\pm(\lambda) = \sinh(2\lambda \pm 2\eta)g_\pm(\lambda + \eta/2)g_\pm(-\lambda + \eta/2).$$

Notations:

$$\mathsf{A}_-(\lambda) = g_-(\lambda)a(\lambda)d(-\lambda), \quad d(\lambda) = a(\lambda - \eta), \quad a(\lambda) = \prod_{n=1}^{N} \sinh(\lambda - \xi_n),$$

$$g_\pm(\lambda) = \frac{\sinh(\lambda + \alpha_\pm \pm \eta/2)\cosh(\lambda \mp \beta_\pm \pm \eta/2)}{\sinh \alpha_\pm \cosh \beta_\pm},$$

$\alpha_\pm$ and $\beta_\pm$ give a different parametrisation for the boundary parameters:

$$\sinh \alpha_\pm \cosh \beta_\pm = \frac{\sinh \zeta_\pm}{2\kappa_\pm}, \qquad \cosh \alpha_\pm \sinh \beta_\pm = \frac{\cosh \zeta_\pm}{2\kappa_\pm}.$$

Gauge transformation

Cao et al (2003): 8-vertex scheme, following Baxter 1972 and Faddeev Takhtadjan 1979. **Gauge transformation** to diagonalize the boundary matrices.

$$\bar{G}(\lambda|\beta) = (X(\lambda|\beta), Y(\lambda|\beta)), \quad \tilde{G}(\lambda|\beta) = (X(\lambda|\beta+1), Y(\lambda|\beta-1)) \tag{1}$$

where we have defined the following columns

$$X(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha+\beta)\eta]} \\ 1 \end{pmatrix}, \qquad Y(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha-\beta)\eta]} \\ 1 \end{pmatrix}.$$

Evidently these matrices depend also on α but as this parameter will not vary in the following computations we omit this argument. For ABA one needs K_+ diagonal and K_- triangular (with **2 parameter family** of the gauge transformations these condition lead to **1 constraint**). For our approach we need K_+ triangular and K_- generic $\to$ **no constraint**, one free parameter

Monodromy matrices

Bulk left to right monodromy matrix:

$$M(\lambda|\beta) = \tilde{G}^{-1}(\lambda - \eta/2|\beta)\, M(\lambda)\tilde{G}(\lambda - \eta/2|\beta + N) = \begin{pmatrix} A(\lambda|\beta) & B(\lambda|\beta) \\ C(\lambda|\beta) & D(\lambda|\beta) \end{pmatrix}$$

right to left monodromy matrix:

$$\widehat{M}(\lambda|\beta) = \bar{G}^{-1}(\eta/2 - \lambda|\beta + N)\, \widehat{M}(\lambda)\bar{G}(\eta/2 - \lambda|\beta) = \begin{pmatrix} \bar{A}(\lambda|\beta) & \bar{B}(\lambda|\beta) \\ \bar{C}(\lambda|\beta) & \bar{D}(\lambda|\beta) \end{pmatrix}$$

two-row monodromy matrix

$$\mathcal{U}_-(\lambda|\beta) = e^{-\lambda+\eta/2}\tilde{G}^{-1}(\lambda - \eta/2|\beta)\, \mathcal{U}_-(\lambda)\, \tilde{G}(\eta/2 - \lambda|\beta) = \begin{pmatrix} \mathcal{A}_-(\lambda|\beta + 2) & \mathcal{B}_-(\lambda|\beta) \\ \mathcal{C}_-(\lambda|\beta + 2) & \mathcal{D}_-(\lambda|\beta) \end{pmatrix}$$

