DIMERS ON RAIL YARD GRAPHS

Sanjay Ramassamy (Brown University)

joint work with Cédric Boutillier
Jérémie Bouttier
Guillaume Chapuy
Sylvie Corteel

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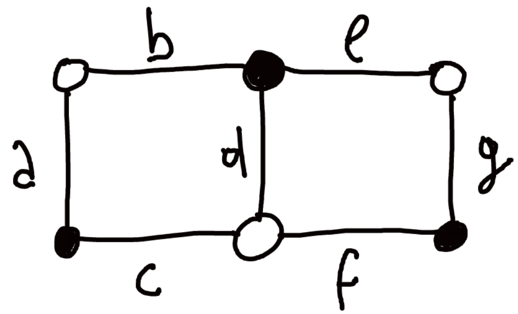
- The dimer model is a well-studied model in statistical mechanics.
- Outstanding examples include the Aztec diamond (dimers on square lattice) and plane partitions (dimers on hexagonal lattice).
- Rail yard graph setting: common generalisation of these 2 models (and several others).
- Realises a general Schur process as a dimer model, extending work of Okounkov-Reshetikhin.

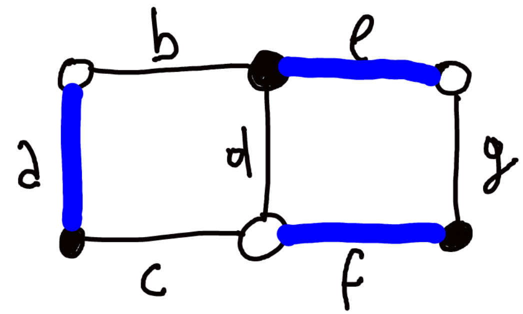
Outline:

1. The dimer model
2. Two examples: Aztec diamond and plane partitions
3. Definition of rail yard graphs
4. Main results
5. Proof ideas
6. Summary and outlook

1. The dimer model



- A dimer configuration is a subset of edges such that each vertex belongs to exactly one edge.

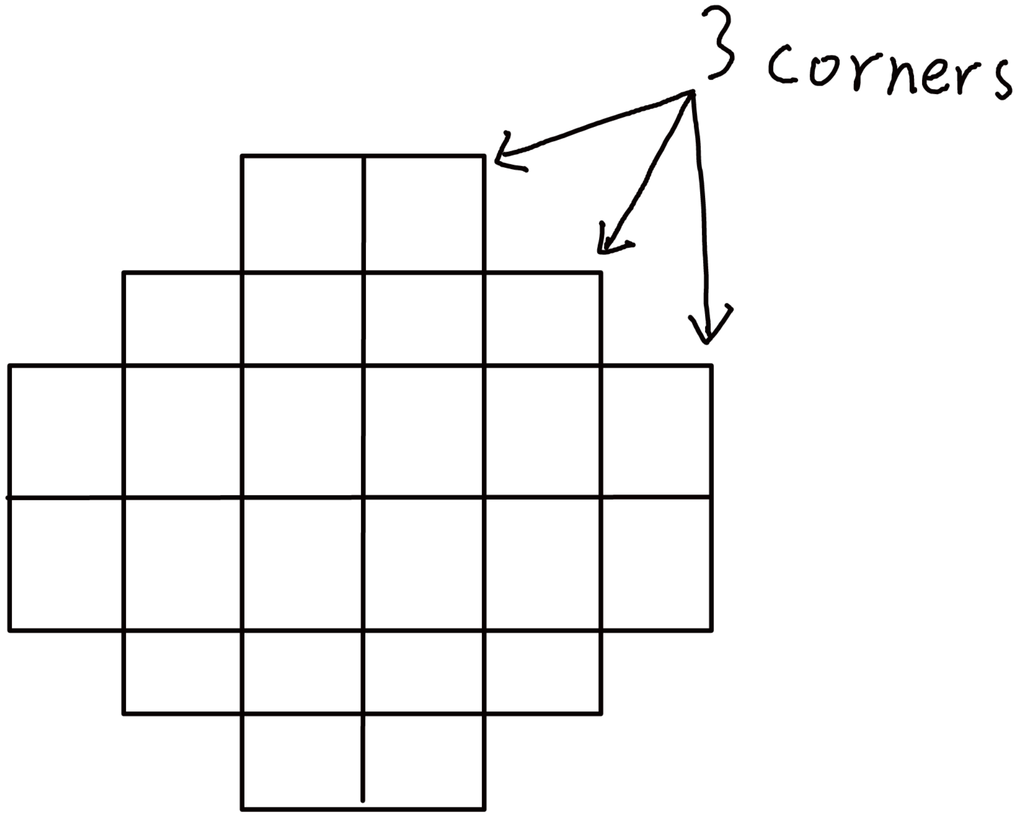


- A dimer configuration is a subset of edges such that each vertex belongs to exactly one edge.
- Weight of a config = product of the weights of the edges used (here weight = aef).
- Partition function Z = sum of the weights of all the dimer configs.

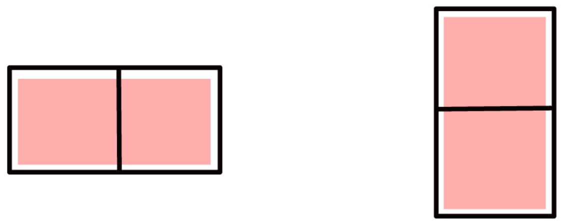
- Given a finite planar bipartite graph, one can compute a Kasteleyn matrix K, of size linear in the number of vertices.
- $Z = |\det K|$ (Kasteleyn / Temperley-Fisher 60's)
- Probabilistic model: draw a dimer config at random, with probability proportional to its weight.
- Dimer correlations have a determinantal structure, with determinantal kernel equal to K^{-1} (Kenyon '01)

