Exact results from the replica Bethe ansatz from KPZ growth and random directed polymers

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Thomas Gueudre (LPTENS,Torino)

most recent: Andrea de Luca (LPTENS,Orsay)

Thimothee Thiery (LPTENS)

- many discrete models in "KPZ class" exhibit universality related to random matrix theory: Tracy Widom distributions: of largest eigenvalue of GUE,GOE..

RBA: method integrable systems (Bethe Ansatz) +disordered systems(replica)

- provides solution directly continuum KPZ eq./DP (at all times)

KPZ eq. is in KPZ class !

- also to discrete models => allowed rigorous replica

Outline:

- KPZ equation, KPZ class, random matrices,Tracy Widom distributions.
- solving KPZ at any time by mapping to directed paths
  then using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => large time TW distrib. for KPZ height

- droplet initial condition

P. Calabrese, PLD, A. Rosso EPL 90 20002 (2010)

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- flat initial condition

P. Calabrese, PLD, PRL 106 250603 (2011)
J. Stat. Mech. P06001 (2012)

- KPZ in half space

T. Gueudre, PLD, EPL 100 26006 (2012).

- Integrable directed polymer models on square lattice

T. Thiery, PLD, J.Stat. Mech. P10018 (2014) and arXiv1506.05006

- Non-crossing probability of directed polymers

A. De Luca, PLD, arXiv1505.04802.

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- other works/perspectives:

V. Dotsenko, H. Spohn, Sasamoto

(math) Amir, Corwin, Quastel, Borodine,..

also G. Schehr, Reymenik, Ferrari, O'Connell,..

not talk about: stationary initial condition

T. Inamura, T. Sasamoto PRL 108, 190603 (2012)

reviews KPZ: I. Corwin, H. Spohn..

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2}$ $x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

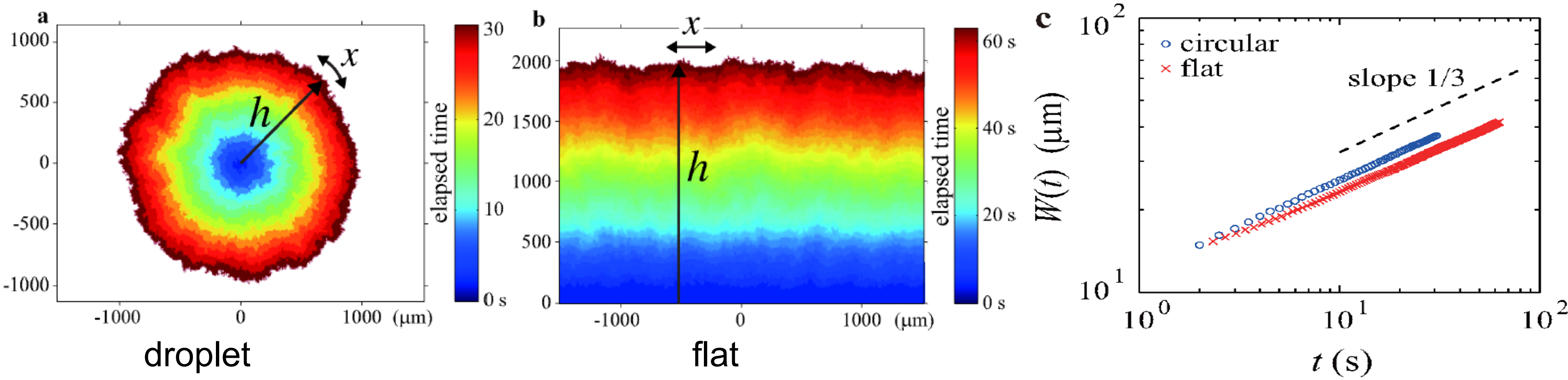
depends on some details of initial condition

flat	h(x,0) =0
wedge (droplet)	h(x,0) = - w |x|

$\lambda_0 = 0$ Edwards Wilkinson P(h) gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



droplet flat

$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t\to+\infty} v_\infty t + \chi\, t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

also reported in:

- slow combustion of paper — J. Maunuksela et al. PRL 79 1515 (1997)
- bacterial colony growth — Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)
- fronts of chemical reactions — S. Atis (2012)
- formation of coffee rings via evaporation — Yunker et al. PRL (2012)

Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $\quad i = 1, ..N$

$$P[\lambda] = c_{N,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

H is:

$\beta =$

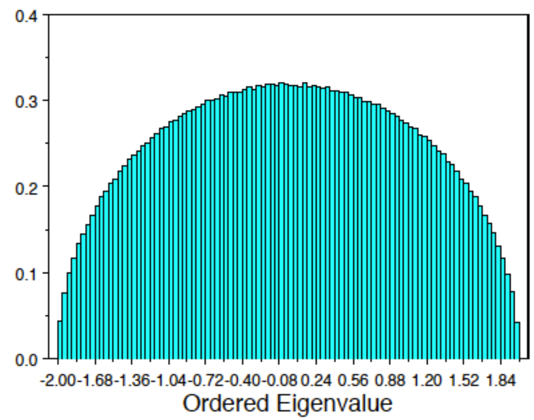
1 (GOE) real symmetric

2 (GUE) hermitian

4 (GSE) symplectic

Universality large N :

