# Elliptic Bethe Ansatz for fermions on a 3-periodic chain

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## Witten, Fendley

A set of operators $Q_i$ with $1 \leq i \leq \mathcal{N}$.
Satisfying $\{Q_i, Q_j\} = H\delta_{i,j}$.

We consider the case $\mathcal{N} = 2$, and define $Q = Q_1 + \mathrm{i}Q_2$.
Then it follows

$$Q^2 = 0$$

and we take

$$H = \{Q, Q^\dagger\} \qquad [F, Q] = Q$$

interpreting $H$ as the Hamiltonian, and $F$ as a fermion number.
This minimal structure has already many consequences

- $\langle\phi|H|\phi\rangle = \|\, Q|\phi\rangle \,\|^2 + \|\, Q^\dagger|\phi\rangle \,\|^2 \geq 0$ only non-negative eigenvalues.
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- $\mathcal{H} = (Q\mathcal{H}) \oplus (Q^\dagger\mathcal{H}) \oplus \{|\phi\rangle : \; Q|\phi\rangle = Q^\dagger|\phi\rangle = 0\}$

  Proof: consider eigenstate of H

$$E_\phi \, |\phi\rangle = H|\phi\rangle \quad \Longleftrightarrow \quad E_\phi \, |\phi\rangle = Q^\dagger Q|\phi\rangle + QQ^\dagger|\phi\rangle$$

- Take $|\phi\rangle \in Q\mathcal{H}$ and $E_\phi \, |\phi\rangle = H|\phi\rangle$.
  Then $Q^\dagger|\phi\rangle \in Q^\dagger\mathcal{H}$ has the same eigenvalue.
  non-zero energy states form doublets connected by $Q$ and $Q^\dagger$.

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  independent of $\beta$. $W$ is lower bound of number of $E$=0 states.

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Fendley, Schoutens

# Explicit construction

Assume some lattice (any graph), and fermion operators $c_i$ on the sites $\{c_i^\dagger, c_j\} = \delta_{i,j}$ and $\{c_i, c_j\} = 0$.
We define the $Q$ operator as

$$Q = \sum_j c_j^\dagger p_j \qquad \text{where} \qquad p_j = \prod_{k \smile j} (1 - c_k^\dagger c_k)$$

$p_j$ projects on the conditions that all neighbors of $j$ are empty.

Clearly $Q^2 = 0$, because $\{c_k^\dagger p_k, c_j^\dagger p_j\} = 0$.

$$H = \{Q^\dagger, Q\} = \sum_{i \smile j} p_i \; c_i^\dagger c_j \; p_j \; + \; \sum_i p_i$$

Fairly realistic model for interacting, itinerant fermions:
Hopping, excluded neighbors, attracted second neighbors.

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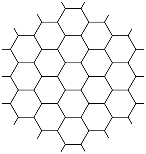
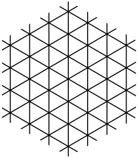
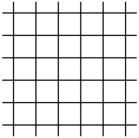






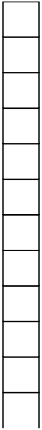
This construction is possible on any lattice.
Dimension, regular, irregular, $\cdots$

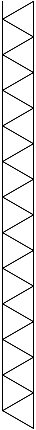
Ground state degeneracy exponential (generically)

Natural way to represent dimers

Quantum criticality

Integrability





Here we focus on the simplest case: the chain

$$H = \sum_x p_x \; c_x^\dagger c_{x+1} \; p_{x+1} + p_{x+1} \; c_{x+1}^\dagger c_x \; p_x + p_x$$

Number of zero-energy states:

| $L \bmod 3$ | open | periodic |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 0 | 1 |
| 2 | 1 | 1 |

In the sector with $\sim L/3$ fermions.

This model can be mapped to a XXZ chain:

with appropriate Jordan-Wigner factors.

$$\mathcal{H} = -\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z\right)$$

at the *combinatoric* point: $\Delta = -1/2$.

The XXZ model is integrable, and so is this fermion chain.