2. Two examples: Aztec diamonds and plane partitions

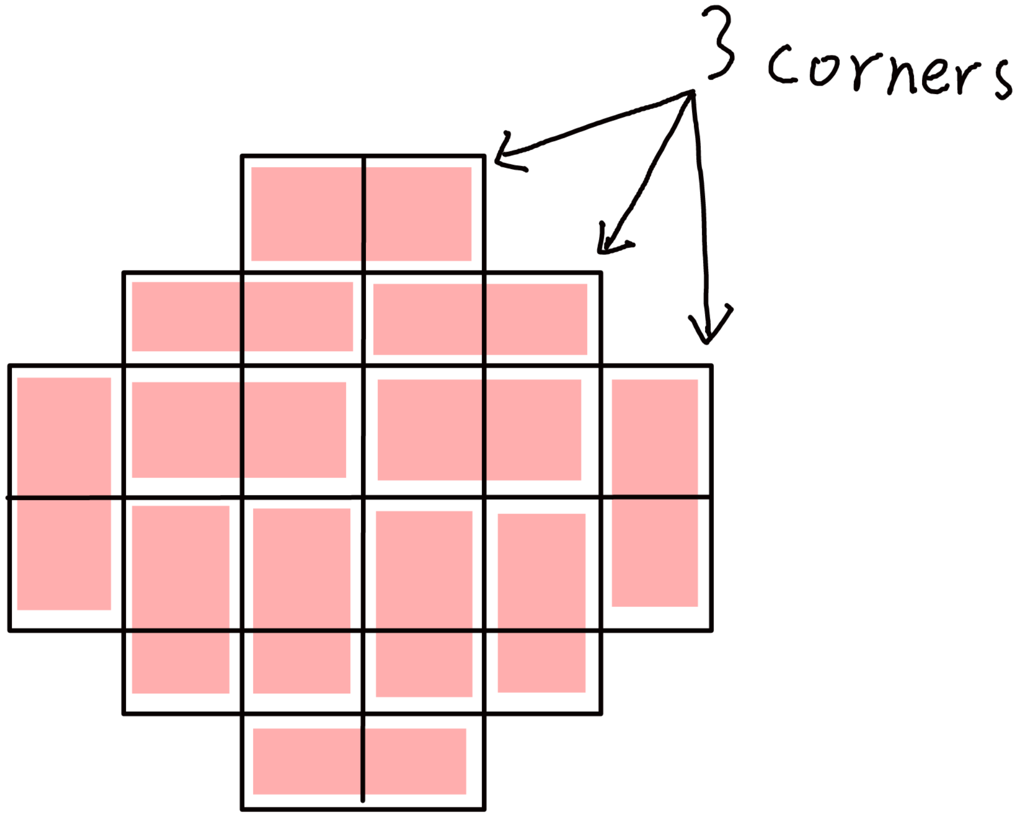
Domino tilings of the Aztec diamond



- Aztec diamond of size 3
- tiled by dominos



Domino tilings of the Aztec diamond



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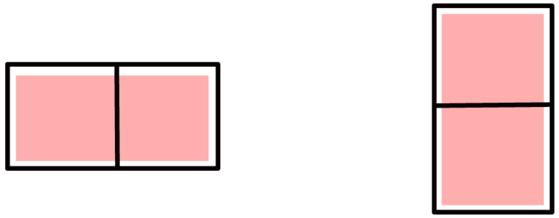
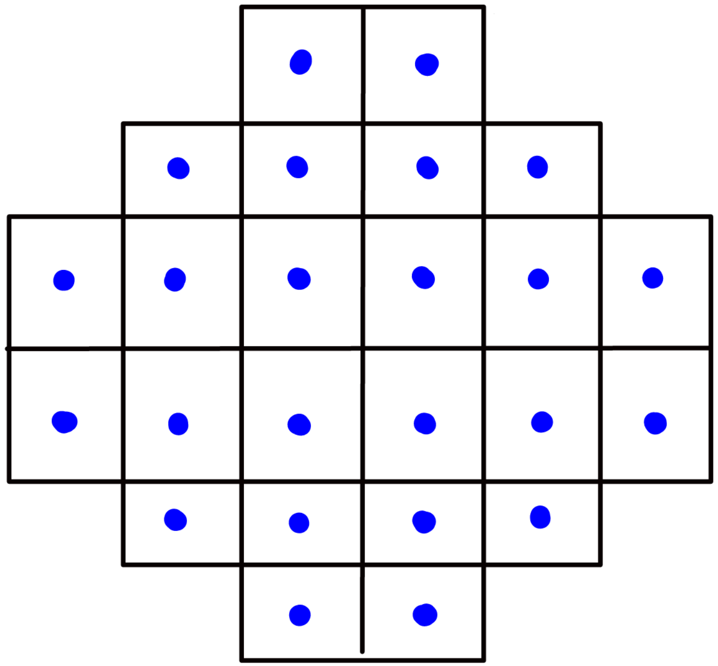
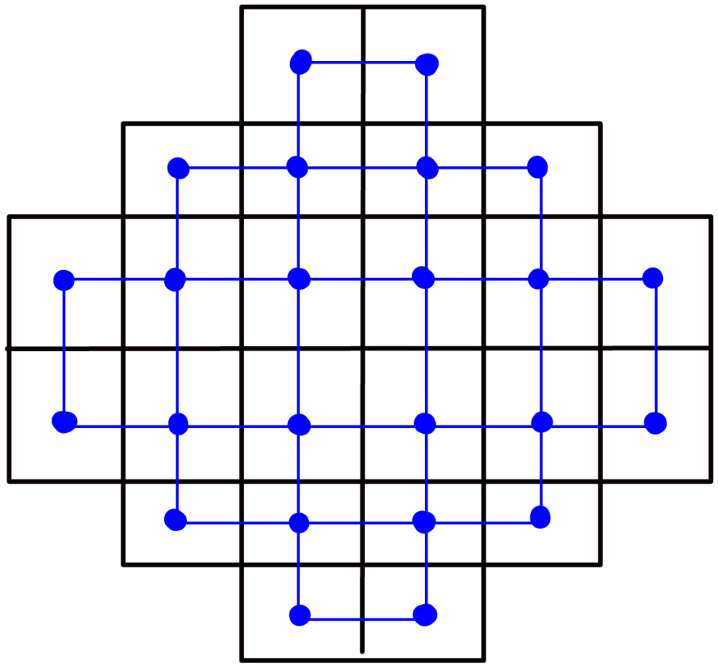


Domino tilings of the Aztec diamond

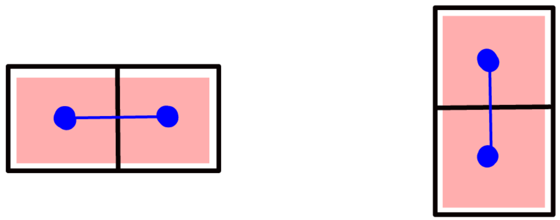


- equivalent to a dimer model on the dual graph

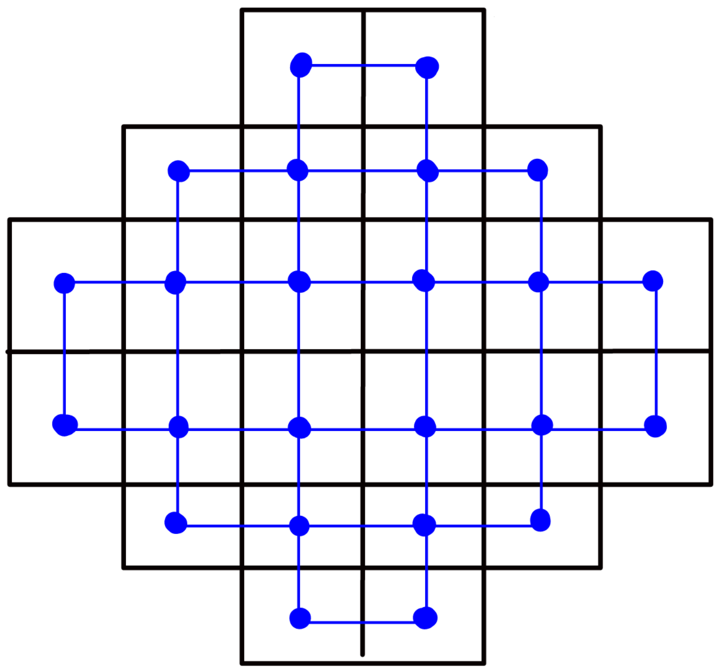
Domino tilings of the Aztec diamond



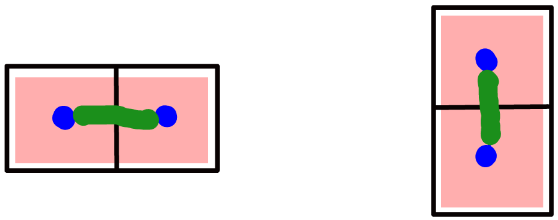
- equivalent to a dimer model on the dual graph
- domino $\longleftrightarrow$ dimer