- DOS: semi-circle law



histogram of eigenvalues N=25000

- distribution of the largest eigenvalue

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

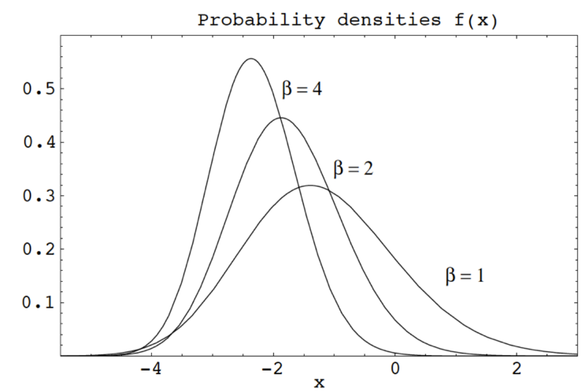
Tracy-Widom distributions (largest eigenvalue of RM)

Fredholm determinants

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$

GOE $$F_1(s) = Det[I - K_1]$$

$$K_1(x,y) = \theta(x) Ai(x+y+s)\theta(y)$$



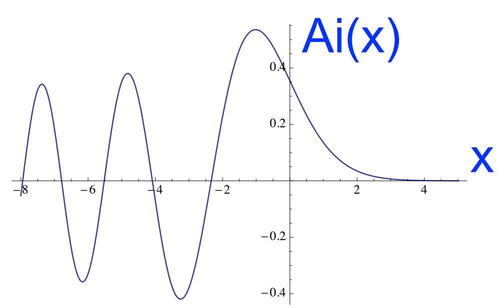
GUE $$F_2(s) = Det[I - K_2]$$

$$K_2(x,y) = K_{Ai}(x+s, y+s)$$

$$K_{Ai}(x,y) = \int_{v>0} Ai(x+v)Ai(y+v)$$

Ai(x-E)

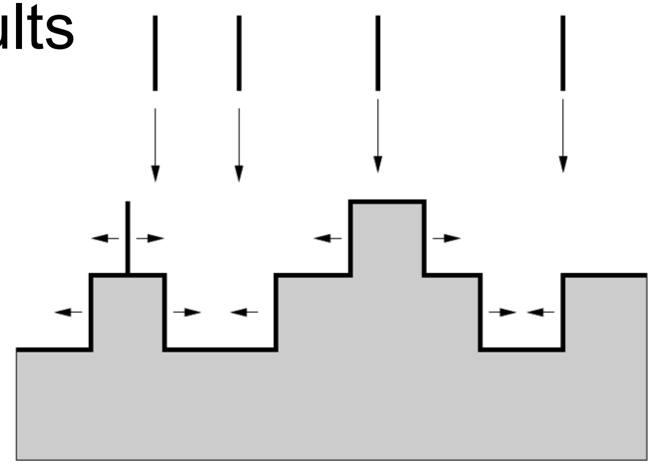
is eigenfunction E
particle linear potential



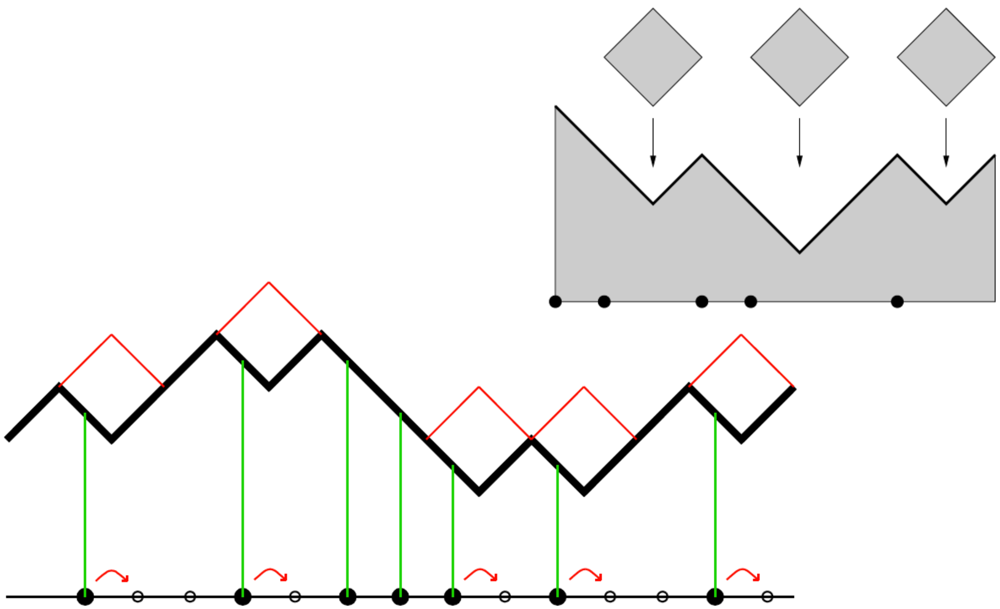
discrete models in KPZ class/exact results

- polynuclear growth model (PNG)