Many observables are surprisingly simple:
(Periodic, for $L = 3f$)

$$\langle n_x \; n_{x+2} \rangle = \frac{f^2 - 1}{3(4f^2 - 1)}$$

$$\langle n_x \; n_{x+3} \rangle = \frac{(4 - f^2 + 45f^4)}{16(4f^2 - 1)^2}$$

$$\langle n_x \; n_{x+4} \rangle = \frac{(-864 + 660f^2 - 2903f^4 + 1223f^6)(f^2 - 1)}{64(4f^2 - 1)^3(4f^2 - 9)}$$

Emptiness formation: $\langle p_x \; p_{x+1} \; \dots \; p_{x+m-1} \rangle =$

$$\prod_{k=1}^{m-1} \frac{k! \, (3k+1)! \, (2f-k)! \, (f+k)!}{(2k)! \, (2k+1)! \, (3f+k+1)! \, (f-k-1)!}$$

and a similar expression for $\langle n_x \; n_{x+2} \; n_{x+4} \cdots \; n_{x+2(m-1)} \rangle$

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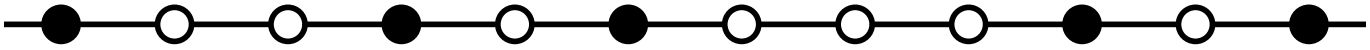
Most notable, when $L = 3f$, the ground state sector can break translational symmetry with period 3 and amplitude:

$$\langle n_{3x} - n_{3x+1} \rangle \ \propto \ \prod_{j=1}^{f} \frac{(3j-2)}{(3j-1)} \ \approx \ L^{-1/3}$$

What happens if this periodicity is enhanced with a periodic potential?



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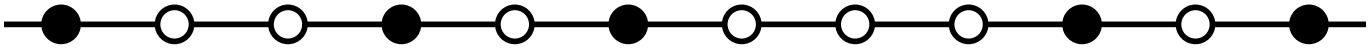

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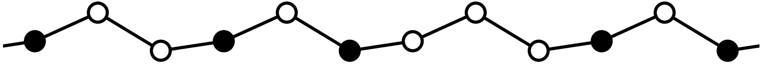

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First order (quantum) phase transition, sharp in a finite system.

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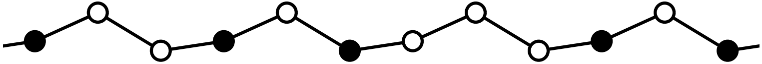

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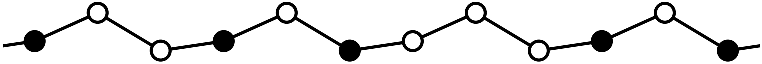

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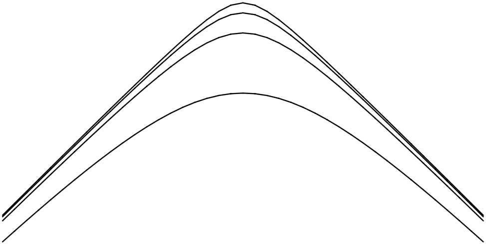
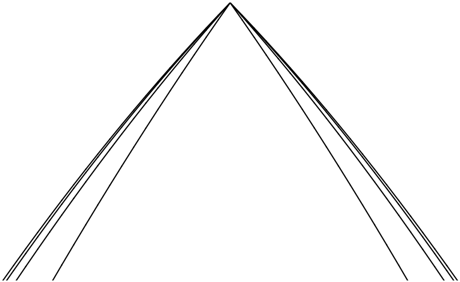


The groundstate energy per site near a typical first order transition. (for a sequence of system sizes)



The groundstate energy per site for this fermion chain in a 3-periodic potential.

## Fendley, Hagendorf

But it is also possible to break the translation invariance without breaking the SuperSymmetry: Take

$$Q = \sum_x \lambda_{x \bmod 3} \; c_x^\dagger \, p_x$$

The derivation of SUSY goes through unaffected by the spatial variation of $\lambda_x$.

Fendley and Hagendorf made many interesting observations about this model.

Observables for finite $L$ are polynomial in the parameters $\lambda_x$ and sometimes can be guessed from finite $L$ results.

But is the model still integrable when $\lambda_x$ varies with $x$?

It is tempting to transform it to a spin model.

But the interactions depend on the position, and after the transformation your position depends on the number of up-spins to your left.

The corresponding spin model has complicated multispin interactions

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Construction of an eigenstate of $H$ using coordinate Bethe Ansatz.

$$H = \sum_{x=1}^{L} \lambda_x \lambda_{x+1} (p_x \; c_x^\dagger c_{x+1} \; p_{x+1} + p_{x+1} \; c_{x+1}^\dagger c_x \; p_x) + \lambda_x^2 p_x$$

General principle:

1. Particles behave as plane waves as long as they are distant
2. 2 particles colliding can only exchange momenta
3. M-particle collisions can be described as a sequence of 2-particle collisions

These result in the following ansatz for the eigenstates of $H$:

$$\langle x_1, x_2, \cdots, x_f | \psi \rangle = \sum_{\pi} C_\pi \prod_j z_{\pi_j}^{x_j}$$

The plane wave assumption has to be modified:

$$z^x \;\rightarrow\; A_{x \bmod 3} \; z^x$$

# A Bethe Ansatz detector

Numerical approach:

- Construct an eigenstate numerically.
- Investigate if it is of the B.A. form

Generic B.A. form:

$$\psi(x_1, x_2, \cdots) = \sum_{\text{permutations } p} \sum_{\text{complications } \gamma} A_{p,\gamma} \prod_{j=1}^{N} z_{p_j}^{x_j}$$

(quasi-)excitations living at $x_j$ can have internal structure, there can be nesting with further sets of B.A. variables besides the $z_p$.