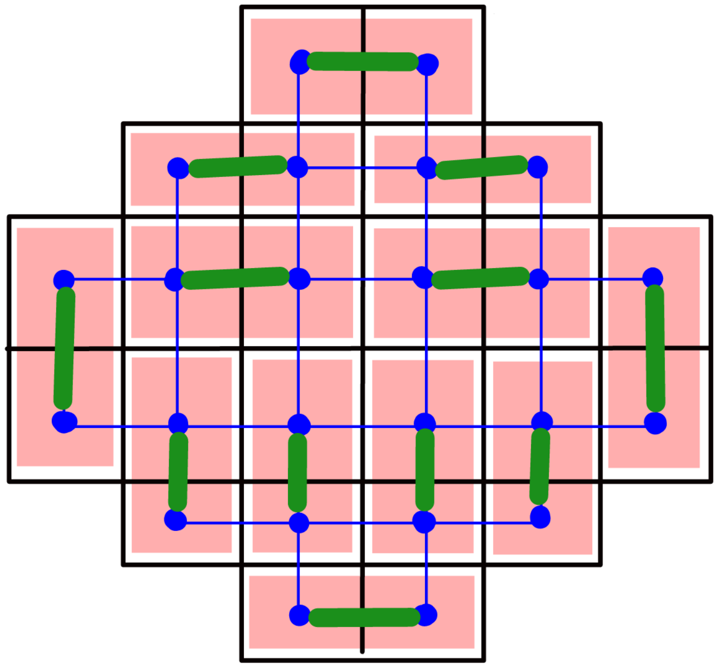
Domino tilings of the Aztec diamond



- equivalent to a dimer model on the dual graph
- domino ⟷ dimer



Domino tilings of the Aztec diamond



- equivalent to a dimer model on the dual graph

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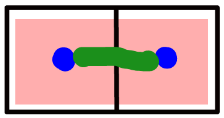
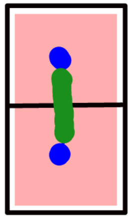
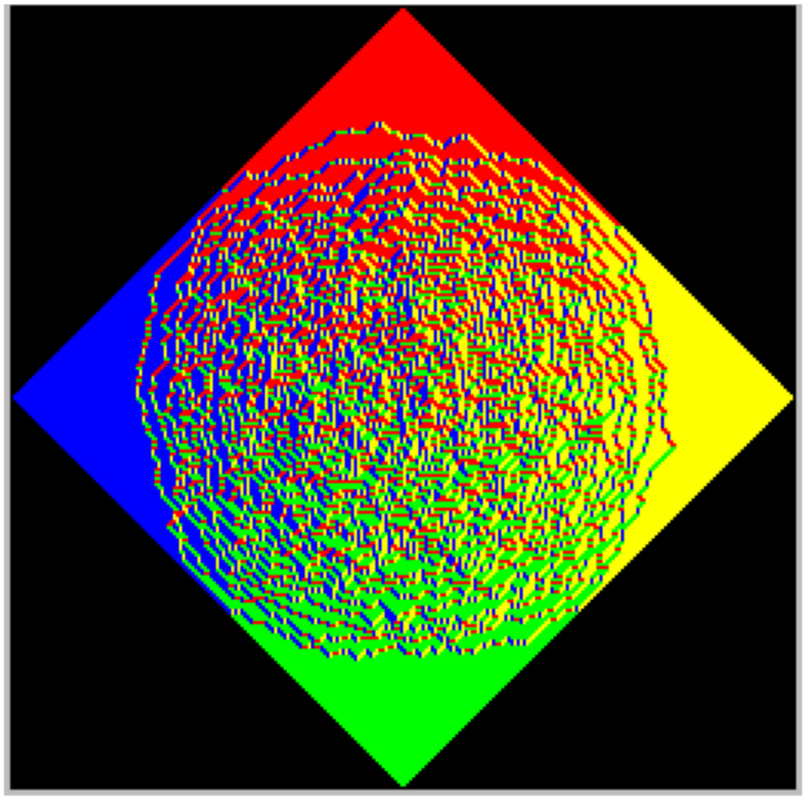


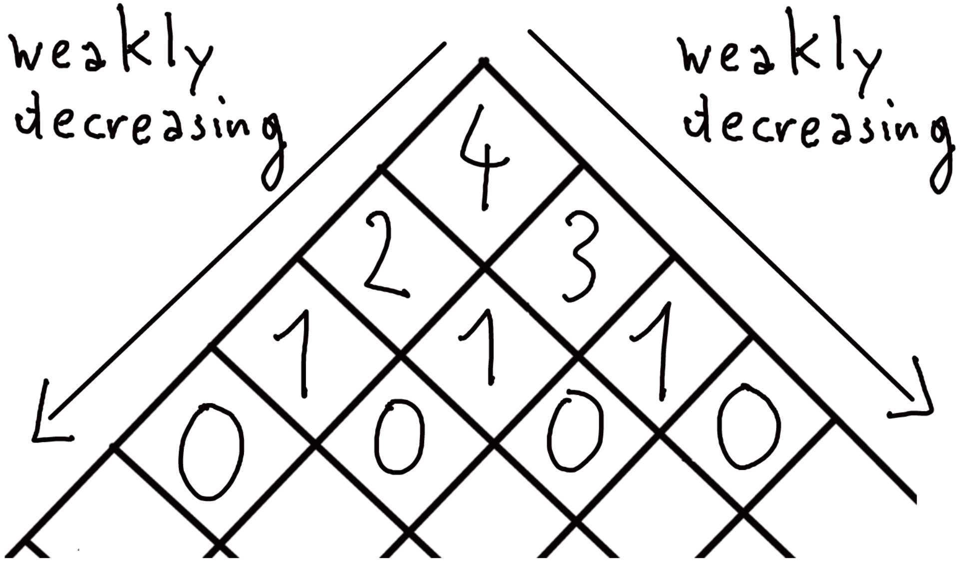


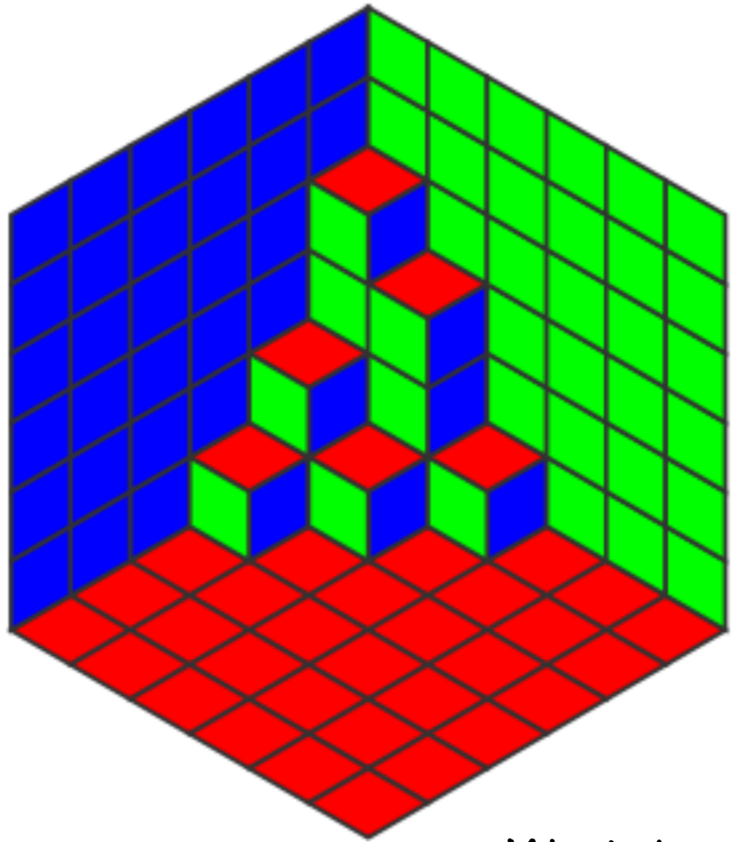
- There are $2^{\frac{n(n+1)}{2}}$ tilings of the Aztec diamond of size n (Elkies-Kuperberg-Larsen-Propp '91).
- Correlation kernel computed by Chhita-Young ('14).
- Arctic circle phenomenon (Jockusch-Propp-Shor '98):



picture by Cris Moore,
available at:
tuvalu.santafe.edu/~moore/aztec256.gif

Plane partitions

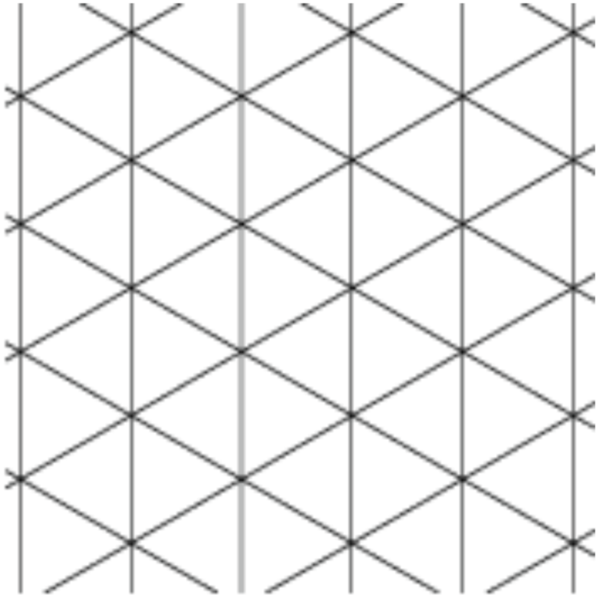


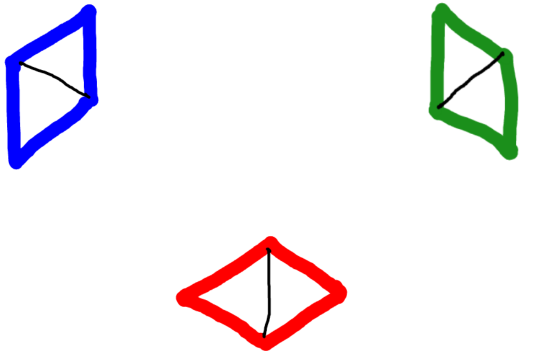


(courtesy of Sevak Mkrtchyan)

volume = sum of the numbers = number of cubes

- Stack of cubes (3D) $\longleftrightarrow$ tiling of the infinite triangular lattice (2D) by rhombi of 3 kinds:





- Corresponds to a dimer model on the infinite hexagonal lattice (dual of the triangular lattice).