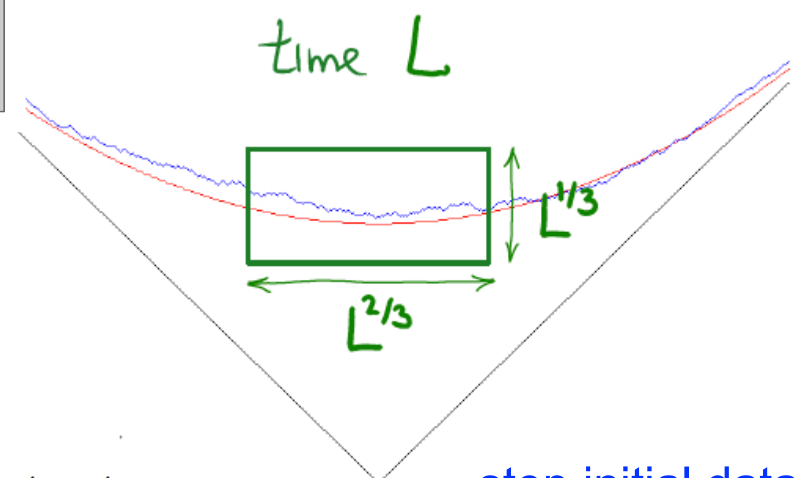
Prahofer, Spohn, Baik, Rains (2000)



- totally asymmetric exclusion process (TASEP)



Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as $\exp(-x)dx$.



step initial data

Johansson (1999)

Exact results for height distributions for some discrete models in KPZ class

- PNG model

Baik, Deft, Johansson (1999)

$$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$$

droplet IC GUE

Prahofer, Spohn, Ferrari, Sasamoto,.. (2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$A_1(y)$ GOE

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3} A_n(y)$$

- similar results for TASEP

Johansson (1999), ...

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Johansson (1999), ...

question: is KPZ equation in KPZ class ?

Cole Hopf mapping

KPZ equation ⟷ Continuum
Directed paths (polymers)
in a random potential

⇕

Quantum mechanics
of bosons
(imaginary time)

Kardar 87

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
  Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
  Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

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- **Flat** KPZ/Continuum DP one free endpoint (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Ortmann, J. Quastel and D. Remenik arXiv1407.8484 and arXiv 1503.05626

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- **Stationary** KPZ => Patrik Ferrari's talk

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

define:

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

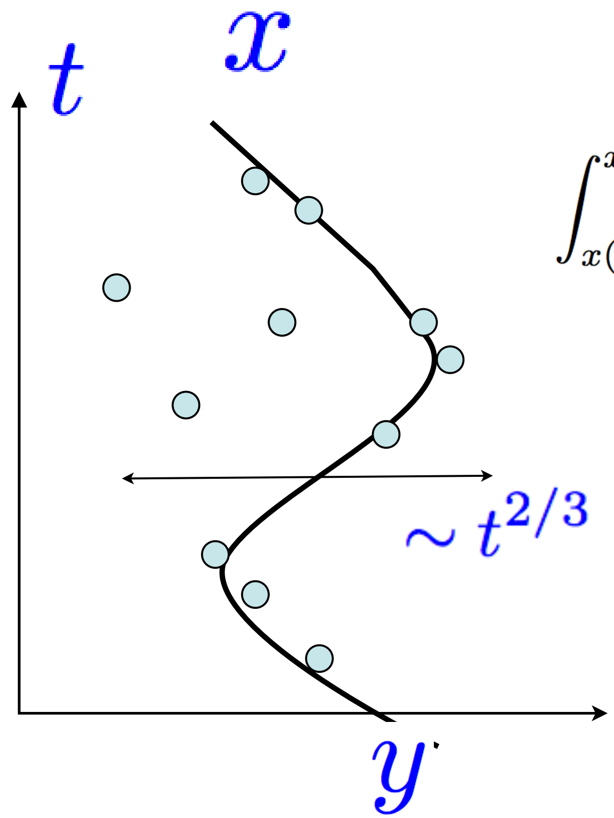
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)



$$Z(x,t|y,0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau \, \frac{\kappa}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau),\tau)}$$

$$\overline{V(x,t)V(x',t')} = \bar{c} \;\; \delta(t-t')\delta(x-x')$$

Feynman Kac

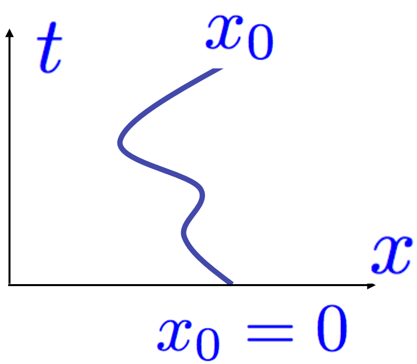
$$Z(x,y,t=0) = \delta(x-y)$$

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

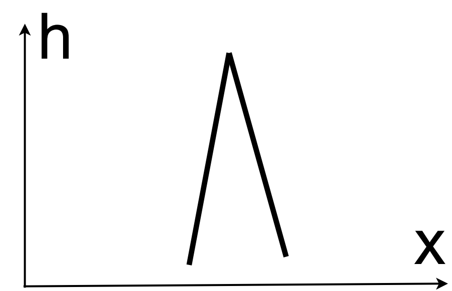
1) DP both fixed endpoints $Z(x_0,t|x_0,0)$