“Dynamical” reflection algebra

Examples of commutation relations for the gauged transformed reflection algebra generators:

$$\mathcal{B}_-(\lambda_2|\beta)\mathcal{B}_-(\lambda_1|\beta-2)=\mathcal{B}_-(\lambda_1|\beta)\mathcal{B}_-(\lambda_2|\beta-2),$$

$$\begin{aligned}
&\mathcal{A}_-(\lambda_2|\beta+2)\mathcal{B}_-(\lambda_1|\beta)\\
&\quad=\frac{\sinh(\lambda_1-\lambda_2+\eta)\sinh(\lambda_2+\lambda_1-\eta)}{\sinh(\lambda_1-\lambda_2)\sinh(\lambda_1+\lambda_2)}\mathcal{B}_-(\lambda_1|\beta)\mathcal{A}_-(\lambda_2|\beta)\\
&\quad+\frac{\sinh(\lambda_1+\lambda_2-\eta)\sinh(\lambda_1-\lambda_2+(\beta-1)\eta)\sinh\eta}{\sinh(\lambda_2-\lambda_1)\sinh(\lambda_1+\lambda_2)\sinh(\beta-1)\eta}\mathcal{B}_-(\lambda_2|\beta)\mathcal{A}_-(\lambda_1|\beta)\\
&\quad+\frac{\sinh\eta\sinh(\lambda_1+\lambda_2-\beta\eta)}{\sinh(\lambda_1+\lambda_2)\sinh(\beta-1)\eta}\mathcal{B}_-(\lambda_2|\beta)\mathcal{D}_-(\lambda_1|\beta),
\end{aligned}$$

Quantum determinant:

$$\mathcal{U}_-(\lambda+\eta/2|\beta)\,\mathcal{U}_-(\eta/2-\lambda|\beta)=\frac{\det_q \mathcal{U}_-(\lambda)}{\sinh(2\lambda-2\eta)},$$

$$\mathcal{U}_-(\eta/2-\lambda|\beta)\,\mathcal{U}_-(\lambda+\eta/2|\beta)=\frac{\det_q \mathcal{U}_-(\lambda)}{\sinh(2\lambda-2\eta)}.$$

Transfer matrix:

$$\begin{aligned} e^{-\lambda+\eta/2}\mathcal{T}(\lambda) &= K_+^{(L)}(\lambda|\beta-1)_{11}\mathcal{A}_-(\lambda|\beta)+K_+^{(L)}(\lambda|\beta-1)_{22}\mathcal{D}_-(\lambda|\beta)\\ &+K_+^{(L)}(\lambda|\beta-1)_{21}\mathcal{B}_-(\lambda|\beta-2)+K_+^{(L)}(\lambda|\beta-1)_{12}\mathcal{C}_-(\lambda|\beta+2), \end{aligned}$$

where $K_+^{(L)}(\lambda|\beta-1)$ gauge transformed boundary matrix

More convenient form:

$$\mathcal{T}(\lambda) = \mathsf{a}_+(\lambda|\beta-1)\mathcal{A}_-(\lambda|\beta) + \mathsf{a}_+(-\lambda|\beta-1)\mathcal{A}_-(-\lambda|\beta) + K_+^{(L)}(\lambda|\beta-1)_{21}\mathcal{B}_-(\lambda|\beta-2) + K_+^{(L)}(\lambda|\beta-1)_{12}\mathcal{C}_-(\lambda|\beta+2).$$

where

$$\mathsf{a}_+(\lambda|\beta) = \frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda \sinh(\beta-1)\eta \sinh\zeta_+}\Big[\sinh\zeta_+ \cosh(\lambda-\eta/2)\sinh(\lambda+\eta/2+\beta\eta) - \cosh\zeta_+ \sinh(\lambda-\eta/2)\cosh(\lambda+\eta/2+\beta\eta) - \kappa_+ \sinh(2\lambda-\eta)\sinh(\tau_+ + \alpha\eta + 2\eta)\Big]$$

Gauge fixing

We need a triangular $K_+^{(L)}(\lambda|\beta-1)$ matrix. If we set

$$(\alpha-\beta+2)\eta=-\tau_+ + (-1)^k(\alpha_+-\beta_+)+i\pi k,$$

then $K_+^{(L)}(\lambda|\beta-1)_{12}=0$ and

$$\det_q K_+(\lambda-\eta/2)=\sinh(2\lambda+\eta)\mathsf{a}_+(\lambda|\beta-1)\mathsf{a}_+(-\lambda+\eta|\beta-1)$$

We introduce

$$\mathbf{A}(\lambda)\equiv \mathsf{a}_+(\lambda|\beta-1)\mathsf{A}_-(\lambda)=(-1)^N\frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda}g_+(\lambda)g_-(\lambda)a(\lambda)d(-\lambda)$$

