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Height fluctuations for the stationary KPZ equation

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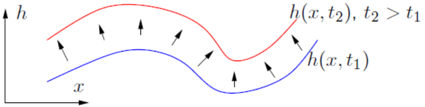
- Surface described by a height function $h(x,t)$, $x \in \mathbb{R}^d$ the space, $t \in \mathbb{R}$ the time



- Models with local growth + smoothing mechanics

$\Rightarrow$ macroscopic growth velocity v is a function of the slope only:

$$\frac{\partial h}{\partial t} = v(\nabla h)$$

- Example: Isotropic growth

$$v(\nabla h) = v(0)\sqrt{1 + (\nabla h)^2}$$

Nematic liquid crystals: stable (black) vs metastable (gray) cluster

Takeuchi,Sano'10: PRL 104, 230601 (2010)

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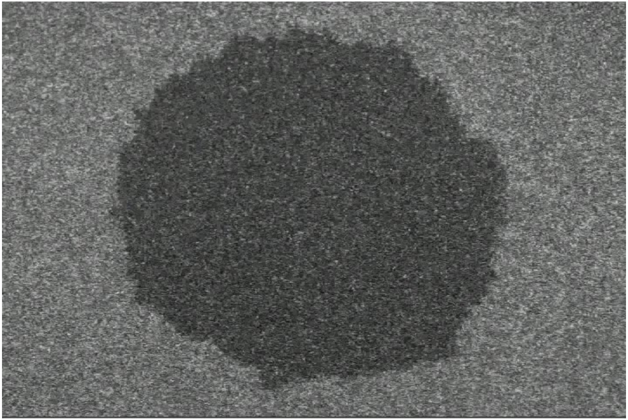

The KPZ equation

- The Kardar-Parisi-Zhang (KPZ) equation is one of the models in the KPZ universality class, class of irreversible stochastic random growth models. Kardar,Parisi,Zhang'86

- The KPZ equation writes (by a choice of parameters) in one-dimension is

$$\partial_T h = \frac{1}{2}\partial_X^2 h + \frac{1}{2}(\partial_X h)^2 + \dot{W}$$

where $\dot{W}$ is the space-time white noise

- Stationary initial conditions are any two-sided Brownian motion with drift fixed $b \in \mathbb{R}$.

- KPZ equation

$$\partial_T h = \tfrac{1}{2}\partial_X^2 h + \tfrac{1}{2}(\partial_X h)^2 + \dot{W}$$

⇒ Problem in defining the object $(\partial_X h)^2$.
For a way of doing it, see Hairer's work Hairer'11

- Setting $h = \ln \mathcal{Z}$ (and ignoring the Itô-correction term) one gets the (well-defined) Stochastic Heat Equation (SHE):

$$\partial_T \mathcal{Z} = \tfrac{1}{2}\partial_T^2 \mathcal{Z} + \mathcal{Z}\dot{W}$$

- Given the solution of the SHE with initial condition $\mathcal{Z}(0,X) := e^{h(0,X)}$, one calls

$$h(T,X) = \ln(\mathcal{Z}(T,X))$$

the Cole-Hopf solution of the KPZ equation.

- KPZ equation

$$\partial_T h = \tfrac{1}{2}\partial_X^2 h + \tfrac{1}{2}[(\partial_X h)^2 - \infty] + \dot{W}$$

$\Rightarrow$ Problem in defining the object $(\partial_X h)^2$.
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- The Feynmann-Kac formula gives

$$\mathcal{Z}(T,X) = \mathbb{E}_{T,X}\left(\mathcal{Z}_0(\pi(0)) : \exp : \left\{-\int_0^T ds \dot{W}(\pi(s),s)\right\}\right)$$

where the expectation is with respect Brownian paths, π, backwards in time with $\pi(T) = X$.

- Interpretation: $\mathcal{Z}$ is a partition function of the random directed polymer π with energy given by the white noise "seen" by it. This is called Continuous Directed Random Polymer model (CDRP), the universal scaling limit of directed polymers.

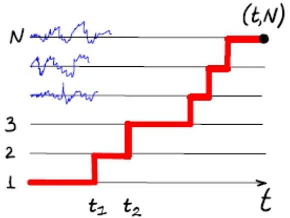
- Goal: obtain a reasonably explicit formula (solved problem) for

$$\mathbb{P}(h(T,X) \leq s)$$

or the law of the process $X \mapsto h(T,X)$ (open problem).

