The six vertex model is can be reformulated as a random stepped surface called heights.

In the thermodynamic limit, the limiting average height function becomes deterministic and can be found by solving a certain boundary value problem.

The six vertex model is quantum integrable in the sense that it admits commuting transfer matrices and can be solved by Bethe ansatz.

What does the quantum integrability imply for the PDE governing the limiting height function?
Introduction

Background

• The six vertex model is can be reformulated as a random stepped surface called heights.

• In the thermodynamic limit, the limiting average height function becomes deterministic and can be found by solving a certain boundary value problem.
Background

- The six vertex model is can be reformulated as a random stepped surface called heights.
- In the thermodynamic limit, the limiting average height function becomes deterministic and can be found by solving a certain boundary value problem.
- The six vertex model is quantum integrable in the sense that it admits commuting transfer matrices and can be solved by Bethe ansatz.
Background

- The six vertex model is can be reformulated as a random stepped surface called heights.
- In the thermodynamic limit, the limiting average height function becomes deterministic and can be found by solving a certain boundary value problem.
- The six vertex model is quantum integrable in the sense that it admits commuting transfer matrices and can be solved by Bethe ansatz.
- What does the quantum integrability imply for the PDE governing the limiting height function?
Outline of Talk

- Quick Review of Six Vertex Model
- Thermodynamic Limit
- Integrability:
  - Transfer Matrices
  - Commuting Hamiltonians
- Examples
- Outlook
Configurations and Weights

- Let $S_T = [0, T] \times [0, 1]$, and let $S_T^\epsilon = \epsilon \mathbb{Z}^2$ be the scaled square lattice centered inside $S_T$. 
Configurations and Weights

- Let $S_T = [0, T] \times [0, 1]$, and let $S_T^\epsilon = \epsilon \mathbb{Z}^2$ be the scaled square lattice centered inside $S_T$.
- A configuration $s$ of the six vertex model is a set of paths that only go right and up.
Configurations and Weights

- Let $S_T = [0, T] \times [0, 1]$, and let $S_T^\epsilon = \epsilon \mathbb{Z}^2$ be the scaled square lattice centered inside $S_T$.
- A configuration $s$ of the six vertex model is a set of paths that only go right and up.

- Each vertex has a weight $\nu(s)$.
- The Boltzmann weight of $s$:

$$w(s) = \prod_{\text{vertex } v} \nu(s)$$
Boundary Conditions

- The state of $s$ at time $t$ is the set of horizontal edges traversed by $s$ at $t$. 

![Diagram of Six Vertex Model]

Review: Six Vertex Model
Boundary Conditions

• The state of $s$ at time $t$ is the set of horizontal edges traversed by paths at $t$.
• Fixed boundary conditions are choice initial and final states $\eta_1$ and $\eta_2$. 
Boundary Conditions

- The state of $s$ at time $t$ is the set of horizontal edges traversed by paths at $t$.
- Fixed boundary conditions are choice initial and final states $\eta_1$ and $\eta_2$.

\[
Z^{\epsilon}_{\eta_1, \eta_2, T} = \sum_{\substack{s(0) = \eta_1 \\ s(1) = \eta_2}} w(s)
\]

\[
f^{\epsilon}_{\eta_1, \eta_2, T} = \epsilon^2 \log (Z^\epsilon_{\eta_1, \eta_2})
\]
Height Function

- A height function is a function on faces satisfying a gradient constraint:
  - $0 \leq h(x, y) - h(x + \epsilon, y) \leq 1$
  - $0 \leq h(x, y + \epsilon) - h(x, y) \leq 1$
Height Function

- A height function is a function on faces satisfying a gradient constraint:
  - \( 0 \leq h(x, y) - h(x + \epsilon, y) \leq 1 \)
  - \( 0 \leq h(x, y + \epsilon) - h(x, y) \leq 1 \)

- Height functions are in bijection with configurations; the level curves of \( h \) are the paths of the configuration.
Height Function