Quantum determinant identity:

$$\frac{\det_q K_+(\lambda-\eta/2)\det_q \mathcal{U}_-(\lambda-\eta/2)}{\sinh(2\lambda+\eta)\sinh(2\lambda-\eta)}=\mathbf{A}(\lambda)\mathbf{A}(-\lambda+\eta).$$

Reference state

Gauge transformation deform the bulk reference states. We define the following left reference state for the **bulk** operators:

$$\langle\beta| \equiv \otimes_{n=1}^{N} \left(-1, e^{-\alpha\eta+(N-n+\beta)\eta-\xi_n}\right)_{(n)} = N_m \langle 0| \prod_{n=1}^{N} \bar{G}_n^{-1}(\xi_n|\beta+N-n),$$

$$\begin{aligned}
\langle\beta|B(\lambda|\beta) &= \langle\beta|\bar{B}(\lambda|\beta) = 0, \\
\langle\beta|A(\lambda|\beta) &= \frac{\sinh(N+\beta)\eta}{\sinh\beta\eta} \prod_{n=1}^{N} \sinh(\lambda-\xi_n)\langle\beta-1| \\
\langle\beta|D(\lambda|\beta) &= \prod_{n=1}^{N} \sinh(\lambda-\xi_n-\eta)\langle\beta+1|
\end{aligned}$$

Pseudo-eigenstates of $\mathcal{B}$

Left $\mathcal{B}_-(\lambda|\beta)$ SOV-basis the states

$$\langle\beta, h_1, \ldots, h_N| = \langle\beta| \prod_{n=1}^{N} \left(\frac{\mathcal{A}_-(-\xi_n|\beta+2)}{\mathsf{A}_-(-\xi_n)} \right)^{h_n}, \quad h_j = 0, 1$$

form a basis and are pseudo-eigenstates of $\mathcal{B}_-(\lambda|\beta)$:

$$\langle\beta, \mathbf{h}|\mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\mathbf{h}}(\lambda|\beta)\langle\beta-2, \mathbf{h}|,$$

$$\mathsf{B}_{\mathbf{h}}(\lambda|\beta) = (-1)^N e^{(\beta+N)\eta} a_{\mathbf{h}}(\lambda) a_{\mathbf{h}}(-\lambda) \\ \times \frac{\sinh(2\lambda-\eta)\left(2\kappa_- \sinh\left[(N+\beta-\alpha-1)\eta-\tau_-\right] - e^{\zeta_-}\right)}{2\sinh\zeta_- \sinh(N+\beta)\eta}.$$

In a similar way (with operators $\mathcal{D}_-$) we construct the right pseudo-eigenstates $|\beta, \mathbf{h}\rangle$

Action of the operators $\mathcal{A}_-$

We define the raising (lowing operators)

$$\langle \beta, h_1, \ldots, h_a, \ldots, h_N | T_a^{\pm} = \langle \beta, h_1, \ldots, h_a \pm 1, \ldots, h_N |.$$

and

$$\langle \beta, h_1, \ldots, h_a = 1, \ldots, h_N | T_a^{+} = \langle \beta, h_1, \ldots, h_a = 0, \ldots, h_N | T_a^{-} = 0$$

Interpolation formula for the action of the operators $\mathcal{A}_-$ in the SOV basis

$$\langle \beta, \mathbf{h} | \mathcal{A}_-(\lambda|\beta+2) = f^0(\lambda)\langle \beta, \mathbf{h}| + \sum_{a=1}^{N} f_a^-(\lambda)\langle \beta, \mathbf{h}|T_a^- + \sum_{a=1}^{N} f_a^+(\lambda)\langle \beta, \mathbf{h}|T_a^+$$

Orthogonality

$$\langle \beta - 2, \mathbf{h} | \beta, \mathbf{k} \rangle = \delta_{\mathbf{h},\mathbf{k}} Z(\beta - 2) \mathcal{N}^{-1}(\mathbf{h})$$

with Sklyanin mesure

$$\mathcal{N}(\mathbf{h}) = \prod_{1 \le b < a \le N} \Big(\cosh 2\,(\xi_a + h_a \eta) - \cosh 2\,(\xi_b + h_b \eta) \Big)$$

and

$$Z(\beta) = \mathcal{N}(1, \ldots 1) \langle \beta | \left(\prod_{n=1}^{N} \frac{\mathcal{A}_-(-\xi_n | \beta + 2)}{\mathsf{A}_-(-\xi_n)} \right) | -\beta \rangle$$