- One possible approach: start with any directed polymer model which converges under an appropriate limit to the CDRP.

We consider now the following semi-discrete directed polymer model at positive temperature O'Connell-Yor'01



- Path measure P_0: Continuous time one-sided simple random walk from $(0,1)$ to (t,N).

- Random media: $B_1, B_2, \ldots, B_N$ be independent standard Brownian motions. The energy is given by

$$-E(\pi) = B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))$$

- Boltzmann weight: $\mathbb{P}(\pi) = Z(t,N)^{-1}e^{-E(\pi)}P_0(\pi)$

$$Z(t,N) := \int_{0<t_1<t_2<\ldots<t_{N-1}<t} e^{B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))}dt_1\ldots dt_{N-1}.$$

- Recall the partition function

$$Z(t,N) = \int_{0<t_1<t_2<\ldots<t_{N-1}<t} e^{B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))} dt_1 \ldots dt_{N-1}.$$

- Law of large numbers: for any $\kappa > 0$,

$$f(\kappa) := \lim_{N\to\infty} \frac{1}{N} \ln Z(\kappa N, N) = \inf_{t>0} (\kappa t - (\ln \Gamma)'(t)).$$

O'Connell-Yor'01;Moriarty,O'Connell'07

- Fluctuations: in agreement with KPZ universality conjecture, for some known $c(\kappa) > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\ln Z(\kappa N, N) - N f(\kappa)}{c(\kappa) N^{1/3}} \leq r \right) = F_{\mathrm{GUE}}(r)$$

where F_{GUE} is the GUE Tracy-Widom distribution function

Borodin,Corwin,Ferrari'12

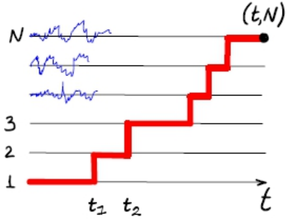
Recall that

$$Z(t,N) := \int_{0<t_1<t_2<\ldots<t_{N-1}<t} e^{B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))} dt_1 \ldots dt_{N-1}.$$

- The quantity $u(t,N) := e^{-t} Z(t,N)$ satisfies

$$\partial_t u(t,N) = (u(t,N-1) - u(t,N)) + u(t,N)\dot{B}_N(t)$$

with initial condition $u(0,N) = \delta_{1,N}$.



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- Its continuous analogue is the CDRP, where P_0 is the law of a Brownian Bridge from $(0,0)$ to (T,X), and the random noise is white noise $\dot{W}$. Its partition function $Z(T,X)$ satisfy

$$\partial_T Z = \frac{1}{2}\partial_X^2 Z + Z\dot{W}$$

  with initial conditions $Z(0,X) = \delta_0(X)$.

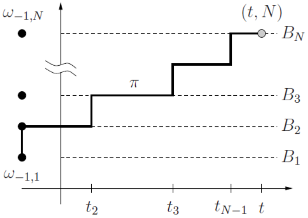
- Q: How to get a stationary situation?
- A: Use Burke-type results

O'Connell,Yor'01; Seppäläinen,Valkó'10

(1) Replace $B_1(t)$ with $B_1(t) + at$

(2) Add boundary weights at $(-1, n)$ given by $\omega_{-1,n} \sim -\ln\Gamma(\alpha)$ for $n \geq 2$ and $\omega_{-1,1} = 1$.

$\Rightarrow$ This gives the partition function $Z(t, N)$



(3) Stationarity is recovered with $a = \alpha$

- To recover the CDRP from the semi-discrete model

Step 1: We find an expression, with $\alpha > a$, for

$$\mathbb{E}(e^{-uZ(t,N)})$$

Step 2: Take the scaling

$$t = \sqrt{TN} + X, \quad a = \sqrt{N/T} + 1/2 + b, \quad \alpha = \sqrt{N/T} + 1/2 + \beta$$

and by Quastel,Remenik,Moreno-Flores

$$\frac{Z(\sqrt{TN} + X, N)}{C(N, X, T)} \Rightarrow \mathcal{Z}_{b,\beta}(T, X)$$

with C an explicit function, $\mathcal{Z}_{b,\beta}(0, X) = \exp(B(X))$ with the Brownian motion B having a drift b on $\mathbb{R}_+$ and β on $\mathbb{R}_-$.