- A height function is a function on faces satisfying a gradient constraint:
  - \( 0 \leq h(x, y) - h(x + \epsilon, y) \leq 1 \)
  - \( 0 \leq h(x, y + \epsilon) - h(x, y) \leq 1 \)

- Height functions are in bijection with configurations; the level curves of \( h \) are the paths of the configuration.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- The boundary conditions determine the height function at the boundary.
Height Function

- A height function is a function on faces satisfying a gradient constraint:
  - $0 \leq h(x, y) - h(x + \epsilon, y) \leq 1$
  - $0 \leq h(x, y + \epsilon) - h(x, y) \leq 1$

- Height functions are in bijection with configurations; the level curves of $h$ are the paths of the configuration.

- The boundary conditions determine the height function at the boundary.

- The normalized height function $\bar{h} = \epsilon h$. The average height function $\langle \bar{h} \rangle$ is the ensemble average of the normalized height function.
• Suppose we have a sequence of six vertex models $S^{\epsilon_i}$ and boundary height functions $\eta^{\epsilon_i}_1, \eta^{\epsilon_i}_2$ with $\epsilon_i \to 0$.

• The boundary conditions are said to be stabilizing if the normalized boundary height functions $\eta^{\epsilon_i}_1, \eta^{\epsilon_i}_2$ converge to $\eta_1, \eta_2$ in the uniform metric as $\epsilon_i \to 0$.

• In this case, there exist limiting free energy and limiting height function:

$$f_{\eta_1, \eta_2, T} = \lim_{\epsilon_i \to 0} f^{\epsilon_i}_{\eta_1^{\epsilon_i}, \eta_2^{\epsilon_i}, T},$$

$$\langle h \rangle_{\eta_1, \eta_2} = \lim_{\epsilon_i \to 0} \langle h \rangle^{\epsilon_i}_{\eta_1^{\epsilon_i}, \eta_2^{\epsilon_i}}.$$
Thermodynamic Limit

**Thermodynamic limit**

- Suppose we have a sequence of six vertex models $S^i_\epsilon$ and boundary height functions $\eta^i_1, \eta^i_2$ with $\epsilon^i \to 0$.
- The boundary conditions are said to be stabilizing if the normalized boundary height functions $\eta^\epsilon_1, \eta^\epsilon_2$ converge to $\eta_1, \eta_2 : [0, 1] \to \mathbb{R}$ in the uniform metric as $\epsilon \to 0$. 
Thermodynamic Limit

Thermodynamic limit

- Suppose we have a sequence of six vertex models $S^i_\epsilon$ and boundary height functions $\eta^i_1, \eta^i_2$ with $\epsilon^i \to 0$.
- The boundary conditions are said to be stabilizing if the normalized boundary height functions $\eta^\epsilon_1, \eta^\epsilon_2$ converge to $\eta_1, \eta_2 : [0, 1] \to \mathbb{R}$ in the uniform metric as $\epsilon \to 0$.
- In this case, there exist limiting free energy and limiting height function:

$$f_{\eta_1, \eta_2, T} = \lim_{\epsilon \to 0} f^\epsilon_{\eta_1, \eta_2, T}$$

$$\langle h \rangle = \lim_{\epsilon \to 0} \langle h \rangle^\epsilon$$
Thermodynamic Limit

Variational Principle

- The limiting free energy and average height function can be computed by variational principle.

\[
f_{\eta_1, \eta_2, T} = \max_{h \in \mathcal{H}} \int_0^1 \int_0^T \sigma_w(\partial_t h, \partial_y h) \, dt \, dy
\]

where \(\sigma\) is called the surface tension function, and \(\mathcal{H}\) is the set of limiting height functions, \(h : S_t \to \mathbb{R}\) satisfying: \(h(0, 0) = 0\), monotonicity, and Lipschitz continuity with constant 1.
Variational Principle

- The limiting free energy and average height function can be computed by variational principle.

\[
f_{\eta_1, \eta_2, T} = \max_{h \in \mathcal{H}} \int_0^1 \int_0^T \sigma_w(\partial_t h, \partial_y h) \, dt \, dy
\]

where \(\sigma\) is called the surface tension function, and \(\mathcal{H}\) is the set of limiting height functions, \(h : S_t \to \mathbb{R}\) satisfying: \(h(0, 0) = 0\), monotonicity, and Lipschitz continuity with constant 1.