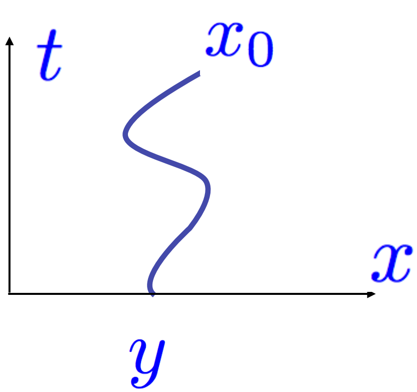
KPZ: narrow wedge <=> droplet initial condition

$$h(x,t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0,t|y,0)$



KPZ: flat initial condition

$$h(x,t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

calculate $$\overline{Z^n} = \int dZ\, Z^n P(Z) \qquad n \in \mathbb{N}$$

"guess" the probability distribution from its integer moments:

$$P(Z) \to P(\ln Z) \to P(h)$$

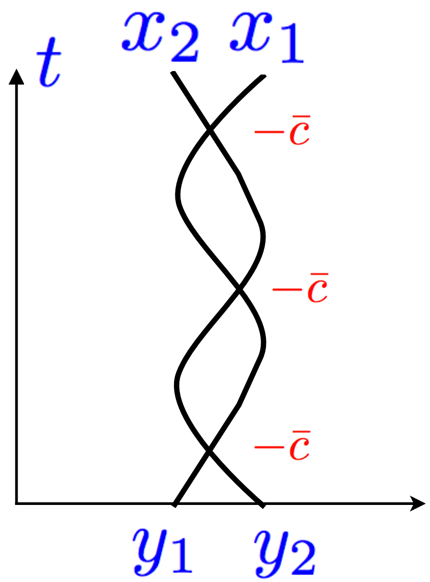
Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1,t|y_1,0)..Z(x_n,t|y_n0)} = \langle x_1,..x_n|e^{-tH_n}|y_1,..y_n\rangle$$

$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

drop the tilde..



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\leq i<j\leq n} \delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963) what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0 ... x_0 | e^{-tH_n} | x_0, .. x_0 >$$

symmetric states = bosons

$$= \sum_\mu \Psi^*_\mu(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2} e^{-E_\mu t}$$

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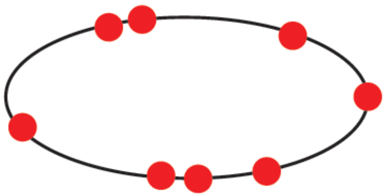
symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

- flat initial condition

$$\overline{(\int_y Z(x_0 t|y0))^n} = \sum_\mu \Psi_\mu^*(x_0, .x_0) \int_{y_1,.y_n} \Psi_\mu(y_1, .y_n)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

LL model: n bosons on a ring with local delta attraction



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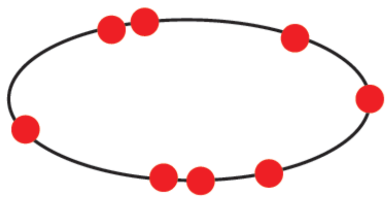

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Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^n \lambda_j^2 \qquad A_P = \prod_{n \ge \ell > k \ge 1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

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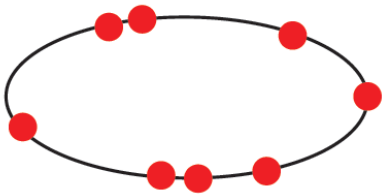

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They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1, ..x_n) \sim \exp(-\frac{\bar{c}}{2} \sum_{i<j} |x_i - x_j|)$$

$$E_0(n) = -\frac{\bar{c}^2}{12} n(n^2 - 1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \sim_{t \to \infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12} n^3 t}$$

exponent 1/3

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exponent 1/3

can it be continued in n ? NO !

information about the tail
of FE distribution

$$P(f) \sim_{f\to-\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

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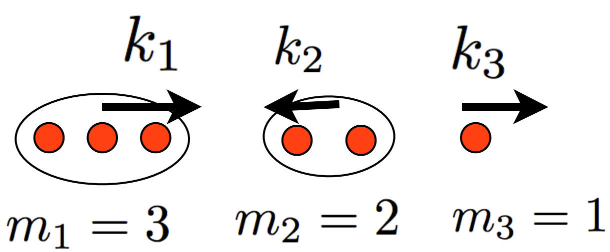
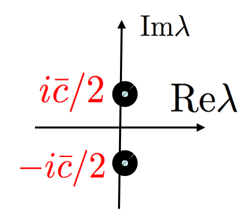
exponent 1/3

need to sum over all eigenstates !

