A class of $(2+1)$-dimensional growth processes with explicit stationary measure

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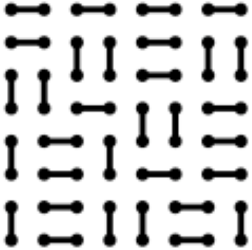
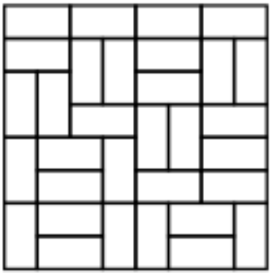
Plan

- Dimer models (perfect matchings) and height function
- Irreversible dynamics: a $(2+1)$-d random growth model
- Speed and fluctuations

Perfect matchings of bipartite planar graphs





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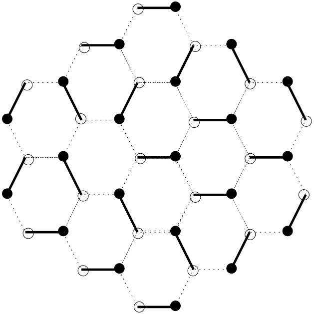
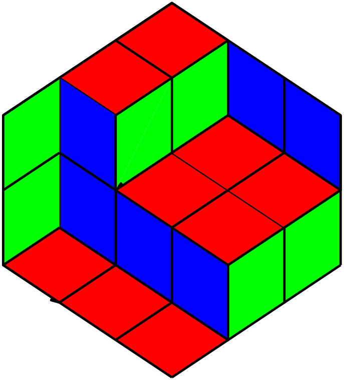
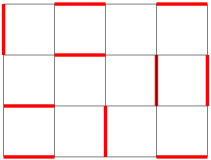
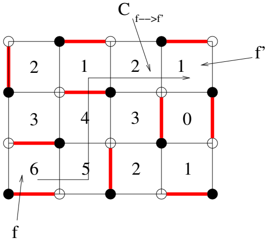




Height function





Height function:

$$h(f') - h(f) = 4 \sum_{e \in C_{f \to f'}} \sigma_e (1_{e \in M} - 1/4)$$

where $\sigma_e = +1/-1$ if e crossed with white on the right/left.

Definition is path-independent.

Ergodic Gibbs measures [Kenyon-Okounkov-Sheffield]

- Choose $\rho = (\rho_1, \rho_2, \rho_3)$ with $\rho_i \in (0, 1), \rho_1 + \rho_2 + \rho_3 = 1$. There exists a unique translation invariant, ergodic Gibbs measure π_ρ s.t. the density of horizontal, NW and NE lozenges are ρ_1, ρ_2, ρ_3.

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$$\pi_\rho(1_{e \in M}; 1_{e' \in M}) \approx |e - e'|^{-2}$$

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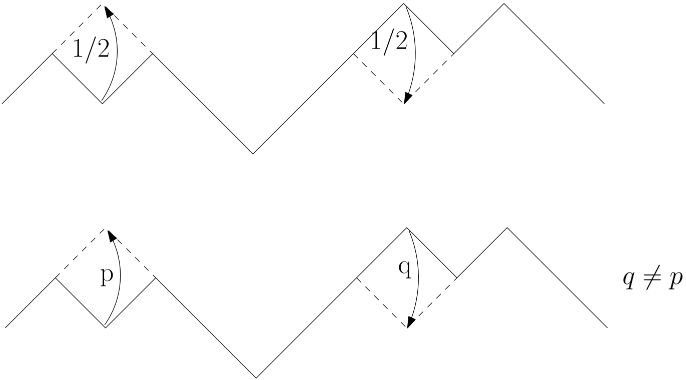
- height function converges to GFF: if $\int_{\mathbb{R}^2} \varphi(x)dx = 0$ then

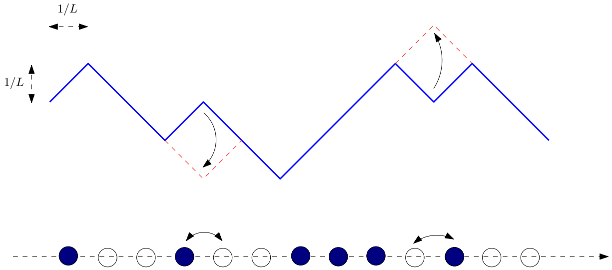
$$\epsilon^2 \sum_x \varphi(\epsilon x) h_x \stackrel{\epsilon \to 0}{\longrightarrow} \int \varphi(x) X(x) dx$$

with $\langle X(x) X(y) \rangle = -\frac{1}{2\pi^2} \log |x - y|$.