- The limiting height function \(\langle h \rangle\) is the maximizer.
Variational Principle

- The limiting free energy and average height function can be computed by variational principle.

\[
f_{\eta_1, \eta_2, T} = \max_{h \in \mathcal{H}} \int_0^1 \int_0^T \sigma_w (\partial_t h, \partial_y h) \, dt \, dy
\]

where \( \sigma \) is called the surface tension function, and \( \mathcal{H} \) is the set of limiting height functions, \( h : S_t \to \mathbb{R} \) satisfying: \( h(0, 0) = 0 \), monotonicity, and Lipschitz continuity with constant 1.

- The limiting height function \( \langle h \rangle \) is the maximizer.

- Euler Lagrange equations:

\[
\partial_{11} \sigma_w \partial_t^2 h + 2 \partial_{12} \sigma_w \partial_t \partial_y h + \partial_{22} \sigma_w \partial_y^2 h = 0
\]
Transfer Matrices

- Let \( \{ e_0, e_1 \} \) be an orthonormal basis for \( \mathbb{C}^2 \), and let \( V = (\mathbb{C}^2)^{\otimes \lceil 1/\epsilon \rceil} \).
Integrability of the Six Vertex Model

Transfer Matrices

- Let \( \{ e_0, e_1 \} \) be an orthonormal basis for \( \mathbb{C}^2 \), and let \( V = (\mathbb{C}^2)^{\otimes \lfloor 1/\epsilon \rfloor} \).
- A state \( s \) of the six vertex model corresponds to a basis vector \( |s\rangle = e_{s_0} \otimes e_{s_1} \cdots e_{s_N} \), where \( s_i = 1 \) is the indicator of the \( i \)th edge.
Transfer Matrices

- Let $\{e_0, e_1\}$ be an orthonormal basis for $\mathbb{C}^2$, and let $V = (\mathbb{C}^2)^{\otimes \lfloor 1/\epsilon \rfloor}$.
- A state $s$ of the six vertex model corresponds to a basis vector $|s\rangle = e_{s_0} \otimes e_{s_1} \cdots e_{s_N}$, where $s_i = 1$ is the indicator of the $i$th edge.
- Define the transfer matrix $T_w : V \rightarrow V$ by its matrix elements:

$$\langle s_1 | T_w | s_2 \rangle = Z_{s_1, s_2, \epsilon}$$

(i.e. the partition function for just one column).
Integrability of the Six Vertex Model

Transfer Matrices

- Let \( \{e_0, e_1\} \) be an orthonormal basis for \( \mathbb{C}^2 \), and let \( V = (\mathbb{C}^2)^{\otimes \lfloor 1/\epsilon \rfloor} \).
- A state \( s \) of the six vertex model corresponds to a basis vector \( |s\rangle = e_{s_0} \otimes e_{s_1} \cdots e_{s_N} \), where \( s_i = 1 \) is the indicator of the \( i \)th edge.
- Define the transfer matrix \( T_w : V \to V \) by its matrix elements:

\[
\langle s_1 | T_w | s_2 \rangle = Z_{s_1,s_2,\epsilon}
\]

(ie. the partition function for just one column).
- Then:

\[
Z_{\eta_1,\eta_2,t} = \langle \eta_1 | T_w^{t/\epsilon} | \eta_2 \rangle
\]
Hamiltonian Formulation of Variational Principle