- all eigenstates are: All possible partitions of n into ns “strings” each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$E_\mu = \sum_{j=1}^{n_s}(m_j k_j^2 - \frac{\bar{c}^2}{12}m_j(m_j^2-1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_{\mu} \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

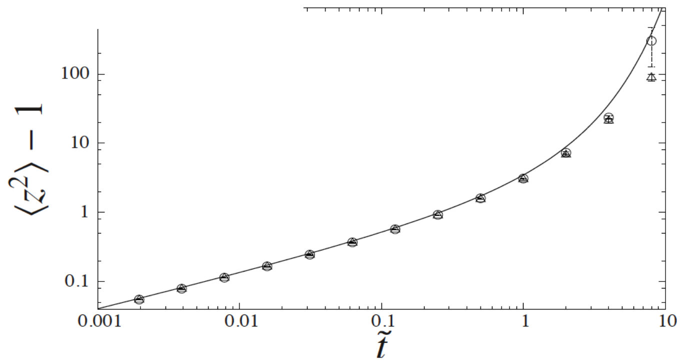
$$\Psi_\mu(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\dots m_{n_s})_n} \qquad n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2/4}$$



how to get P(ln Z) i.e. P(h) ?

$$\ln Z = -\lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3} \qquad f$$

random variable
expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f - x)} = Prob(f > x)$$

how to get P(ln Z) i.e. P(h) ?

$$\ln Z = -\lambda f \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3} \qquad f$$

random variable expected O(1)

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what we aim to calculate= Laplace transform of P(Z)

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \dots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

double Cauchy formula

$$det[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)}]$$

$$= \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s,x)=\prod_{j=1}^{n_s}\int_{v_j>0} dv_j \; det[K(v_j,v_\ell)]$$

$$\lambda=(\frac{\bar{c}^2}{4}t)^{1/3}$$

$$K(v_1,v_2)= \; -\int \frac{dk}{2\pi} dy Ai(y+k^2-x+v_1+v_2)e^{-ik(v_1-v_2)} \; \frac{e^{\lambda y}}{1+e^{\lambda y}}$$

$$g(x)=1+\sum_{n_s=1}^{\infty}\frac{1}{n_s!}Z(n_s,x) \;\; = Det[I+K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1,v_2)\equiv\theta(v_1)K(v_1,v_2)\theta(v_2)$$

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by an equivalent definition of a Fredholm determinant

$$K(v_1,v_2) \equiv \theta(v_1) K(v_1,v_2) \theta(v_2)$$

2) large time limit $\lambda = +\infty$ $\frac{e^{\lambda y}}{1+e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2+v+v') e^{ik(v-v')} = 2^{2/3}\pi Ai(2^{1/3}v) Ai(2^{1/3}v')$$

g(x)= $Prob(f > x = -2^{2/3}s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v,v') = \int_{y>0} Ai(v+y) Ai(v'+y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j+q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i+k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i-m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i+m_i-2ik_j-m_j}{2ik_i+m_i+2ik_j+m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j>0} \mathrm{Pf}[\mathbf{K}(v_i,v_j)]_{2n_s,2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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2) large time limit $\lambda = +\infty$

$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x+y+s)\theta(y)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1+v_i+s+4k^2)Ai(y_2+v_j+s+4k^2)[\frac{e^{-2i(v_i-v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)}) + \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2}\int_y Ai(y+s+v_i)(e^{-2e^{\lambda y}}-1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i-v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\, {}_1F_2\left(1; 2-2ik, 2+2ik; -z\right)}{\sinh\left(2\pi k\right)\Gamma\left(2-2ik\right)\Gamma\left(2+2ik\right)}\,, \qquad (19)$$

$$F(z_i,z_j) = \sinh(z_2-z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$
$$\times J_0(2\sqrt{z_1 z_2(1-u)})[z_1\sinh(z_1 u) - z_2\sinh(z_2 u)]\,.$$

$$g_\lambda(s) = \sqrt{Det(1-2K_{10})}(1+\langle\tilde{K}|(1-2K_{10})^{-1}|\delta\rangle)$$

$$K_{10}(v_1,v_2) = \partial_{v_1}K_{11}(v_1,v_2) \qquad K_{12}(v_1,v_2) = \tilde{K}(v_1)\delta(v_2)$$

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$$\times J_0(2\sqrt{z_1z_2(1-u)})[z_1\sinh(z_1u)-z_2\sinh(z_2u)]\,.$$

large time limit

$$\lim_{\lambda\to+\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda\to+\infty} F(2e^{\lambda y_1},2e^{\lambda y_2}) =$$

$$\theta(y_1+y_2)(\theta(y_1)\theta(-y_2)-\theta(y_2)\theta(-y_1))$$

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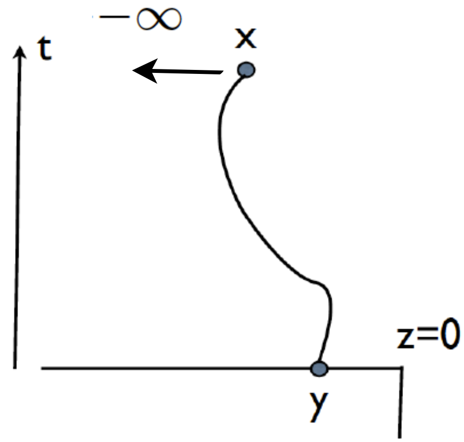
how to calculate $\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$

first method: flat as limit of half-flat (wedge)

$$\lim_{x\to-\infty, w\to 0} Z_{\rm hs}(x,t) \equiv Z_{\rm flat}(x,t)$$

$$Z_{\rm hs,w}(x,t) = \int_{-\infty}^{0} dy e^{wy} Z(x,t|y,0)$$

$$Z(x,t=0) = \theta(-x) e^{wx}$$



$$\Big(\prod_{\alpha=1}^{n} \int_{-\infty}^{0} dy_\alpha e^{wy_\alpha}\Big) \Psi_\mu(y_1 \dots y_n) = \sum_P A_P G_{P\lambda}$$