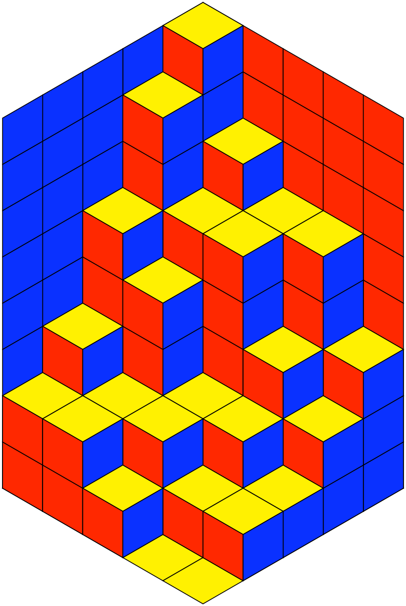
Symmetric vs. asymmetric random dynamics

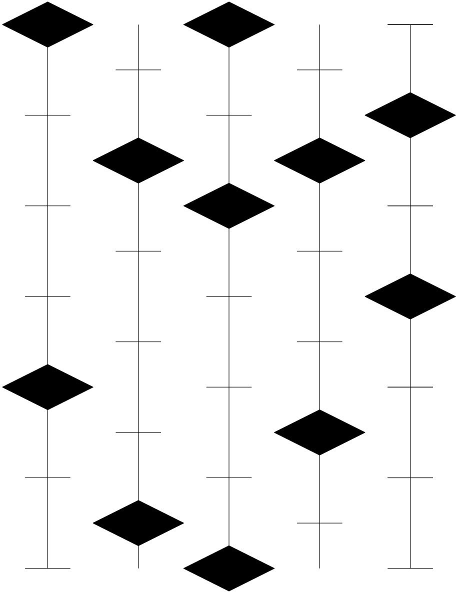


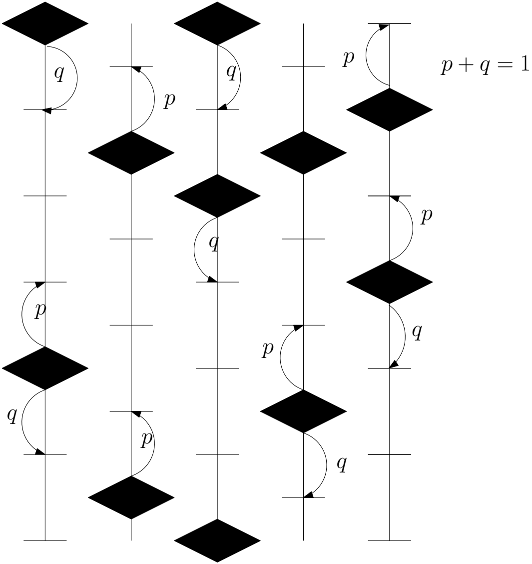
For $d = 1$: Symmetric vs. Asymmetric Simple Exclusion Process



In both SSEP/ASEP, Bernoulli(ρ) are invariant.
For $p \neq q$, irreversibility (particle flux).







q
p
q
p
p + q = 1
q
p
p
q
p
q
p
q

Asymmetric cube deposition/evaporation dynamics

- If $p = q$, Gibbs states are invariant (no surprise; reversibility)
- if $p \neq q$, stationary states presumably very different from π_ρ. Numerical simulations [Forrest-Tang-Wolf 1992] show $\approx t^{0.24\cdots}$ growth of height fluctuations.

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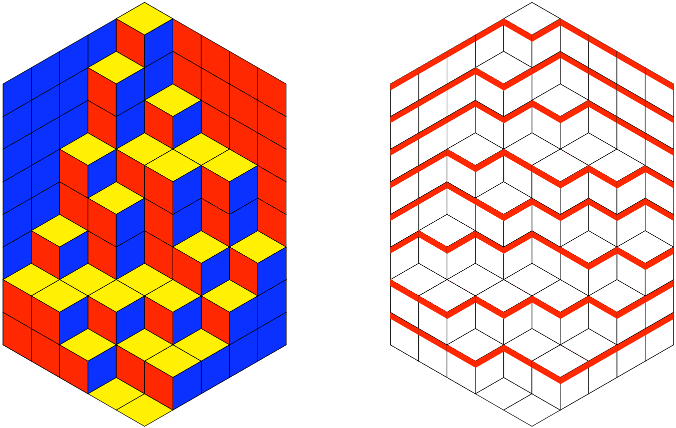
- large-scale dynamics should be described by "isotropic two-dimensional KPZ equation":