Transfer matrix spectrum

1. Preliminaries

Proposition: Transfer matrix is a polynomial of degree $N+1$ of $\cosh(2\lambda)$.
It means that it is sufficient to fix it in $N+2$ points: N points ξ_a and two special points where transfer matrix simplifies: $\frac{\eta}{2}$ and $\frac{\eta-i\pi}{2}$.

$$\mathcal{T}(\eta/2) = 2\cosh\eta\,\mathrm{det}_q\,M(0),$$
$$\mathcal{T}(\eta/2 - i\pi/2) = -2\cosh\eta\coth\zeta_-\coth\zeta_+\,\mathrm{det}_q\,M(i\pi/2).$$

Interpolation formula:

$$\tau(\lambda) = f(\lambda) + \sum_{a=1}^{N} g_a(\lambda) x_a$$

here $x_a \equiv \tau(\xi_a)$

$$f(\lambda) = \frac{(\cosh 2\lambda + \cosh\eta)}{2\cosh\eta} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\xi_b}{\cosh\eta - \cosh 2\xi_b} \tau(\eta/2)$$

$$- (-1)^N \frac{(\cosh 2\lambda - \cosh\eta)}{2\cosh\eta} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\xi_b}{\cosh\eta + \cosh 2\xi_b} \tau(\eta/2 + i\pi/2),$$

and

$$g_a(\lambda) = \frac{\cosh^2 2\lambda - \cosh^2\eta}{\cosh^2 2\xi_a - \cosh^2\eta} \prod_{\substack{b=1 \\ b\neq a}}^{N} \frac{\cosh 2\lambda - \cosh 2\xi_b}{\cosh 2\xi_a - \cosh 2\xi_b} \qquad \text{for } a \in \{1, \ldots, \mathsf{N}\},$$

It remains to determine x_a and **construct the eigenstates**. We use the SOV basis

Transfer matrix spectrum. SOV approach

For any eigenstate $|\tau\rangle$ of the transfer matrix we consider the **wave function**

$$\Psi_\tau(\mathbf{h}) = \langle \beta - 2, \mathbf{h}|\tau\rangle$$

Due to the properties of the SOV basis the spectral problem for $\mathcal{T}(\lambda)$ is reduced to the following discrete system of 2^N Baxter-like equations:

$$\tau(\xi_n + h_n\eta)\Psi_\tau(\mathbf{h}) = \mathbf{A}(\xi_n + h_n\eta)\Psi_\tau(\mathsf{T}_n^-(\mathbf{h})) + \mathbf{A}(-\xi_n - h_n\eta)\Psi_\tau(\mathsf{T}_n^+(\mathbf{h})),$$

here

$$\mathsf{T}_n^\pm(\mathbf{h}) = (h_1, \dots, h_n \pm 1, \dots, h_N).$$

It can be written in a matrix form

$$\begin{pmatrix} \tau(\xi_n) & -\mathbf{A}(-\xi_n) \\ -\mathbf{A}(\xi_n + \eta) & \tau(\xi_n + \eta) \end{pmatrix} \begin{pmatrix} \Psi_{\tau-}(h_1, \dots, h_n = 0, \dots, h_1) \\ \Psi_{\tau-}(h_1, \dots, h_n = 1, \dots, h_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then (up to an overall normalization) the solution for the wave function is unique, and can be computed from the x_a:

$$\frac{\Psi_\tau(h_1, \ldots, h_n = 1, \ldots, h_N)}{\Psi_\tau(h_1, \ldots, h_n = 0, \ldots, h_N)} = \frac{\tau(\xi_n)}{\mathbf{A}(-\xi_n)}$$

This equation means that the wave function is factorized, leading to the following expression for the eigenstates

$$|\tau\rangle = \sum_{h_1,\ldots,h_N=0}^{1} \mathcal{N}(\mathbf{h}) \prod_{a=1}^{N} Q_\tau(\xi_a + h_a\eta)\, |\beta, h_1, \ldots, h_N\rangle,$$