- Measure considered: fix $q<1$ and weigh each plane partition π by $q^{\mathrm{volume}(\pi)}$.

- Partition function (MacMahon 1912)

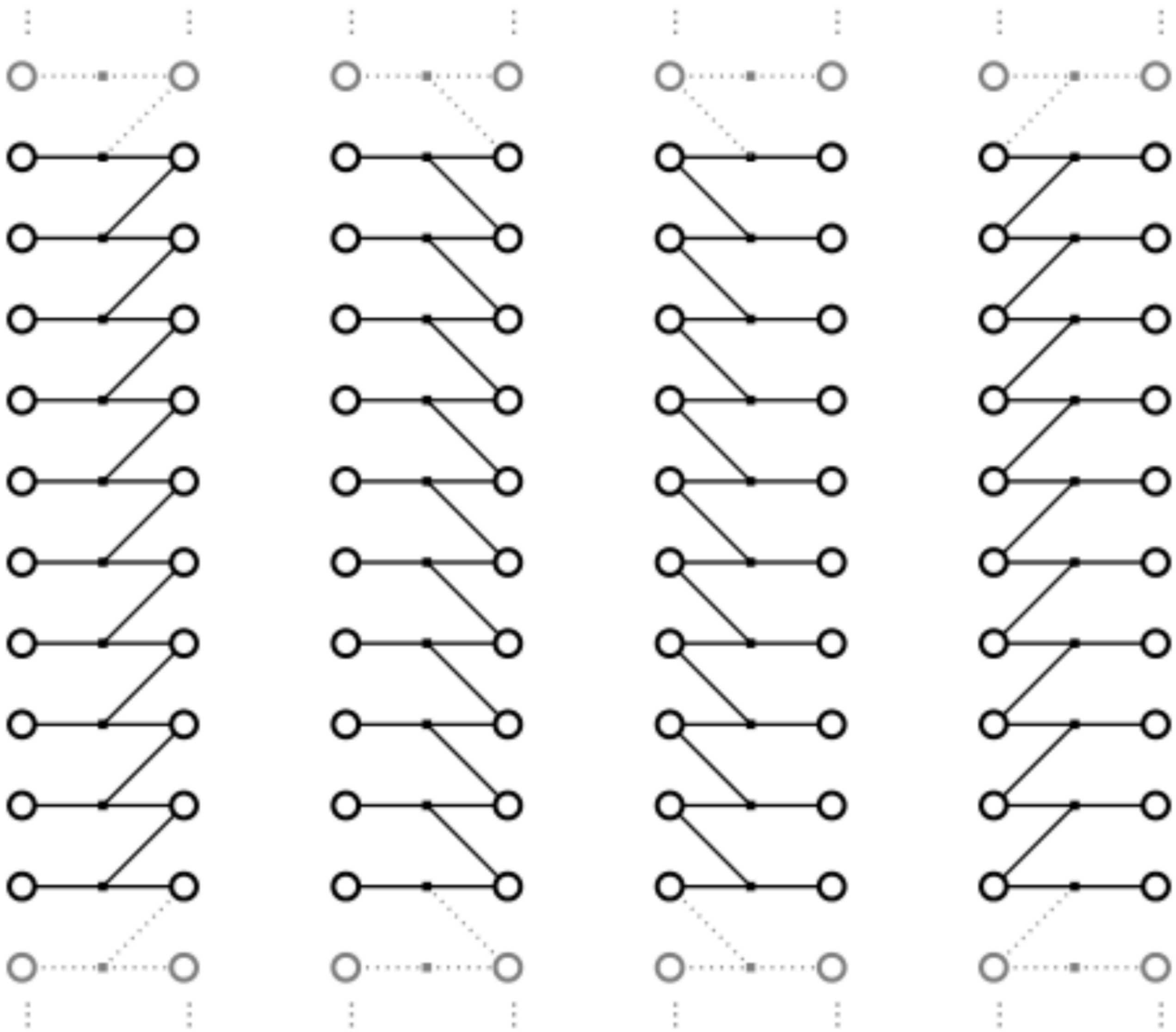
$$Z = \sum_{\pi} q^{\mathrm{vol}(\pi)} = \prod_{k=1}^{\infty} (1-q^k)^{-k}$$

- Determinantal correlation kernel and limit shape computed by Okounkov-Reshetikhin ('03).

3. Definition of rail yard graphs

Four elementary graphs :

(R,+) (R,-) (L,+) (L,-)

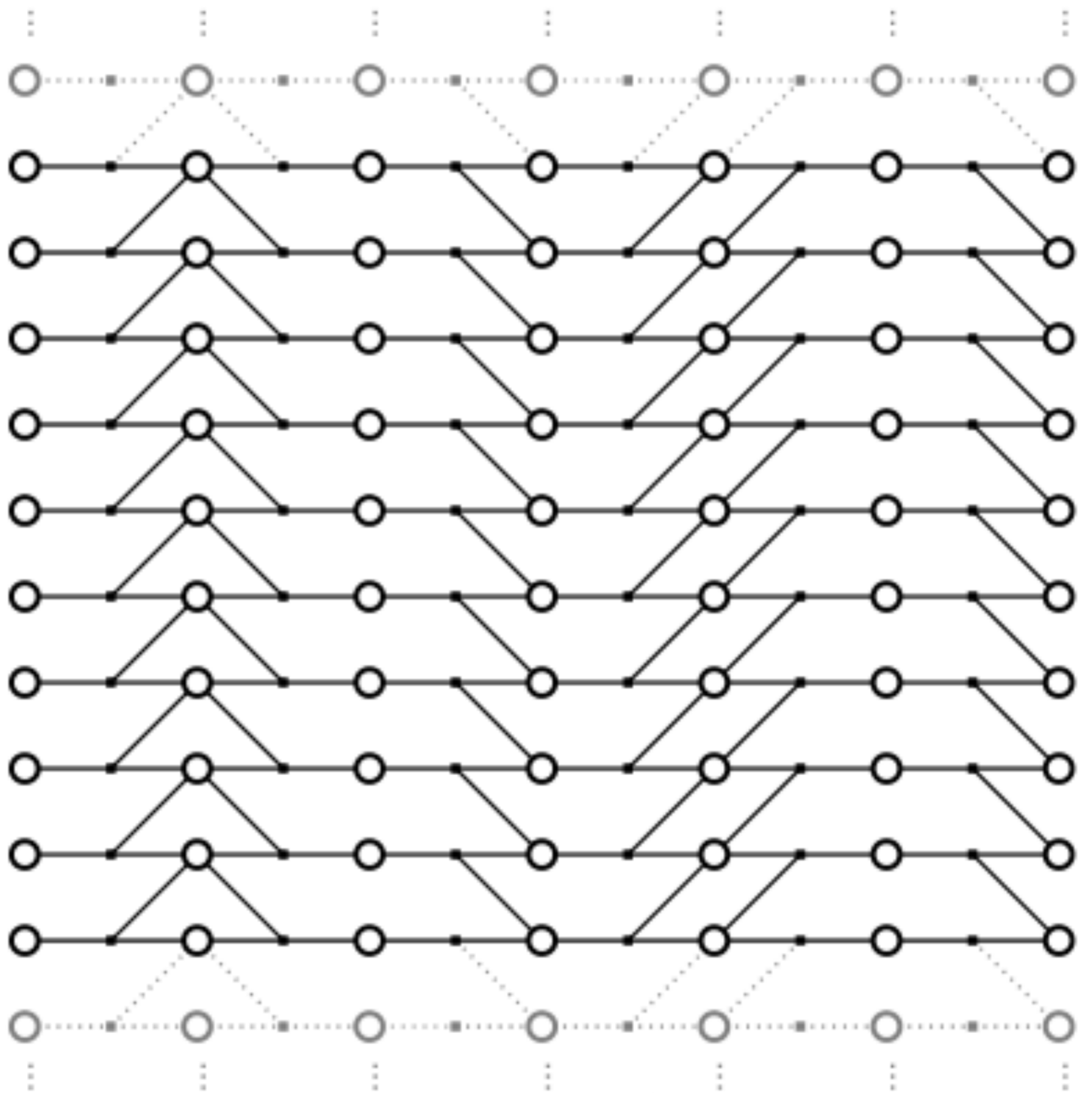


- A rail yard graph (RYG) is obtained by gluing elementary graphs along columns of white vertices.