- Recast the variational problem in the Hamiltonian formulation by Legendre transform:

\[ H_w(\pi, t) = \max_s \pi s - \sigma_w(s, t) \]

The new variables are \( h \) and \( \pi \), where \( \pi \) is conjugate to \( \partial_t h \).
Hamiltonian Formulation of Variational Principle

- Recast the variational problem in the Hamiltonian formulation by Legendre transform:

\[ \mathcal{H}_w(\pi, t) = \max_s \pi s - \sigma_w(s, t) \]

The new variables are \( h \) and \( \pi \), where \( \pi \) is conjugate to \( \partial_t h \).

- The Hamiltonian is:

\[
H_w(\pi(y), h(y)) = \int_0^1 \mathcal{H}(\pi(y), \partial_y h(y)) \, dy
\]
Integrability of the Six Vertex Model

Hamiltonian Formulation of Variational Principle

- Recast the variational problem in the Hamiltonian formulation by Legendre transform:

$$H_w(\pi, t) = \max_s \pi s - \sigma_w(s, t)$$

The new variables are $h$ and $\pi$, where $\pi$ is conjugate to $\partial_t h$.

- The Hamiltonian is:

$$H_w(\pi(y), h(y)) = \int_0^1 H(\pi(y), \partial_y h(y)) \, dy$$

- The variational principle is:

$$\mathcal{S}[\pi, h] = \max_{\pi, h} S[\pi, h]$$

$$S[\pi, h] = \int_0^T \int_0^1 \pi \, \partial_t h - H_w(\pi, \partial_y h) \, dt \, dy$$
Hamiltonian Formulation of Variational Principle

- Recast the variational problem in the Hamiltonian formulation by Legendre transform:

\[ \mathcal{H}_w(\pi, t) = \max_s \pi s - \sigma_w(s, t) \]

The new variables are \( h \) and \( \pi \), where \( \pi \) is conjugate to \( \partial_t h \).

- The Hamiltonian is:

\[ H_w(\pi(y), h(y)) = \int_0^1 \mathcal{H}(\pi(y), \partial_y h(y)) \, dy \]

- The variational principle is:

\[ f_{\eta_1, \eta_2, T} = \max_{\pi, h} S[\pi, h] \]

\[ S[\pi, h] = \int_0^T \int_0^1 \pi \, \partial_t h - H_w(\pi, \partial_y h) \, dt \, dy \]
Integrability of the Six Vertex Model

Hamiltonian Formulation

- The canonical Poisson structure is given by:
  \[ \{ \pi(y), h(y') \} = \delta(y - y') \].
Integrability of the Six Vertex Model

Hamiltonian Formulation

• The canonical Poisson structure is given by:
  \[ \{ \pi(y), h(y') \} = \delta(y - y'). \]

• The equations of motion are:
  \[
  \frac{\partial h}{\partial t}(y) = \{ h(y), H \}
  \]
  \[
  \frac{\partial \pi}{\partial t}(y) = \{ \pi(y), H \}
  \]
Integrability of the Six Vertex Model

Hamiltonian Formulation

• The canonical Poisson structure is given by:
  \[ \{ \pi(y), h(y') \} = \delta(y - y'). \]
• The equations of motion are:
  \[
  \frac{\partial h}{\partial t}(y) = \{ h(y), H \} \\
  \frac{\partial \pi}{\partial t}(y) = \{ \pi(y), H \}
  \]

These are equivalent to the Euler-Lagrange equations.
Integrability of the Six Vertex Model

Commuting Transfer Matrices and Hamiltonians

- Recall $\Delta w = \frac{w_1^2 + w_2^2 - w_3^2}{2w_1w_2}$.
- Quantum Integrability: if $w$ and $\tilde{w}$ satisfy $\Delta w = \Delta \tilde{w}$ then the transfer matrices commute:

$$[T_w, T_{\tilde{w}}] = 0$$
Integrability of the Six Vertex Model

Commuting Transfer Matrices and Hamiltonians

- Recall $\Delta w = \frac{w_1^2 + w_2^2 - w_3^2}{2w_1w_2}$.