The coefficients are characterized by a Baxter-like equation (note that $\mathbf{A}(\xi_a) = 0$):

$$\tau(\xi_a)Q_\tau(\xi_a) = \mathbf{A}(-\xi_a)Q_\tau(\xi_a + \eta) + \mathbf{A}(\xi_a)Q_\tau(\xi_a - \eta).$$

The last step: how to compute $x_a = \tau(\xi_a)$

We established that

- Every eigenstate of $\mathcal{T}(\lambda)$ is written in the factorized form in terms of the x_a
- Corresponding eigenvalue is given by the interpolation formula in terms of x_a

Theorem: (G. Niccoli, 2013): $\tau(\lambda)$ given by the interpolation formula is an eigenvalue of the transfer matrix if and only if it satisfies the q-determinant identities

$$\tau(\xi_a)\tau(\xi_a+\eta) = \mathbf{A}(\xi_a+\eta)\mathbf{A}(-\xi_a), \quad \forall a \in \{1,...,N\}$$

It means that each eigenstate is characterized by the solutions of the following system of quadratic equations which replace the Bethe equations here

$$x_n \sum_{a=1}^{N} g_a(\xi_n+\eta)x_a + x_n f(\xi_n+\eta) = q_n,$$

$$q_n = \frac{\det_q K_+(\xi_n)\det_q \mathcal{U}_-(\xi_n)}{\sinh(2\xi_n+2\eta)\sinh 2\xi_n}, \quad n = 1,\dots N$$

T-Q relation

In general $Q_\tau(\xi_a + h_a\eta)$ are values of the Baxter Q operator satisfying T-Q **relation**.
Related question: are there **Bethe equation**?
Lemma Let boundary parameters be generic (Nepomechie's constraints are not satisfied).

$$\kappa_+ \neq 0, \kappa_- \neq 0, \quad Y^{(i,r)}(\tau_\pm, \alpha_\pm, \beta_\pm) \neq 0 \ \ \forall i \in \{0,1\}, r = 0, \ldots N$$

where

$$Y^{(i,r)}(\tau_\pm, \alpha_\pm, \beta_\pm) \equiv \tau_- - \tau_+ + (-1)^i \left[(N-1-r)\,\eta + (\alpha_- + \alpha_+ + \beta_- - \beta_+)\right]$$

Then for any eigenstate, the homogeneous Baxter equation

$$\tau(\lambda)Q(\lambda) = \mathbf{A}(\lambda)Q(\lambda-\eta) + \mathbf{A}(-\lambda)Q(\lambda+\eta),$$

has no non-trivial polynomial solution.

It's sufficient to compare leading behaviour at $\lambda \to \infty$

Inhomogeneous T-Q relation

We define:

$$F(\lambda) = F_0\,(\cosh^2 2\lambda - \cosh^2\eta)\prod_{b=1}^{N}(\cosh 2\lambda - \cosh 2\xi_b)(\cosh 2\lambda - \cosh 2(\xi_b+\eta))$$

with the obstacle term for the Baxter equation

$$F_0 = \frac{2\kappa_+\kappa_-\,(\cosh(\tau_+ - \tau_-) - \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- - (N+1)\eta))}{\sinh\zeta_+\sinh\zeta_-},$$

Theorem: Let the boundary parameters be generic. Then $\tau(\lambda)$ is an eigenvalue of the transfer matrix if and only if there is the unique polynomial solution $Q(\lambda)$ of the **inhomogeneous Baxter equation**

$$\tau(\lambda)Q(\lambda) = \mathbf{A}(\lambda)Q(\lambda-\eta) + \mathbf{A}(-\lambda)Q(\lambda+\eta) + F(\lambda).$$

$Q(\lambda)$ is a polynomial of degree N of $\cosh(2\lambda)$. Solving the corresponding Bethe equations for the roots of Q we obtain the **complete set of eigenstates!**.