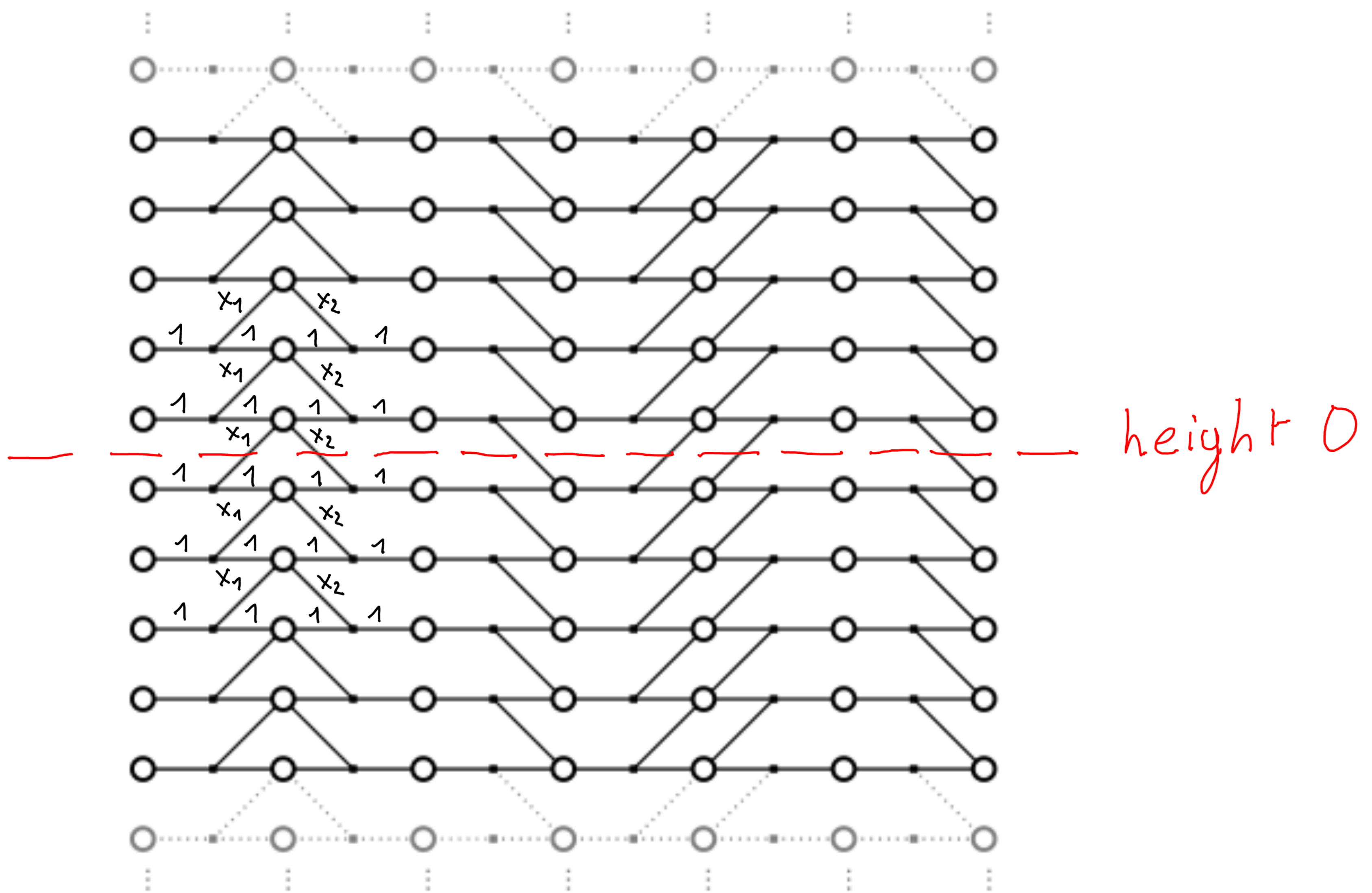
- For every $n \geqslant 1$ and every couple of words $(a_1, \dots, a_n) \in \{L, R\}^n$ and $(b_1, \dots, b_n) \in \{+, -\}^n$ we define an associated RYG.

$n = 6$ $\quad (a_1, a_2, a_3, a_4, a_5, a_6) = (R, L, R, R, L, R)$

$(b_1, b_2, b_3, b_4, b_5, b_6) = (+, +, -, +, -, -)$



weights : 1 on horizontal edges
x_i on diagonal edges encoded by (a_i, b_i) $(x_i < 1)$



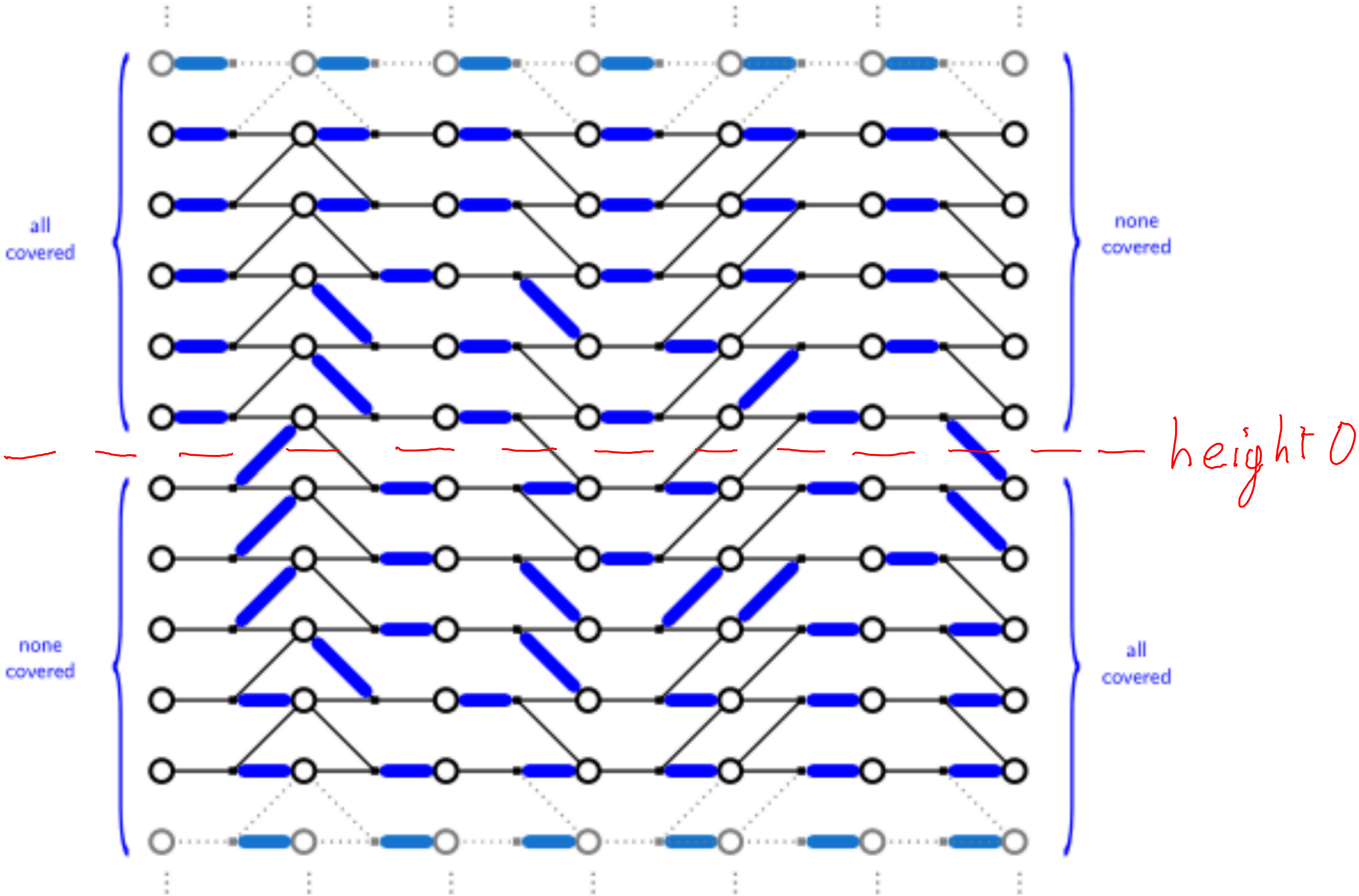
A dimer config on a RYG is a subset of edges such that:

(1) On the left boundary, all the vertices below height 0 are unmatched

(2) On the right boundary, all the vertices above height 0 are unmatched

(3) All other vertices are matched to exactly one neighbour

(4) The dimer config contains finitely many diagonal edges.


all covered
none covered
none covered
all covered
height 0

4. Main results

Theorem 1 [Boutillier-Bouttier-Chapuy-Corteel-R '15]

The partition function for dimer configs on a RYG specified by n, $\underline{a}$ and $\underline{b}$ is

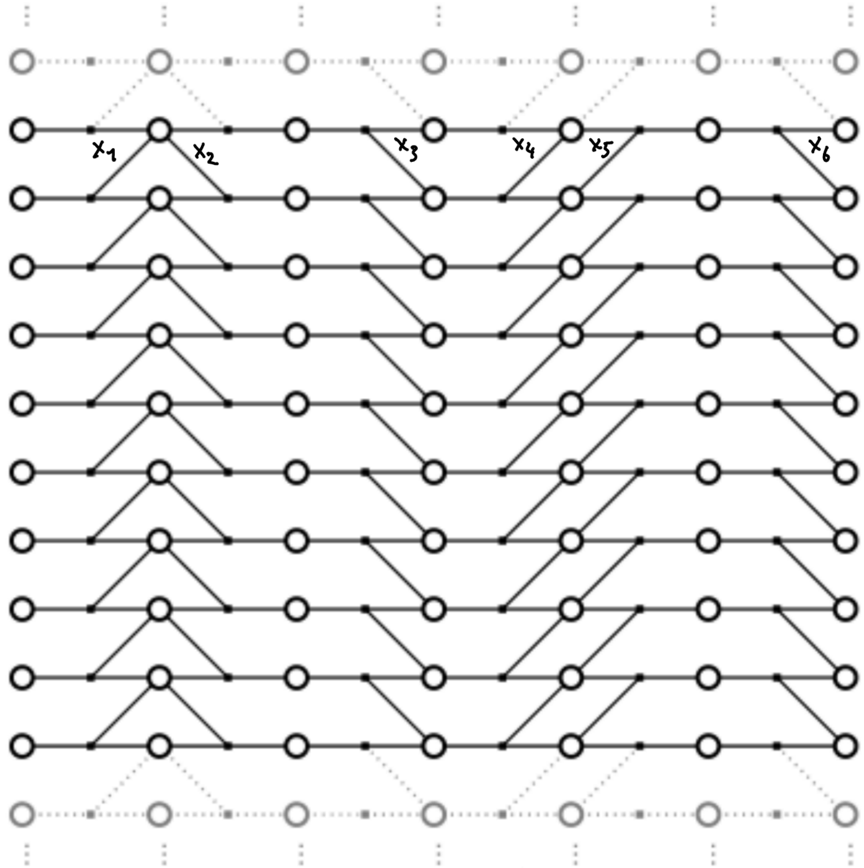
$$Z(n, \underline{a}, \underline{b}) = \prod_{\substack{1 \leq i < j \leq n \\ b_i = +,\, b_j = -}} z_{ij}$$

where $$z_{ij} = \begin{cases} 1 + x_i x_j & \text{if } a_i \neq a_j \\ \dfrac{1}{1 - x_i x_j} & \text{if } a_i = a_j \end{cases}$$

$n = 6 \qquad (a_1, a_2, a_3, a_4, a_5, a_6) = (R, L, R, R, L, R)$

$(b_1, b_2, b_3, b_4, b_5, b_6) = (+, +, -, +, -, -)$

$$Z = \frac{(1+x_1x_5)(1+x_2x_3)(1+x_2x_6)(1+x_4x_5)}{(1-x_1x_3)(1-x_1x_6)(1-x_2x_5)(1-x_4x_6)}$$



Theorem 2 [Boutillier - Bouttier - Chapuy - Corteel - R '15]

Let $E = \{e_1, \dots, e_s\}$ be s edges, with $e_i = (\alpha_i, \beta_i)$

black white

The probability that all the edges in E appear in a random dimer configuration is

$$\mathbb{P}(E) = (-1)^{H(E)} \left(\prod_{i=1}^{s} \text{weight}(e_i) \right) \det_{1 \le i,j \le s} \left(C_{\alpha_i, \beta_j} \right)$$

where $H(E)$ is the number of horizontal edges in E with a black right endpoint.

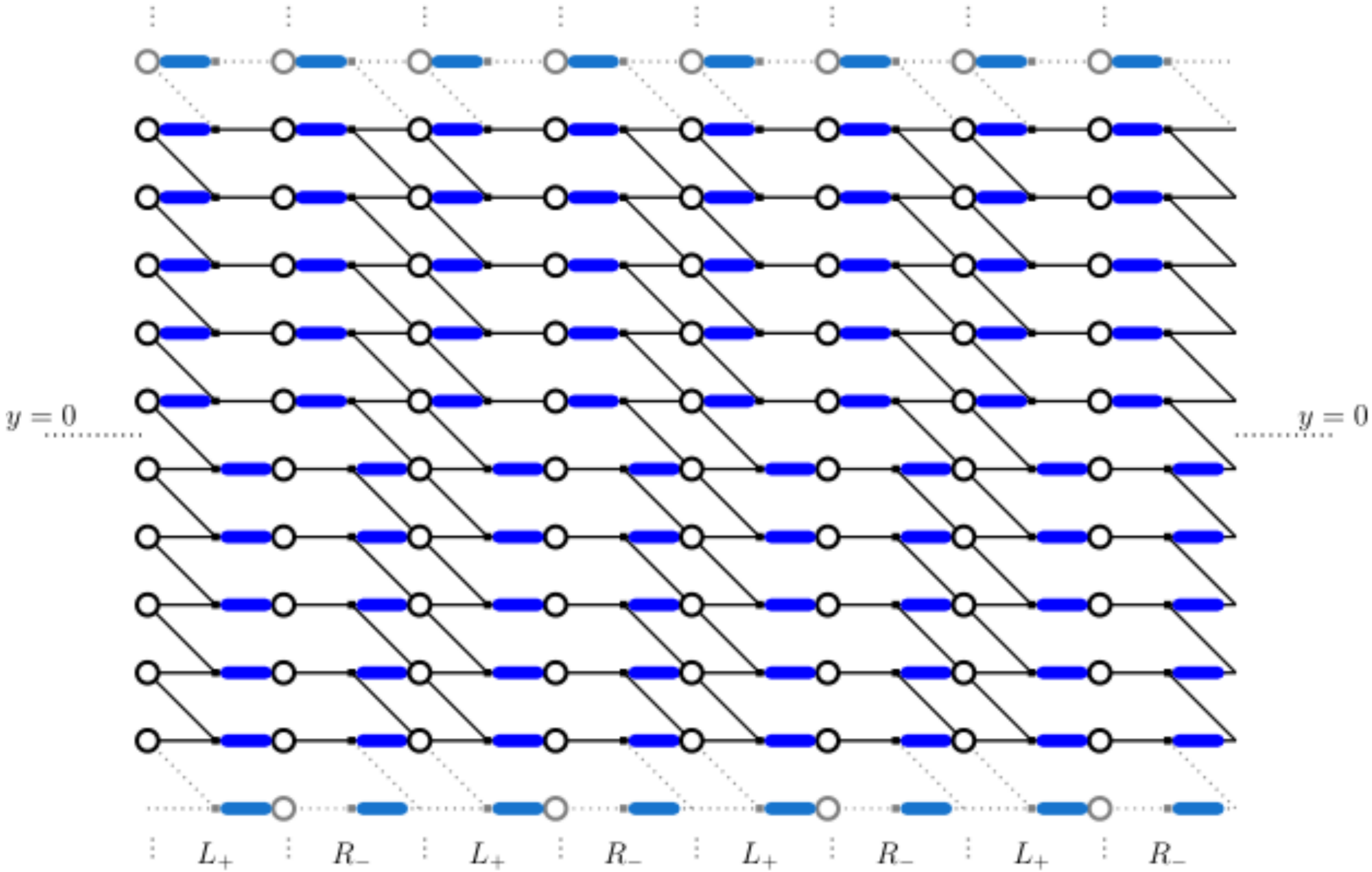
If α is a black vertex at position (α^x, α^y) and β is a white vertex at position (β^x, β^y)

$$C_{\alpha\beta} = \left[z^{\alpha^y} w^{-\beta^y} \right] \frac{F_{\alpha^x}(z)}{F_{\beta^x}(w)} \cdot \frac{\sqrt{zw}}{z-w}$$