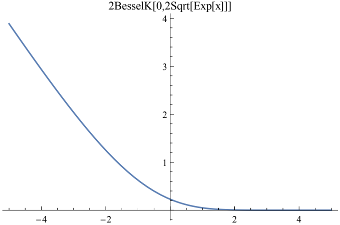
Step 3: Take the $\beta \to b$ limit through analytic continuation

Theorem (For simplicity, case of drift $b = 0$, position $X = 0$.)

Let $h(T, X)$ be the stationary solution to the KPZ equation and let K_0 denote the modified Bessel function. Then, for $T > 0$, $\sigma = (2/T)^{1/3}$ and $S \in \mathbb{C}$ with positive real part,

$$\mathbb{E}\left[2\sigma K_0\left(2\sqrt{S\,\exp\left\{\tfrac{T}{24} + h(T,0)\right\}}\right)\right] = f\left(S, \sigma\right),$$

where the function f is explicit.



Define on $\mathbb{R}_+$ the function

$$Q(x) = \frac{-1}{2\pi \mathrm{i}} \int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}} \mathrm{d}w \frac{\sigma\pi S^{-\sigma w}}{\sin(\pi\sigma w)} e^{-w^3/3+wx} \frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)},$$

and the kernel

$$\bar{K}(x,y) = \frac{1}{(2\pi\mathrm{i})^2} \int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}} \mathrm{d}w \int_{\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}} \mathrm{d}z \frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))} \frac{e^{z^3/3-zy}}{e^{w^3/3-wx}} \frac{\Gamma(-\sigma z)}{\Gamma(\sigma z)} \frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)}.$$

Let $\gamma_{\mathrm{E}} = 0.577\ldots$ be the Euler constant, define

$$\boxed{\begin{aligned} f(S,\sigma) = &-\det(\mathbb{1}-\bar{K})\Big[\sigma(2\gamma_{\mathrm{E}}+\ln S) \\ &+ \big\langle(\mathbb{1}-\bar{K})^{-1}(\bar{K}1+Q),1\big\rangle + \big\langle(\mathbb{1}-\bar{K})^{-1}(1+Q),Q\big\rangle\Big]. \end{aligned}}$$

where the determinants and scalar products are all meant in $L^2(\mathbb{R}_+)$.

Corollary

For any $r \in \mathbb{R}$, we have

$$\mathbb{P}\left(h(T,0) \le -\frac{T}{24} + r\,(T/2)^{1/3}\right) = \frac{1}{\sigma^2}\frac{1}{2\pi \mathrm{i}}\int_{-\delta+\mathrm{i}\mathbb{R}} \frac{\mathrm{d}\xi}{\Gamma(-\xi)\Gamma(-\xi+1)}\int_{\mathbb{R}} \mathrm{d}x\, e^{x\xi/\sigma} f\left(e^{-\frac{x+r}{\sigma}}, \sigma\right)$$

for any $\delta > 0$ and where $\sigma = (2/T)^{1/3}$.

There is another representation obtained in Sasamoto,Imamura'12. It is obtained by (non-rigorous) replica approach, but equality after the replica step of the computation has been verified.

Corollary (For simplicity, here just $b = 0$ and $X = 0$)

For any $r \in \mathbb{R}$,

$$\lim_{T\to\infty} \mathbb{P}\left(h(T,0) \leq -\frac{T}{24} + r(T/2)^{1/3} \right) = F_0(r),$$

where F_0 is the Baik-Rains distribution given by

$$F_0(r) = \frac{\partial}{\partial r}\left(g(r) F_{\mathrm{GUE}}(r) \right),$$

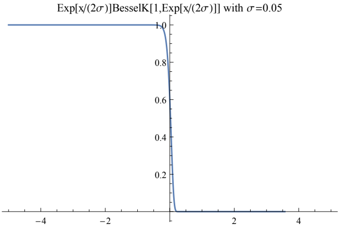
with F_{GUE} is the GUE Tracy-Widom distribution and $g(r)$ is an explicitly known function.