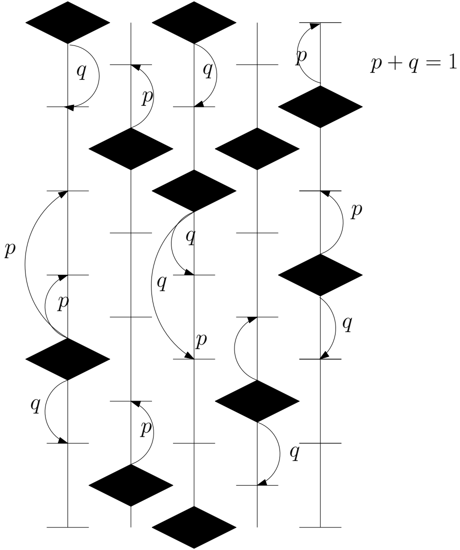
$$\partial_t h = \nu \Delta h + Q(\nabla h) + \text{white noise}$$

with Q a positive-definite quadratic form (whatever mathematical sense this equation has...)

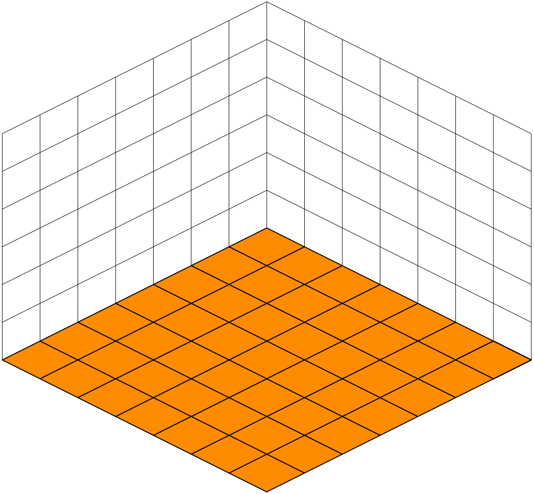
Coupled simple exclusions with constraints



A two-dimensional generalization of Hammersley process



Dynamics well defined?



Particles can leave to ∞ in infinitesimal time

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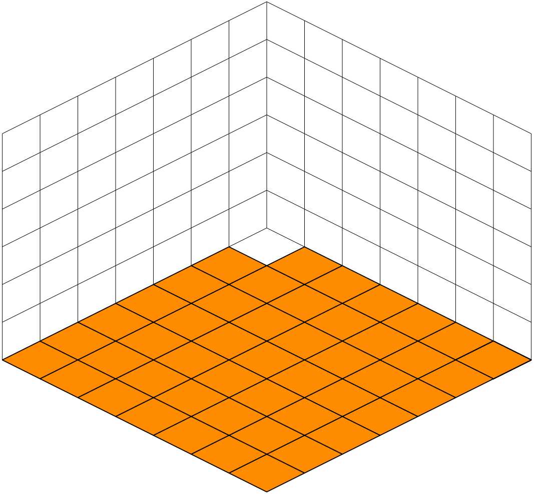

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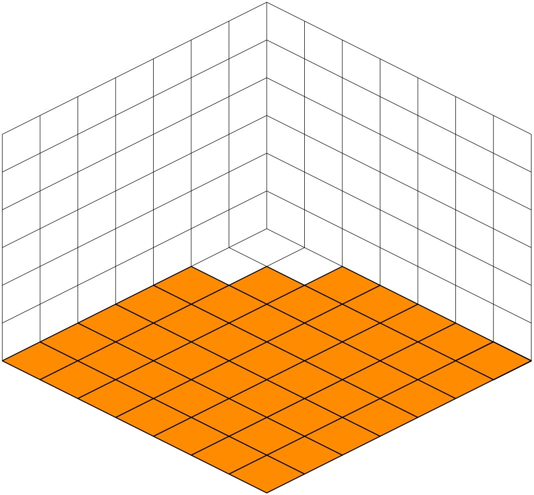

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- Generalization to domino tilings

Comments

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  Exact computations (explicit kernel for some time-space correlations)

- large-scale dynamics should be described by "**anisotropic** two-dimensional KPZ equation":

$$\partial_t h = \nu \Delta h + Q(\nabla h) + \text{white noise}$$

  with Q a $(+,-)$-definite quadratic form.
  Physics literature [Wolf '91]: non-linearity irrelevant.