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$G_\lambda = \prod_{j=1}^{n} \frac{1}{jw + i\lambda_1 + .. + i\lambda_j}$$

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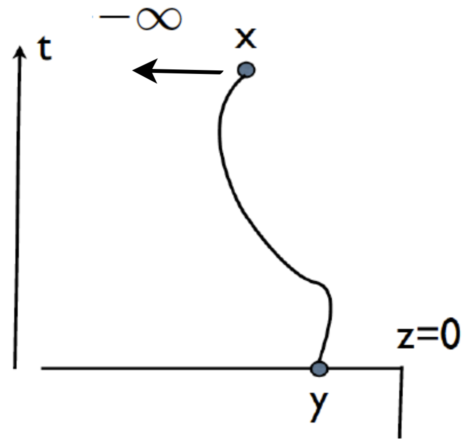

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miracle !

$$= \frac{n!}{\prod_{\alpha=1}^n (w+i\lambda_\alpha)} \prod_{1\le\alpha<\beta\le n} \frac{2w + i\lambda_\alpha + i\lambda_\beta - 1}{2w + i\lambda_\alpha + i\lambda_\beta}$$

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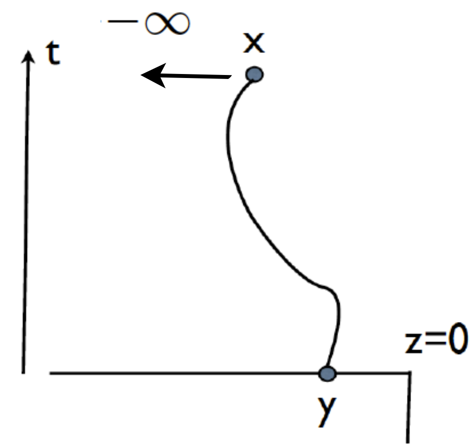

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strings:

$$\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j+1-2a)$$

$$a = 1,\dots,m_j$$

$$\int^{w} \Psi_{\mu} = n!(-2)^{n} \prod_{i=1}^{n_s} S^{w}_{m_i,k_i} \prod_{1 \le i < j \le n_s} D^{w}_{m_i,k_i,m_j,k_j}$$

$$D^{w}_{m_1,k_1,m_2,k_2} = (-1)^{m_2} \frac{\Gamma(1-z+\frac{m_1+m_2}{2})\Gamma(z+\frac{m_1-m_2}{2})}{\Gamma(1-z+\frac{m_1-m_2}{2})\Gamma(z+\frac{m_1+m_2}{2})} \qquad z = ik_1 + ik_2 + 2w$$

$$S^{w}_{m,k} = \frac{(-1)^m \Gamma(z)}{\Gamma(z+m)} \qquad z = 2ik + 2w.$$

in double limit $\lim_{x \to -\infty, w \to 0}$

expand the product $\prod_i S_i \prod_{i<j} D_{ij}$

each momentum k_ℓ appears only in exactly one pole

$$S^{w}_{m_i,k_i} \to \frac{(-1)^{m_i}}{2\Gamma(m_i)} 2\pi\delta(k_i) + s^{0}_{m_i,k_i}$$

$$D^{w}_{m_i,k_i,m_j,k_j} \to (-1)^{m_i} m_i \delta_{m_i,m_j} \; 2\pi\delta(k_i+k_j) + d^{w}_{m_i,k_i,m_j,k_j}$$

pairing of string momenta and pfaffian structure emerges second method:

calculate: $$\prod_{\alpha=1}^{n} \int_0^L dy_\alpha \Psi_\mu(y_1, .., y_n) = \langle \Phi_0 | \mu \rangle$$

is the overlap with uniform state

$$\Phi_0(x_1, ..x_n) = 1$$

use Bethe equations: $$e^{i\lambda_j L} = \prod_{\ell \neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

=> integral vanishes for generic state
oberve: requires pairs opposite rapidities

Can be seen as interaction quench in
Lieb-Liniger model with initial state BEC (c=0)

de Nardis et al., arXiv 1308.4310

overlap is non zero only for parity invariant states $\{\lambda_1, -\lambda_1, .., \lambda_{n/2}, -\lambda_{n/2}\}$

$$\langle \Phi_0 | \mu \rangle = n! c^{n/2} \prod_{\alpha=1}^{n/2} \frac{1}{\lambda_\alpha^2} \prod_{1 \le \alpha < \beta \le n/2} \frac{(\lambda_\alpha - \lambda_\beta)^2 + c^2}{(\lambda_\alpha - \lambda_\beta)^2} \frac{(\lambda_\alpha + \lambda_\beta)^2 + c^2}{(\lambda_\alpha + \lambda_\beta)^2} \times \det G^Q.$$

$$G^Q_{\alpha\beta} = \delta_{\alpha\beta}(L + \sum_{\gamma=1}^{n/2} K^Q(\lambda_\alpha, \lambda_\gamma)) - K^Q(\lambda_\alpha, \lambda_\beta)$$

$$K^Q(x,y) = K(x-y) + K(x+y),$$

$$K(x) = \frac{2c}{x^2 + c^2}.$$

Brockmann, arXiv1402.1471.