**The question:**
Suppose you have the LHS numerically,
can you verify if it can be written as the RHS?

The problem is typically ill posed (many more unknowns than equations) unless $L$ big enough and $N$ small enough.

Consider $\psi(x_1, x_2, x_3, \cdots)$ with $x_2 \leq x_3 \leq \cdots$ all fixed as large as possible. Then $x_1$ (of one given type) is free to play in a large field.


Consider this as a function of $x_1$. If this has BA form then this

$$\phi_x \equiv \psi(x, x_2, x_3, \cdots) = \sum_{k=1}^{N} B_k z_k^x$$

for some $N$. (the sum on complications has been performed, and the sum on permutations, with the restriction $p_1 = k$.)

i.e. this sequence of elements of the state vector can be written as a linear combination of geometric series.

It is useless to try to solve the non-linear equations for $z_k$ and $B_k$.

**But**

Consider the $n \times n$ matrix $\phi_{i+j}$:

$$\Delta_n = \det \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots \\ \phi_2 & \phi_3 & \phi_4 & \cdots \\ \phi_3 & \phi_4 & \phi_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if $\psi$ Bethe state,
all rows are linear combination of series $\{z_k^1, z_k^2, \cdots, z_k^N\}$,

$$\Longrightarrow \text{determinant vanishes if matrix } n > N$$

$\Delta_n$ suddenly drops to zero (in machine precision) as $n > N$.

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Consider now the $(N+1)\times(N+1)$ determinant

$$\Delta(z) = \det \begin{pmatrix} 1 & z & z^2 & z^3 & \cdots \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots \\ \phi_2 & \phi_3 & \phi_4 & \cdots & \\ \phi_3 & \phi_4 & \phi_5 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

It can be expanded in powers of $z$:

$$\Delta(z) = \sum_{k=0}^{N} D_k (-z)^k$$

where $D_k$ is the determinant of the $N\times(N+1)$ matrix $\phi_{i,j}$ with $k+1$-th column omitted.
Since $\Delta(z_j)$ vanishes, the $z_j$ are the roots of the equation $\Delta(z) = 0$.

Conclusion:

Given an eigenstate, one can decide numerically if it has the form of a B.A. state with a given number of momenta

One can determine the BA momenta involved

requirements:
relatively dilute and large system.
high precision computations

This method confirmed to us that the SUSY fermion chain is solvable for any $\lambda_{x \bmod 3}$.

And gave us confidence to find the B.A. solution

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This method confirmed to us that the SUSY fermion chain is solvable for any $\lambda_{x \bmod 3}$.

And gave us confidence to find the B.A. solution

Conclusion:

Given an eigenstate, one can decide numerically if it has the form of a B.A. state with a given number of momenta

One can determine the BA momenta involved

requirements:

relatively dilute and large system.

high precision computations

This method confirmed to us that the SUSY fermion chain is solvable for any $\lambda_{x \bmod 3}$.

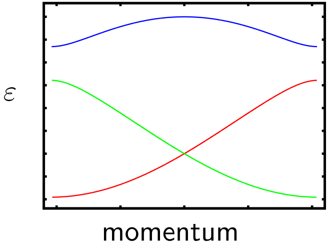
And gave us confidence to find the B.A. solution

We normalize the $\lambda_x$ by $\lambda_1^2+\lambda_2^2+\lambda_3^2=3$, and introduce $E=L+\varepsilon$, then the eigenvalue equation for one particle reads

$$(\varepsilon+3)A_x=\lambda_x\sum_{y=x-1}^{x+1}\lambda_y A_y\, z^{y-x}$$

solved by the dispersion relation:

$$\varepsilon(\varepsilon+3)^2+\Lambda^3(1-z^3)(1-z^{-3})=0, \quad \text{with} \quad \Lambda^3=\lambda_1^2\lambda_2^2\lambda_3^2$$

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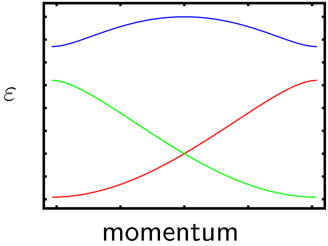


For every value of the momentum the particle can have three values of the energy.