- Quantum Integrability: if $w$ and $\tilde{w}$ satisfy $\Delta w = \Delta \tilde{w}$ then the transfer matrices commute:

  $$[T_w, T_{\tilde{w}}] = 0$$

- Main result is semiclassical integrability: if $\Delta w = \Delta \tilde{w}$ then the corresponding Hamiltonians Poisson commute:

  $$\{H_w, H_{\tilde{w}}\} = 0$$
Brief Sketch of Proof

- The proof is relies on two calculations:
  - Lemma 1: If $\sigma_w$ and $\tilde{\sigma}_w$ have equal Hessian, i.e., $\det(\frac{\partial}{\partial i} \frac{\partial}{\partial j} \sigma_w) = \det(\frac{\partial}{\partial i} \frac{\partial}{\partial j} \tilde{\sigma}_w)$, then the corresponding Hamiltonians Poisson commute $\{H_w, H_{\tilde{\sigma}_w}\} = 0$.
  - Lemma 2: The Hessian of the surface tension $\sigma_w$ of the six vertex model $\sigma$ depends on $w$ only via $\Delta(w)$. 


Brief Sketch of Proof

- The proof is relies on two calculations:
- Lemma 1: If $\sigma_w$ and $\tilde{\sigma}_w$ have equal Hessian, i.e. $\det(\partial_i \partial_j \sigma_w) = \det(\partial_i \partial_j \tilde{\sigma}_w)$, then the corresponding Hamiltonians Poisson commute

$$\{H_w, H_{\tilde{w}}\} = 0$$
Integrability of the Six Vertex Model

Brief Sketch of Proof

• The proof is relies on two calculations:

• Lemma 1: If $\sigma_w$ and $\sigma_{\tilde{w}}$ have equal Hessian, ie. $\det(\partial_i \partial_j \sigma_w) = \det(\partial_i \partial_j \sigma_{\tilde{w}})$, then the corresponding Hamiltonians Poisson commute

$$\{H_w, H_{\tilde{w}}\} = 0$$

• Lemma 2: The Hessian of the surface tension $\sigma_w$ of the six vertex model $\sigma$ depends on $w$ only via $\Delta(w)$. 
Easy Example: Dimer Model

- For a dimer model on a bipartite graph, the surface tension takes a particular form.
Easy Example: Dimer Model

- For a dimer model on a bipartite graph, the surface tension takes a particular form.

- By diagonalizing the Kasteleyn matrix, the free energy with magnetic field \((H, V)\) takes the form:

\[
f(H, V) = \int_0^{2\pi} \int_0^{2\pi} \log(A + Be^{ik-H} + Ce^{im+V}) \, dk \, dm
\]

for some constants \(A, B, C\).
Examples

Easy Example: Dimer Model

- For a dimer model on a bipartite graph, the surface tension takes a particular form.

- By diagonalizing the Kasteleyn matrix, the free energy with magnetic field $(H, V)$ takes the form:

\[
f(H, V) = \int_0^{2\pi} \int_0^{2\pi} \log(A + B e^{i(k+H)} + C e^{i(m+V)}) \, dk \, dm
\]

for some constants $A, B, C$.