P. Calabrese, P. Le Doussal, arXiv 1402.1278

large L limit, overlap for strings

partially recovers the moments Z^n for flat

Summary: we found

for droplet initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + (\frac{t}{t^*})^{1/3}\chi$$

similar (more involved)

χ at large time has the same distribution as the largest eigenvalue of the GOE

$$t^* = \frac{8(2\nu)^5}{D^2\lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

large for weak noise, large diffusivity

GSE ?

Summary:

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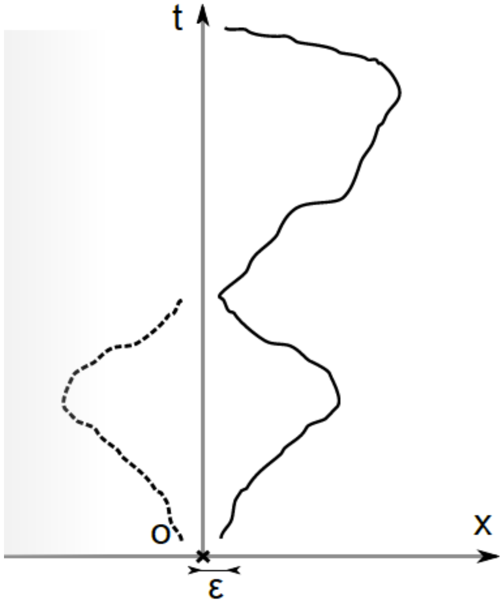
GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal,
EPL 100 26006 (2012)



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$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right) - {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

$\nabla h(0, t)$ fixed

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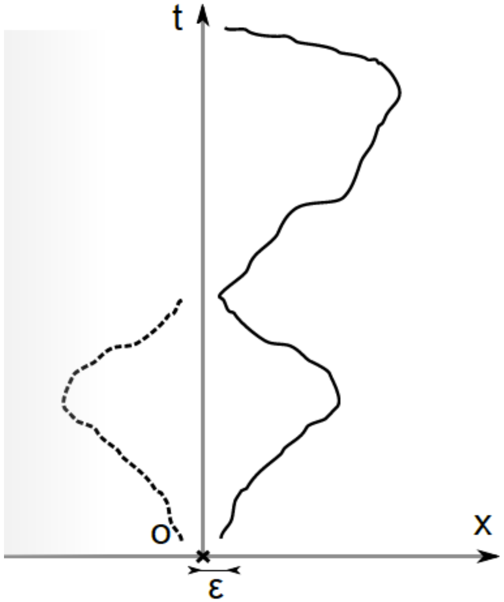

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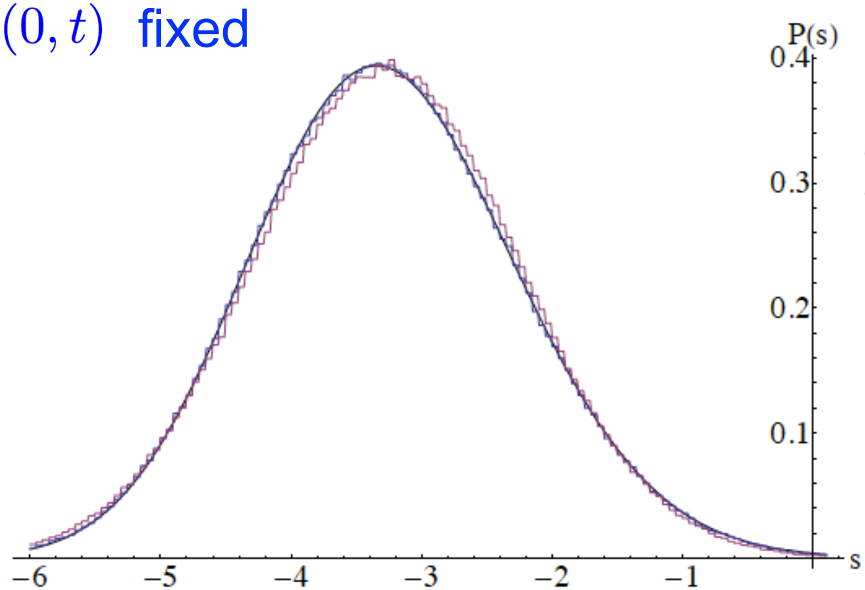
$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$