Constrained case

1. Let the boundary parameters satisfy the following constraint:

$$\kappa_+ \neq 0, \kappa_- \neq 0, \quad \exists i \in \{0,1\} \quad : Y^{(i,2N)}(\tau_\pm, \alpha_\pm, \beta_\pm) = 0$$

Then $\tau(\lambda)$ is an eigenvalue of the transfer matrix if and only if there is the unique (up to overall normalization) polynomial solution $Q(\lambda)$ of the **homogeneous Baxter equation**

$$\tau(\lambda)Q(\lambda) = \mathbf{A}(\lambda)Q(\lambda - \eta) + \mathbf{A}(-\lambda)Q(\lambda + \eta),$$

$Q(\lambda)$ is a polynomial of degree N of $\cosh(2\lambda)$. Solving the corresponding Bethe equations for the roots of Q we obtain again the complete set of eigenstates.
2. More general constraint (for any integer $M = 0, \ldots N - 1$):

$$\kappa_+ \neq 0, \kappa_- \neq 0, \quad \exists i \in \{0,1\}, M \in \{0, ..., N-1\} : Y^{(i,2M)}(\tau_\pm, \alpha_\pm, \beta_\pm) = 0,$$

Then there are two sectors: one with **homogeneous** Baxter equations and $Q(\lambda)$ polynomial of degree M of $\cosh(2\lambda)$ and the second one with **inhomogeneous** Baxter equation and $Q(\lambda)$ polynomial of degree N.

Discrete symmetries

Boundary parameters $\alpha_\pm,\ \beta_\pm,\ \tau_\pm$

Proposition: Discrete transformations

$$\tau_+,\ \alpha_+,\ \beta_+,\ \tau_-,\ \alpha_-,\ \beta_- \longrightarrow \epsilon_\tau\tau_+,\ \epsilon_\alpha\alpha_+,\ \epsilon_\beta\beta_+,\ \epsilon_\tau\tau_-,\ \epsilon_\alpha\alpha_-,\ \epsilon_\beta\beta_-$$

with $\epsilon_\alpha, \epsilon_\beta, \epsilon_\tau = \pm 1$ don't change the transfer matrix spectrum (while they change, Hamiltonian, the eigenstates and the T-Q equation)

It means that the same eigenvalues can be written in terms of different Bethe roots, moreover this transformation can lead from inhomogeneous Baxter equation to the usual one and vice-versa.

Example: if $Y^{(i,2M)}(\tau_\pm, \alpha_\pm, \beta_\pm) = 0$ then $Y^{(i,2(N-M-1))}(-\tau_\pm, -\alpha_\pm, -\beta_\pm) = 0$.

Conjecture: Sector described by the **inhomogeneous** Baxter equation before transformation is the sector described by the **homogeneous** Baxter equation after the transformation (based on the numerical analysis of Nepomechie, Ravanini).

Scalar products

The scalar product of any two states constructed by the separation of variables:

$$\langle\omega| = \sum_{h_1,\dots,h_\mathsf{N}=0}^{1} \prod_{a=1}^{\mathsf{N}} \omega(\xi_a + h_a\eta)\mathcal{N}(\mathbf{h})\langle\beta, h_1, \dots, h_\mathsf{N}|,$$

$$|\rho\rangle = \sum_{h_1,\dots,h_\mathsf{N}=0}^{1} \prod_{a=1}^{\mathsf{N}} \rho(\zeta_a^{(h_a)})\mathcal{N}(\mathbf{h})|\beta+2, h_1, \dots, h_\mathsf{N}\rangle,$$

Then

$$\langle\omega|\rho\rangle = Z(\beta-2)\,\det\nolimits_\mathsf{N} \mathcal{M}_{a,b}^{(\omega,\rho)}$$

$$\mathcal{M}_{a,b}^{(\omega,\rho)} = \sum_{h=0}^{1} \omega(\xi_a + h_a\eta)\rho(\xi_a + h_a\eta)\Big(\cosh 2\,(\xi_a + h_a\eta)\Big)^{(b-1)}.$$

Typical SOV result, now we know it can be rewritten as Izergin (Tsuchiya) determinants

Scalar products. Toy example

XXX chain with anti-periodic boundary conditions $\mathcal{T}(\lambda) = B(\lambda) + C(\lambda)$