- Then $\sigma$ is the Legendre transform of $f$

\[
\sigma(s, t) = \max_{H, V} s \, H + t \, V - f(H, V)
\]
Easy Example: Dimer Model

- For a dimer model on a bipartite graph, the surface tension takes a particular form.
- By diagonalizing the Kasteleyn matrix, the free energy with magnetic field $(H, V)$ takes the form:

$$f(H, V) = \int_0^{2\pi} \int_0^{2\pi} \log(A + B e^{i k + H} + C e^{i m + V}) \, dk \, dm$$

for some constants $A, B, C$.
- Then $\sigma$ is the Legendre transform of $f$

$$\sigma(s, t) = \max_{H, V} s \, H + t \, V - f(H, V)$$

- Lemma: The hessian of $\sigma$ is $\pi^2$, independent of weights.
Hexagonal Dimer Model

• The six vertex model with weights

\[ w_1 = 0 \quad w_2 = a \quad w_3 = b \quad w_4 = c \quad w_5 = \sqrt{bc} \quad w_6 = \sqrt{bc} \]

Corresponds to the dimer model on the hexagonal lattice with edge weights \((a, b, c)\).
Hexagonal Dimer Model

- The six vertex model with weights

\[ w_1 = 0 \quad w_2 = a \quad w_3 = b \quad w_4 = c \quad w_5 = \sqrt{bc} \quad w_6 = \sqrt{bc} \]

Corresponds to the dimer model on the hexagonal lattice with edge weights \((a, b, c)\).

- The Euler-Langrange equations for the limiting height function can be transformed to the Burger’s equation, \( \partial_t u + u \partial_y u = 0 \), which admits many integrals of motion: \( \int u(y)^n dy \).
Hexagonal Dimer Model

- The six vertex model with weights

\[ w_1 = 0 \quad w_2 = a \quad w_3 = b \quad w_4 = c \quad w_5 = \sqrt{bc} \quad w_6 = \sqrt{bc} \]

Corresponds to the dimer model on the hexagonal lattice with edge weights \((a, b, c)\).

- The Euler-Langrange equations for the limiting height function can be transformed to the Burger’s equation, \( \partial_t u + u \partial_y u = 0 \), which admits many integrals of motion: \( \int u(y)^n dy \).

- The surface tension function \( \sigma \) can be calculated in closed form, and the Hamiltonians can be shown directly to commute.
Free Fermion Point

• More generally, when $\Delta w = 0$, the six vertex model is equivalent to the dimer model on the graph:

for certain choice of edge weights.
Examples

Free Fermion Point

- More generally, when $\Delta w = 0$, the six vertex model is equivalent to the dimer model on the graph:

  for certain choice of edge weights.

- The surface tension can be computed in closed form, and the Hamiltonians commute.
Generalities

• The semiclassical limit of $[T_w, \tilde{T}_w] = 0$ is as follows:
Further Work

Generalities

- The semiclassical limit of $[T_w, T_{\tilde{w}}] = 0$ is as follows:
- Fix $t, \tilde{t}$ and let

$$Z^\varepsilon_{\eta_1, \eta_2, t, \tilde{t}} = \langle \eta_1 | T_w^{[t/\varepsilon]} T_{\tilde{w}}^{[\tilde{t}/\varepsilon]} | \eta_2 \rangle$$
Further Work

Generalities

- The semiclassical limit of $[T_w, \tilde{T}_w] = 0$ is as follows:
- Fix $t, \tilde{t}$ and let

$$Z^\epsilon_{\eta_1, \eta_2, t, \tilde{t}} = \langle \eta_1 | T_w^{[t/\epsilon]} T_{\tilde{w}}^{[\tilde{t}/\epsilon]} | \eta_2 \rangle$$

- This corresponds to gluing two regions together:

$$Z^\epsilon_{\eta_1, \eta_2, t, \tilde{t}} = \sum_\eta Z^\epsilon_{\eta_1, \eta, t} \tilde{Z}^\epsilon_{\eta, \eta_2, \tilde{t}}$$
Further Work

Generalities

- The semiclassical limit of $[T_w, \tilde{T}_w] = 0$ is as follows:
- Fix $t, \tilde{t}$ and let

$$Z^\epsilon_{\eta_1, \eta_2, t, \tilde{t}} = \langle \eta_1 | T_w^{[t/\epsilon]} T_{\tilde{w}}^{[\tilde{t}/\epsilon]} | \eta_2 \rangle$$