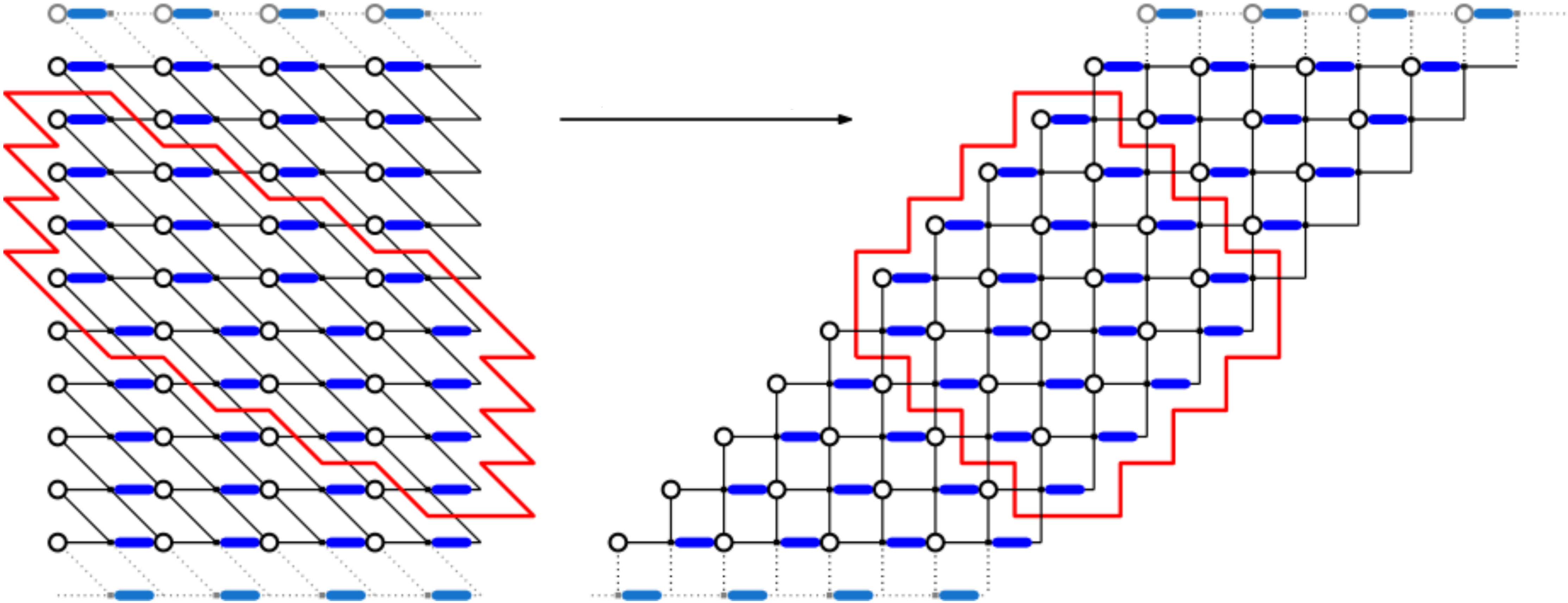
where
$$F_k(z) = \frac{\displaystyle\prod_{\substack{i:(a_i,b_i)=(R,+)\\ 2i<k}} (1+x_i z) \prod_{\substack{j:(a_j,b_j)=(L,-)\\ 2j>k}} \left(1-\frac{x_j}{z}\right)}{\displaystyle\prod_{\substack{i:(a_i,b_i)=(L,+)\\ 2i\le k}} (1-x_i z) \prod_{\substack{j:(a_j,b_j)=(R,-)\\ 2j\geq k}} \left(1+\frac{x_j}{z}\right)}$$

- (Skew) plane partitions (hexagonal lattice), Aztec diamond, pyramid partitions and more generally steep tilings of Bouttier-Chapuy-Correel (square grid) arise as particular instances of RYG.
- We recover the results of Okounkov-Reshetikhin (resp. Chhita-Young) for correlations in skew plane partitions (resp. Aztec diamond).
- We show that the correlation kernel is an inverse to the Kasteleyn operator on the RYGs.

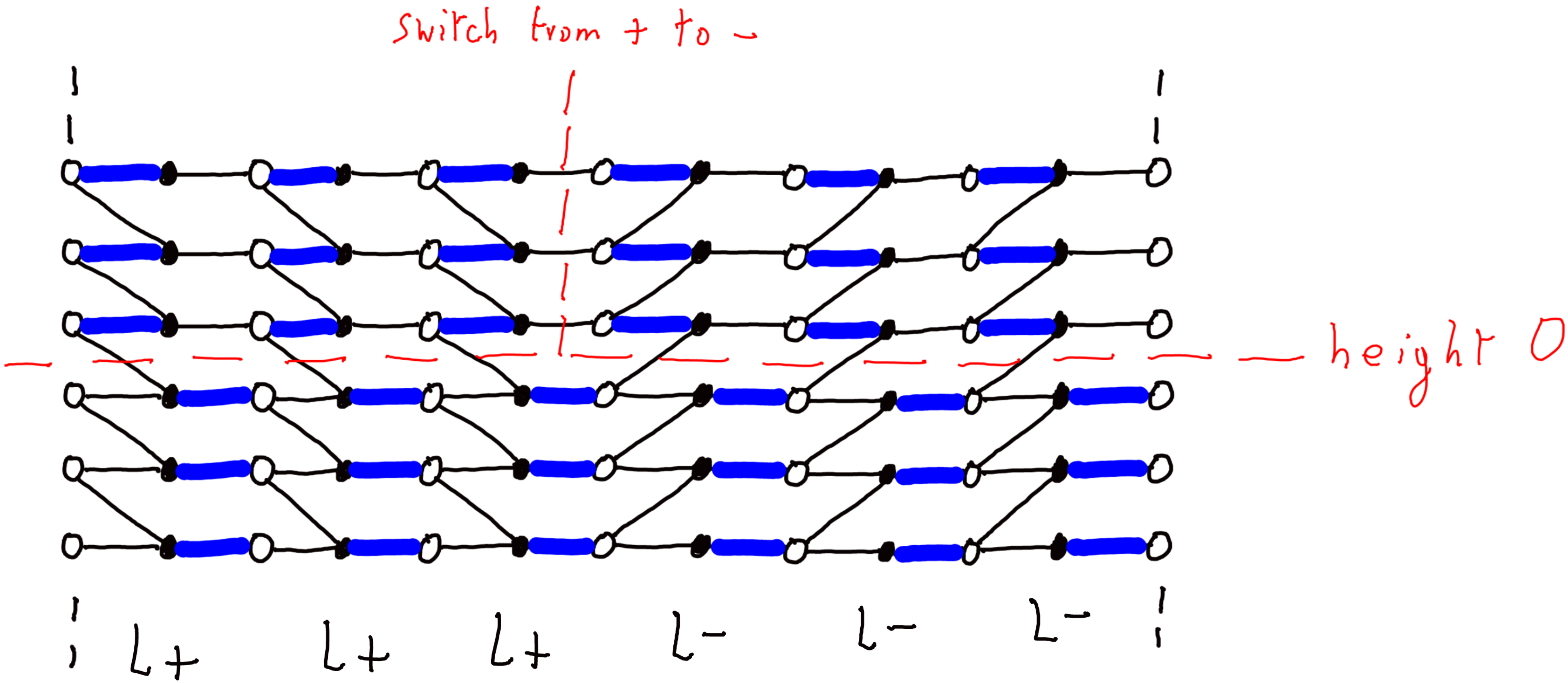
- Aztec diamond of size $n \longleftrightarrow$ alternate $(L,+)$ and $(R,-)$ (n times each)



- Contract vertices of degree 2.