SOV-states

$$\langle\alpha| = \sum_{h_1=0}^{1} \cdots \sum_{h_N=0}^{1} \prod_{a=1}^{N} \alpha(\xi_a - h_a\eta)\, V(\{\xi - h\eta\})\langle h_1, \dots, h_N|,$$

$$|\beta\rangle = \sum_{h_1=0}^{1} \cdots \sum_{h_N=0}^{1} \prod_{a=1}^{N} \bar{\beta}(\xi_a - h_a\eta)\, V(\{\xi - h\eta\})|h_1, \dots, h_N\rangle,$$

Here $V(\{\xi\})$ is the Vandermonde determinant. $\alpha(\lambda)$ and $\beta(\lambda)$ arbitrary functions

Eigenstates $\langle Q|$ is an eigenstate if Baxter equation is satisfied

$$\tau(\lambda)Q(\lambda) = -a(\lambda)Q(\lambda - \eta) + d(\lambda)Q(\lambda + \eta)$$

Scalar products. Toy example

Scalar product:

$$\langle \alpha | \beta \rangle = \frac{\det_N \mathcal{M}^{(\alpha,\beta)}}{V(\{\xi\})},$$

with

$$\mathcal{M}^{(\alpha,\beta)}_{a,b} = \xi_a^{b-1}\alpha(\xi_a)\bar{\beta}(\xi_a) + (\xi_a - \eta)^{b-1}\alpha(\xi_a - \eta)\bar{\beta}(\xi_a - \eta).$$

Main results: Let $\alpha(\lambda) = \prod_{j=1}^{M}(\lambda - \alpha_j)$ and $Q(\lambda) = \prod_{j=1}^{R}(\lambda - \lambda_j)$ then

- if $M < R$

$$\langle \alpha \,| Q \rangle = 0.$$

- If $M = R$ then the Slavnov formula can be applied:

$$\langle \alpha \,| Q \rangle = 2^{N-2M} \left(\prod_{n=1}^{M} d(\alpha_n) d(\lambda_n) \right) \mathcal{S}_M(\{\lambda\}, \{\alpha\}).$$

- If $M > R$ then we obtain the generalized Slavnov formula

$$\langle \alpha \,| Q \rangle = (-1)^{M+R} 2^{N-M-R} \left(\prod_{n=1}^{M} d(\alpha_n) \prod_{k=1}^{R} d(\lambda_k) \right) \mathcal{S}_{R,M}(\{\lambda\}, \{\alpha\}).$$

$$\mathcal{S}_{M,M+S}(\{\lambda\}, \{\alpha\}) = \frac{\det_{M+S} \mathcal{H}}{V(\lambda_1, \ldots, \lambda_M) V(\alpha_{M+S}, \ldots, \alpha_1)}$$

$$\mathcal{H}_{jk} = Q(\alpha_k - \eta) \frac{a(\alpha_k)}{d(\alpha_k)} t(\lambda_j - \alpha_k) + Q(\alpha_k + \eta) t(\alpha_k - \lambda_j), \quad \text{for} \quad j \leqslant M,$$

$$\mathcal{H}_{jk} = Q(\alpha_k - \eta) \frac{a(\alpha_k)}{d(\alpha_k)} \alpha_k^{j-M-1} + Q(\alpha_k + \eta) \, (\alpha_k + \eta)^{j-M-1}, \quad \text{for} \quad j > M,$$

where $t(x) = \frac{\eta}{x(x+\eta)}$

Conclusion and outlook

Main result: **Complete** characterization of the spectrum from the **inhomogeneous Bethe ansatz** + construction of the **eigenstates** and scalar products. Direct way to the form factors, overlaps, correlation functions.

Similar technique can be applied to the open **XYZ chain** (S. Faldella, G. Niccoli 2013).

Open questions:

- Connection between the **homogeneous** and **inhomogeneous** Baxter equations. Can we sacrifice polynomiality and retrieve **homogeneity**?
- Inhomogeneous Baxter equation appear in different frameworks: off-diagonal Bethe ansatz, separation of variables, modified algebraic Bethe ansatz (Belliard, Crampé). In the **classical limit**, what is the meaning of the inhomogeneous Baxter equation?
- **ASEP** dynamics