- This corresponds to gluing two regions together:

$$Z^\epsilon_{\eta_1, \eta_2, t, \tilde{t}} = \sum_\eta Z^\epsilon_{\eta_1, \eta, t} \tilde{Z}^\epsilon_{\eta, \eta_2, \tilde{t}}$$
Further Work

Generalities

• The semiclassical limit of $[T_w, \tilde{T}_w] = 0$ is as follows:

  Fix $t, \tilde{t}$ and let

  $$Z^\epsilon_{\eta_1, \eta_2, t, \tilde{t}} = \langle \eta_1 | T_w^{[t/\epsilon]} T_{\tilde{w}}^{[\tilde{t}/\epsilon]} | \eta_2 \rangle$$

• This corresponds to gluing two regions together:

  $$Z^\epsilon_{\eta_1, \eta_2, t, \tilde{t}} = \sum_\eta Z^\epsilon_{\eta_1, \eta, t} Z^\epsilon_{\eta, \eta_2, \tilde{t}}$$

• In the limit $\epsilon \to 0$, by large deviation principle:

  $$f_{\eta_1, \eta_2, t, \tilde{t}} = \max_\eta f_{\eta_1, \eta, t} + \tilde{f}_{\eta, \eta_2, \tilde{t}}$$
Generalities

- The commutation of the transfer matrices implies:

$$\max_{\eta} f_{\eta_1, \eta, t} + \tilde{f}_{\eta, \eta_2, \tilde{t}} = \max_{\eta} \tilde{f}_{\eta_1, \eta, \tilde{t}} + f_{\eta, \eta_2, t}$$

for all $t, \tilde{t}$ and boundary conditions $\eta_1, \eta_2$. 

Recall that $f$ is the Hamilton-Jacobi action.

Generally:

If the Hamilton-Jacobi actions of $H$ and $\tilde{H}$ commute in the above sense, then does $\{H, \tilde{H}\}$?

Generally no, but under mild assumptions then yes.
Further Work

Generalities

- The commutation of the transfer matrices implies:

\[
\max_{\eta} f_{\eta_1, \eta, t} + \tilde{f}_{\eta, \eta_2, \tilde{t}} = \max_{\eta} \tilde{f}_{\eta_1, \eta, \tilde{t}} + f_{\eta, \eta_2, t}
\]

for all \( t, \tilde{t} \) and boundary conditions \( \eta_1, \eta_2 \).

- Recall that \( f \) is the Hamilton-Jacobi action.
Generalities

- The commutation of the transfer matrices implies:

\[
\max_{\eta} f_{\eta_1, \eta, t} + f_{\eta, \eta_2, \tilde{t}} = \max_{\eta} f_{\eta_1, \eta, \tilde{t}} + f_{\eta, \eta_2, t}
\]

for all \( t, \tilde{t} \) and boundary conditions \( \eta_1, \eta_2 \).

- Recall that \( f \) is the Hamilton-Jacobi action.

- Generally:
  If the Hamilton-Jacobi actions of \( H \) and \( \tilde{H} \) commute in the above sense, then does \( \{ H, \tilde{H} \} \)?

- Generally no, but under mild assumptions then yes,
Further Work

Integrability

- The existence of commuting transfer matrices underlies the solvability of the six vertex model by Bethe Ansatz.
Further Work

Integrability

- The existence of commuting transfer matrices underlies the solvability of the six vertex model by Bethe Ansatz.
- In the infinite dimensional setting, the Liouville integrability (the existence of many commuting Hamiltonians) is not enough to have the complete solvability.
Further Work

Integrability

• The existence of commuting transfer matrices underlies the solvability of the six vertex model by Bethe Ansatz.
• In the infinite dimensional setting, the Liouville integrability (the existence of many commuting Hamiltonians) is not enough to have the complete solvability.
• The existence of commuting hamiltonians is first step towards showing the integrability of the limit shape PDE.