Results for one-point distribution in other KPZ models
Baik,Rains'00; Sasamoto,Imamura'04; Prähofer,Spohn'04; Ferrari,Spohn'05 Results for multi-point distributions
Baik,Ferrari,Péché'10; Ferrari,Spohn,Weiss'15

- To get the large time limit we do not employ the inversion formula.
- Let $\sigma = (2/T)^{1/3}$ and the rescaled height function $\tilde{h} = \sigma(h(T,0) + T/24)$. Let $S = e^{-r/\sigma}$. Then

$$\mathbb{E}\left(2\sigma K_0(e^{(\tilde{h}-r)/(2\sigma)})\right) = \int_{\mathbb{R}} dx \mathbb{P}(\tilde{h} \le x) e^{(x-r)/(2\sigma)} K_1(e^{(x-r)/(2\sigma)})$$

$$\simeq \int_{-\infty}^{r} \mathbb{P}(\tilde{h} \le x) \to F_{\mathrm{GUE}}(r) g(r).$$



For $s \in \mathbb{R}$, define

$$\begin{aligned}\mathcal{R} &= s + \int_s^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \mathrm{Ai}(x+y),\\ \Psi(y) &= 1 - \int_0^\infty \mathrm{d}x \mathrm{Ai}(x+y),\\ \Phi(x) &= \int_0^\infty \mathrm{d}\lambda \int_s^\infty dy \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \int_0^\infty \mathrm{d}y \mathrm{Ai}(y+x).\end{aligned}$$

Let $P_s(x) = \mathbb{1}_{\{x>s\}}$ and the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda).$$

Define the function

$$g(s) = \mathcal{R} - \big\langle (\mathbb{1} - P_s K_{\mathrm{Ai}} P_s)^{-1} P_s \Phi, P_s \Psi \big\rangle.$$

- How to get the main result: consider the semidirected polymer model.

Step 1: Start with $\alpha > a$ and add an extra (independent) weight $\omega(-1,1) \sim -\ln\Gamma(\alpha - a)$. Thus

$$\widetilde{Z}(t,N) \equiv Z(t,N) e^{\omega(-1,1)}$$

In this setting we get first a formula of the form (see later)

$$\mathbb{E}\left[e^{-u\widetilde{Z}(t,N)}\right] = \det(\mathbb{1} + K_u)$$

Step 2: An elementary explicit computation (recall $\widetilde{Z}(t,N) \equiv Z(t,N)e^{\omega(-1,1)}$) gives then

Corollary

For $\alpha > a$,

$$\mathbb{E}\left[2\big(u\,Z(t,N)\big)^{\frac{\alpha-a}{2}} K_{-(\alpha-a)}\Big(2\sqrt{u\,Z(t,N)}\Big)\right] = \Gamma(\alpha-a)\,\mathbb{E}\left[e^{-u\widetilde{Z}(t,N)}\right],$$

where K_ν is the modified Bessel function of order ν.

Step 3: In

$$\mathbb{E}\left[2\big(u\,Z(t,N)\big)^{\frac{\alpha-a}{2}}K_{-(\alpha-a)}\Big(2\sqrt{u\,Z(t,N)}\Big)\right] = \Gamma(\alpha-a)\,\mathbb{E}\left[e^{-u\widetilde{Z}(t,N)}\right],$$

taking $N\to\infty$ under the scaling

$$t=\sqrt{TN}+X,\quad a=\sqrt{N/T}+1/2+b,\quad \alpha=\sqrt{N/T}+1/2+\beta$$

leads to ...

Theorem

Let us denote by $\mathcal{Z}_{b,\beta}(T,0)$ the solution to the SHE/KPZ equation with initial data $\mathcal{Z}_0(X)=\exp(B(X))$, where $B(X)$ is a two-sided Brownian motion with drift β to the left of 0 and drift b to the right of 0, with $\beta > b$. Then, for $S>0$,

$$\mathbb{E}\left[2\left(Se^{\frac{T}{24}}\mathcal{Z}_{b,\beta}(T,0)\right)^{\frac{\beta-b}{2}} K_{-(\beta-b)}\left(2\sqrt{Se^{\frac{T}{24}}\mathcal{Z}_{b,\beta}(T,0)}\right)\right] = \Gamma(\beta-b)\det(\mathbb{1}-K_{b,\beta})_{L^2(\mathbb{R}_+)}$$

where $K_\nu(z)$ is the modified Bessel function of order ν and

$$K_{b,\beta}(x,y)=\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathcal{C}_w}\mathrm{d}w\int_{\mathcal{C}_z}\mathrm{d}z\,\frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zy}}{e^{w^3/3-wx}}\frac{\Gamma(\beta-\sigma z)}{\Gamma(\sigma z-b)}\frac{\Gamma(\sigma w-b)}{\Gamma(\beta-\sigma w)}$$

where $\sigma=(2/T)^{1/3}$.