Plane partition (with $n \times n$ floor) $\longleftrightarrow$ n times $(L, +)$ followed by n times $(L, -)$



5. Proof ideas

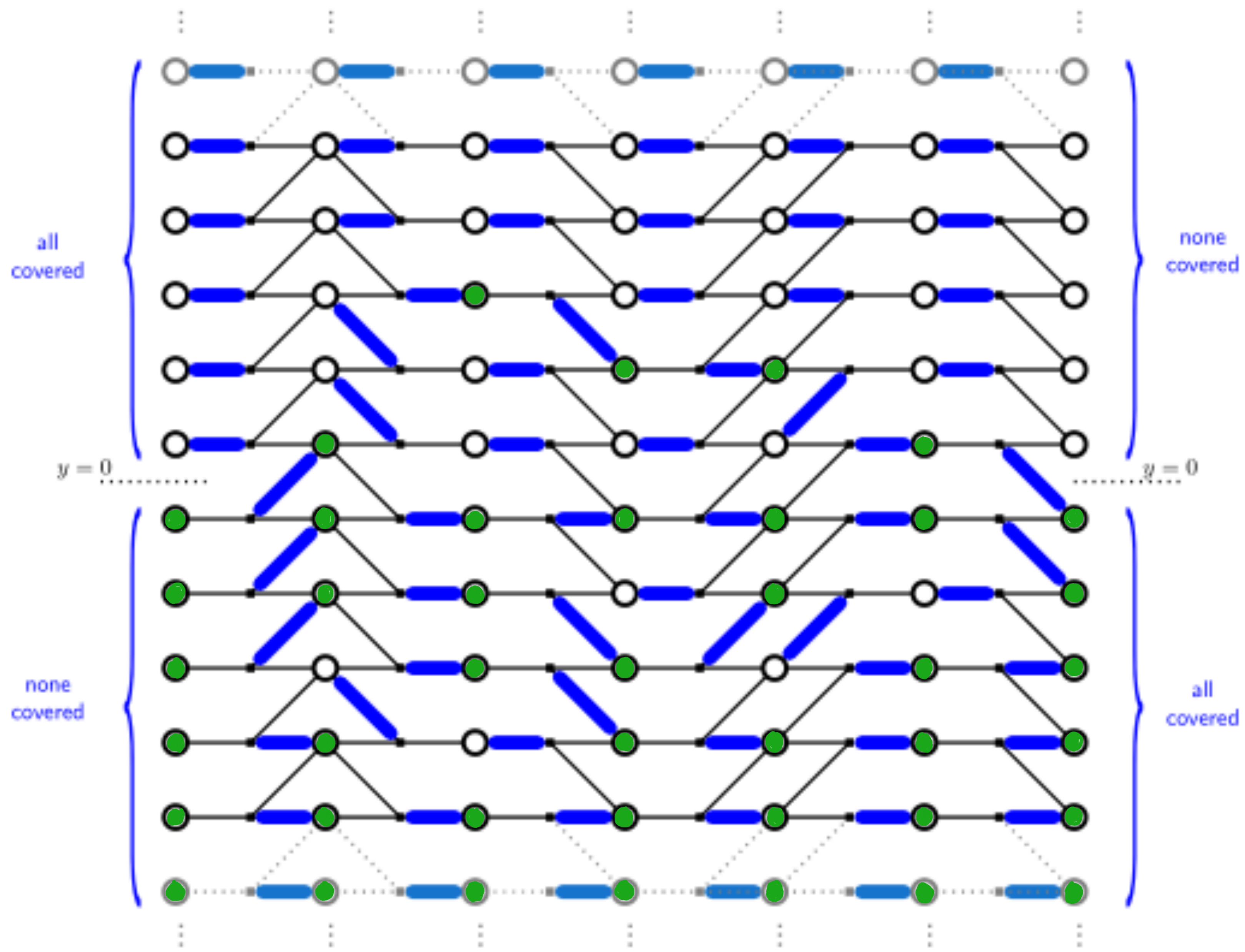
- Key idea: realise the RYG dimer model as a Schur process (Okounkov-Reshetikhin '03, Bouttier-Chapuy-Corteel '14).

- Bijection between dimer configs and configs of particles/holes on each column of white vertices.

Dimers $\longrightarrow$ particles/holes

On each white vertex v, put:

- a particle if v is matched to its left
- a hole if v is matched to its right
- a particle if v is unmatched and on the left boundary
- a hole if v is unmatched and on the right boundary


all
covered
none
covered
$y=0$
$y=0$
none
covered
all
covered

- A dimer config can be seen as the evolution of a particle system on a vertical line, between times 0 and n. The initial and final states are imposed.

- The evolution between times i and $i+1$ is described by a transfer "matrix".

- Four kinds of transfer "matrices", corresponding respectively to $R+$, $R-$, $L+$ and $L-$.

- R_+ (resp. R_-): particles can move down (resp. up) by one unit if possible.
- L_+ (resp. L_-): holes can move up (resp. down) by one unit if possible.

$$\Gamma_{R_-}(x_3)\left|\begin{matrix}\circ\\ \circ\\ \bullet\\ \circ\\ \bullet\\ \circ\\ \bullet\\ \bullet\end{matrix}\right\rangle = \left|\begin{matrix}\circ\\ \circ\\ \bullet\\ \circ\\ \bullet\\ \circ\\ \bullet\\ \bullet\end{matrix}\right\rangle + x_3\left|\begin{matrix}\circ\\ \bullet\\ \circ\\ \circ\\ \bullet\\ \circ\\ \bullet\\ \bullet\end{matrix}\right\rangle + x_3\left|\begin{matrix}\circ\\ \circ\\ \bullet\\ \bullet\\ \circ\\ \circ\\ \bullet\\ \bullet\end{matrix}\right\rangle + x_3\left|\begin{matrix}\circ\\ \circ\\ \bullet\\ \circ\\ \bullet\\ \bullet\\ \circ\\ \bullet\end{matrix}\right\rangle$$

$$+ x_3^2\left|\begin{matrix}\circ\\ \bullet\\ \circ\\ \bullet\\ \circ\\ \circ\\ \bullet\\ \bullet\end{matrix}\right\rangle + \cdots$$

- Dimer partition function expressed as a product of n operators, evaluated between $\langle \emptyset |$ and $| \emptyset \rangle$, where "$\emptyset$" denotes "$\begin{array}{c}\circ\\ \circ\\ \circ\\ \bullet\\ \bullet\\ \bullet\\ \bullet\end{array}$".

- Using
  * $\Gamma_{R+}(x) | \emptyset \rangle = \Gamma_{L+}(x) | \emptyset \rangle = | \emptyset \rangle$
  * $\langle \emptyset | \Gamma_{R-}(x) = \langle \emptyset | \Gamma_{L-}(x) = \langle \emptyset |$
  * commutation relations between Γ's

we obtain Theorem 1.

- The presence of a particle can be expressed using the Γ's and some fermionic operators ψ_k and ψ_k^* (Okounkov - Reshetikhin '03).

- To localize dimers on RYG's, we need to come up with other operators, also constructed from the Γ's and the ψ's.

- Using Wick's formula, express n-point correlations for dimers as the determinant of a matrix, whose entries are 2-point correlation functions.

- To evaluate these 2-point correlations, use commutation relations between the Γs, ψs and ψ^*s.

6. Summary and outlook

Summary

- RYG model provides a dimer realisation of a general Schur process.
- It is a common generalization of skew plane partitions, Aztec diamond and steep tilings.
- Using transfer-matrix approach and dimer-localizing operators, we can compute the partition function and the determinantal correlation kernel.

Open questions

- Using sampling algorithm in [Betea-Boutillier-Bouttier-Chapuy-Corteel-Vuletić '14], we observe some limit shapes. Can we compute them?

- Study more general weights?

- Study more general graphs?

THANK YOU !