Comments

- BF '08 obtain hydrodynamic limit and $\sqrt{\log t}$ Gaussian fluctuations

$$\lim_{L\to\infty} \frac{1}{L}\mathbb{E}h(xL, yL, \tau L) = \mathbf{h}(x, y, \tau)$$

with

$$\partial_\tau \mathbf{h} = v(\nabla \mathbf{h})$$

and

$$\frac{1}{\sqrt{\log L}}[h(xL, yL, \tau L) - \mathbb{E}(h(xL, yL, \tau L))] \Rightarrow \mathcal{N}(0, \sigma^2);$$

moreover, convergence of local statistics to that of a Gibbs measure.

Invariance on the torus

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Call I^+_n set of available positions above/below for particle n.

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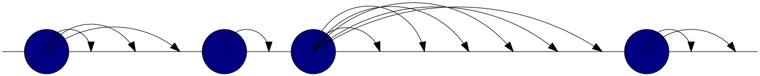
Key fact:
Lemma: The probability of seeing an inter-particle gap $\geq \log R$ within distance R from the origin before time 1 is $O(R^{-K})$ for every K.

Comparison with the Hammersley process (HP)



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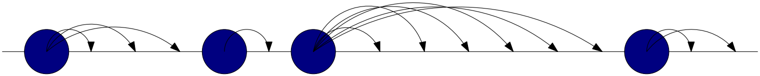

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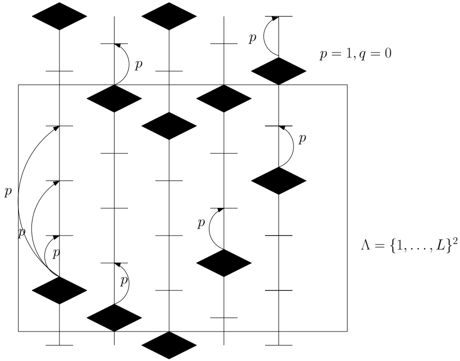
Lozenge dynamics $\sim$ infinite set of coupled Hammersley processes. Comparison: lozenges move less than HP particles

Fluctuations



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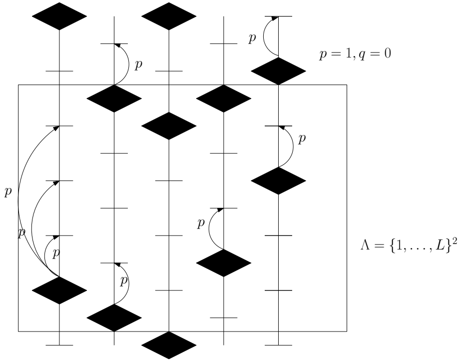

Let $Q_\Lambda(t) = \sum_{x \in \Lambda}(h_x(t) - h_x(0))$.

$$\frac{d}{dt}\langle Q_\Lambda(t)\rangle = \langle K_\Lambda(\sigma_t)\rangle := \langle \sum_x |V(x,\uparrow) \cap \Lambda|(t)\rangle = v|\Lambda|$$

Fluctuations

Similarly, one can prove

$$\frac{d}{dt}\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)^2\rangle = 2\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)(K_\Lambda(\sigma_t) - \pi_\rho(K_\Lambda))\rangle$$

$$+\pi_\rho(\sum_x |V(x,\uparrow) \cap \Lambda|^2)$$

$$\leq 2\sqrt{\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)^2\rangle}\sqrt{Var_{\pi_\rho}(K_1)} + O(L^2)$$

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Equilibrium estimate:

$Var_{\pi_\rho}(K_1) = O(L^{2+\delta})$ or $= O(L^2 \log L)$ for some slopes.

Fluctuations

Therefore,

$$\frac{d}{dt}\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)^2\rangle \leq 2\sqrt{\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)^2\rangle} L^{1+\delta} + O(L^2)$$

so that

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If we choose $L = T$ we get instead $\psi(T) = O(T^\delta)$ as wished.

Thanks!