For two particles the proposed wave function is:

$$\langle x_1, x_2|\psi\rangle = C_{12}\ A_{x_1}(z_1)z_1^{x_1}\ A_{x_2}(z_2)z_2^{x_2} + C_{21}\ A_{x_1}(z_2)z_2^{x_1}\ A_{x_2}(z_1)z_1^{x_2}$$

The eigenvalue equation for $x_2 > x_1 + 2$ leads to:

$$E = L + \varepsilon(z_1) + \varepsilon(z_2) \qquad .$$

An additional equation comes from the events that the particles are close together: $x_2 = x_1 + 2$.
This should determine the ratio $C_{21}/C_{12}$.

But this can happen at three inequivalent positions $x_1 \bmod 3$.

$\longrightarrow$ we have three equations for $C_{21}/C_{12}$.

Luckily, with the appropriate $\varepsilon(z)$ and $A_x(z)$ they agree.

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The solution:

$$S(z_1, z_2) = \frac{C_{12}}{C_{21}} = -\frac{(\varepsilon(z_1)+3)\left[\varepsilon(z_2)(1-z_1^3)+\varepsilon(z_1)+3\right]}{(\varepsilon(z_2)+3)\left[\varepsilon(z_1)(1-z_2^3)+\varepsilon(z_2)+3\right]}$$

The fact that the three equations for $S$ have a common solution, implies that the internal state of the particles is conserved during a collision.

Final test: Multiple-particle collision:

When there is a particle on site $x-2$ and $x$,
some exceptional terms in the eigenvalue equation must cancel:
A missing hop from $x-2$ to $x-1$, and from $x$ to $x-1$ as well as the contribution from $p_x$.

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Summary:

1. plane waves modulated by a 3-periodic factor
2. internal state is miraculously conserved under collision
3. M-particle collisions factorizes automatically

The equations now read:

$$E = L + \sum_{j=1}^{f} \varepsilon(z_j)$$

where $\varepsilon(z)$ solves

$$\varepsilon(z)[\varepsilon(z)+3]^2 + \Lambda^3(1-z^3)(1-z^{-3}) = 0$$

and the $z_j$ satisfy

$$z_j^L = \prod_{k=1}^{f} -S(z_j, z_k)$$

with

$$S(z,w) = -\frac{[\varepsilon(z)+3]\left[\varepsilon(w)(1-z^3)+\varepsilon(z)+3\right]}{[\varepsilon(w)+3]\left[\varepsilon(z)(1-w^3)+\varepsilon(w)+3\right]}$$

We made an attempts to resolve the multivaluedness of the dispersion relation

$$\varepsilon(z)[\varepsilon(z)+3]^2+\Lambda^3(1-z^3)(1-z^{-3})=0$$

seeking an analytic $\varepsilon(t)$ and $z(t)$ such that their relation is automatically satisfied.

What finally worked is the ansatz

$$S(z(t_1),z(t_2))=\frac{z(t_2)}{z(t_1)}\tilde{S}(t_2-t_1)$$

Because this leads to the differential equation

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)\tilde{S}(t_2-t_1)=0$$

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In this way we could derive an equation for the derivative of $\varepsilon(z(t_1))$ w.r.t. $t_1$, from which the $t_2$ dependence completely disappears.

$$\left(\frac{\partial\varepsilon}{\partial t}\right)^2 = \varepsilon\left(\varepsilon[\varepsilon+3]^2 + 4\Lambda^3\right)$$

This can be turned into a standard differential equation for an elliptic integral:

$$\dot{u}^2 = (1-u^2)(1-m^2u^2)$$

by positioning the roots of the RHS by a Möbius transformation.

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In the end the B.A. equations are almost the same as those for the XYZ model

$$\mathcal{H}_{\mathrm{XYZ}} = -\sum_i \left( J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right)$$

with

$$J_x J_y + J_x J_z + J_y J_z = 0$$

(The three parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ collapse onto this two-dimensional subspace)

$$S(t_1, t_2) = \frac{z(t_2)}{z(t_1)} S_{\mathrm{XYZ}}(t_2 - t_1)$$

## Summary

- The 3-periodic SUSY fermion chain is integrable
- The Bethe Ansatz gives an explicit expression for the wave functions (unlike in the XYZ-model)
- Numerical method to determine if a model is integrable
- Question: is this applicable for $\Delta \neq -\frac{1}{2}$?