Step 4: Recover the stationary initial condition by taking the $\beta \downarrow b$ limit:

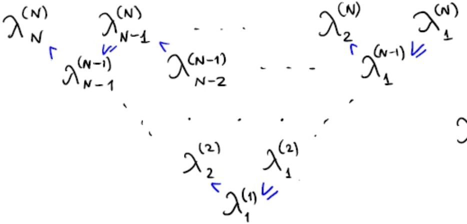
- r.h.s.: analytic continuation (to be singled out: a factor $1/(\beta - b)$ from the Fredholm determinant)
- l.h.s.: analytic continuation and a-priori bound on the left-tail of $\ln \mathcal{Z}_{b,\beta}$ Corwin, Hammond'13

- Q: How to get the starting formula, namely

$$\mathbb{E}\left[e^{-u\widetilde{Z}(t,N)}\right] = \det(\mathbb{1} + K_u)$$

for the semi-directed polymer?

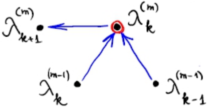
- The configurations are elements on



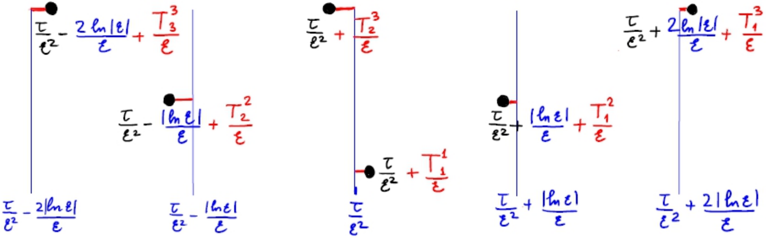
$$\lambda_j^{(m)} \in \mathbb{Z}$$

- Let $q \in (0,1)$ be fixed. Particle $\lambda_k^{(m)}$ jumps to the right with rate

$$\mathrm{rate}\left(\lambda_k^{(m)}\right) = \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}-1}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}+1}\right)}$$



- Set $q = e^{-\varepsilon}$ and look at time $t = \tau/\varepsilon^2$.
- As $\varepsilon \to 0$,



- In particular, $T_1^N = \ln Z(\tau, N)$ in distribution.

Borodin,Corwin'11

- Goal: get a generating function for the q-Whittaker with specialization $\rho(\alpha, 0, \gamma)$ with $q \in (0,1)$.

Step 1: Start with specialization $\rho(0, \beta, 0)$ with β having a finite number of non-zero entries

$$\mathbb{E}\left(\frac{1}{(\zeta q^{-\lambda_1^N}; q)_\infty}\right) = \mathbb{E}\left(\sum_{k\geq 0} \frac{\zeta^k q^{-k\lambda_1^N}}{(q;q)_k}\right)$$

$$= \sum_{k\geq 0} \frac{\zeta^k \mathbb{E}(q^{-k\lambda_1^N})}{(q;q)_k} = \det(1 + K_\zeta)$$

- Remark: For the $\rho(\alpha, 0, \gamma)$ case, our model, $\mathbb{E}(q^{-k\lambda_1^N}) = \infty$ for $k \geq k_0(q)$.
- The key step which is non-rigorous in the replica-type approach is the exchange of $\mathbb{E}$ and $\sum_{k\geq 0}$.

Step 2: See that for general specializations, both lhs/rhs can be expanded in formal power series

$$\det(1+K_\zeta) = \sum_\lambda R_\lambda p_\lambda(\rho(\alpha,\beta,\gamma)),$$

and by the full power of Macdonald processes

Borodin, Corwin '11

$$\mathbb{E}\left(\frac{1}{(\zeta e^{-\lambda_1^N};q)_\infty}\right) = \sum_\lambda L_\lambda p_\lambda(\rho(\alpha,\beta,\gamma))$$

with R_λ and L_λ independent of the $\rho(\alpha,\beta,\gamma)$.

Step 3: By Step 1, we have $R_\lambda = L_\lambda$. Using this and Step 2 for $\rho(\alpha,0,\gamma)$ one gets the result.