$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

χ_4 distributed as $F_4(s)$

Gaussian Symplectic Ensemble

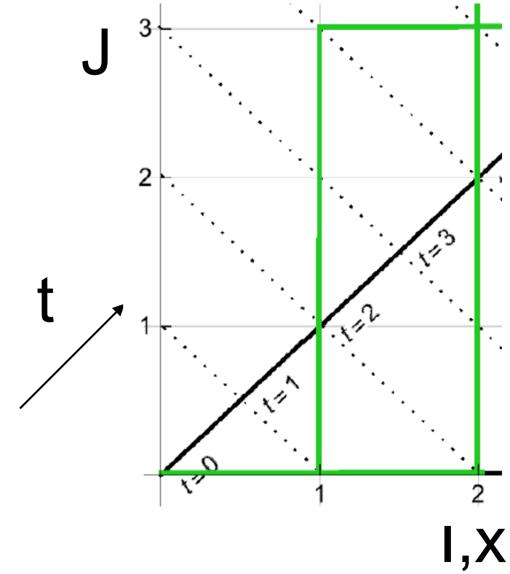
Integrable directed polymer (DP) on square lattice

$$Z_t(x) = \sum_{\pi:(0,0)\to(x,t)} \prod_{(x',t')\in\pi} w_{x',t'}$$

- log-Gamma DP   on-site weights $w \in [0,+\infty[$

inverse Gamma distribution $P(w) = \frac{1}{\Gamma(\gamma)} w^{-1-\gamma} e^{-1/w}$

Seppalainen (2012)   Brunet
COSZ(2011) BCR(2013), Thiery, PLD(2014)



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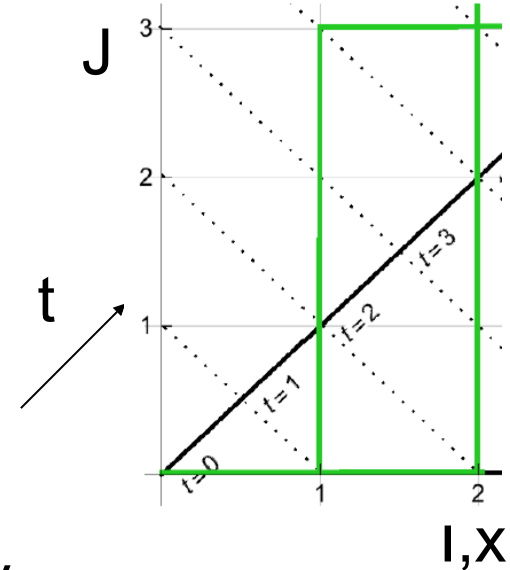

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  Seppalainen (2012) Brunet

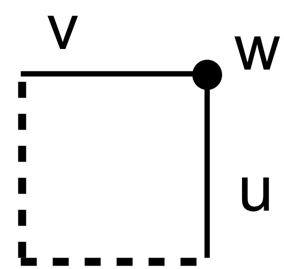
  COSZ(2011) BCR(2013), Thiery, PLD(2014)

- Strict-Weak DP $u \in [0,+\infty[\quad v = 1$

  Gamma distribution $P(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u}$

  Corwin,Seppalainen,Shen(2014)

  O'Connell,Ortmann(2014)



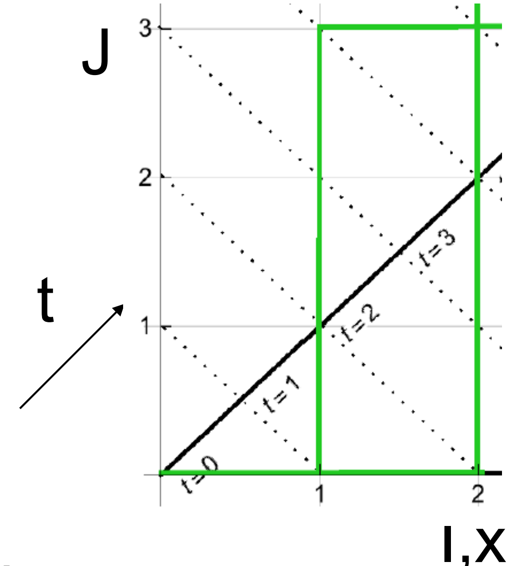
- Beta DP $u, v \in [0,1] \qquad v = 1-u$

  Beta distribution $p_{\alpha,\beta}(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \qquad \alpha > 0 \text{ and } \beta > 0$

  Barraquand,Corwin(2014)

Integrable directed polymer (DP) on square lattice

$$Z_t(x) = \sum_{\pi:(0,0)\to(x,t)} \prod_{(x',t')\in\pi} w_{x',t'}$$



- log-Gamma DP

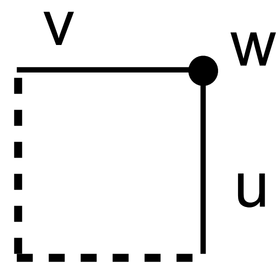
on-site weights $w \in [0,+\infty[$

inverse Gamma distribution $P(w) = \frac{1}{\Gamma(\gamma)} w^{-1-\gamma} e^{-1/w}$

Seppalainen (2012) Brunet
COSZ(2011) BCR(2013), Thiery, PLD(2014)

- Strict-Weak DP

$u \in [0,+\infty[\quad v = 1$



Gamma distribution $P(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u}$

Corwin,Seppalainen,Shen(2014)
O'Connell,Ortmann(2014)

- Beta DP

$u, v \in [0,1] \quad v = 1 - u$

Beta distribution $p_{\alpha,\beta}(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \qquad \alpha > 0 \text{ and } \beta > 0$

Barraquand,Corwin(2014)

- Inverse-Beta DP

$u \in [1,+\infty[\qquad v \in [0,+\infty[\qquad v = u - 1$

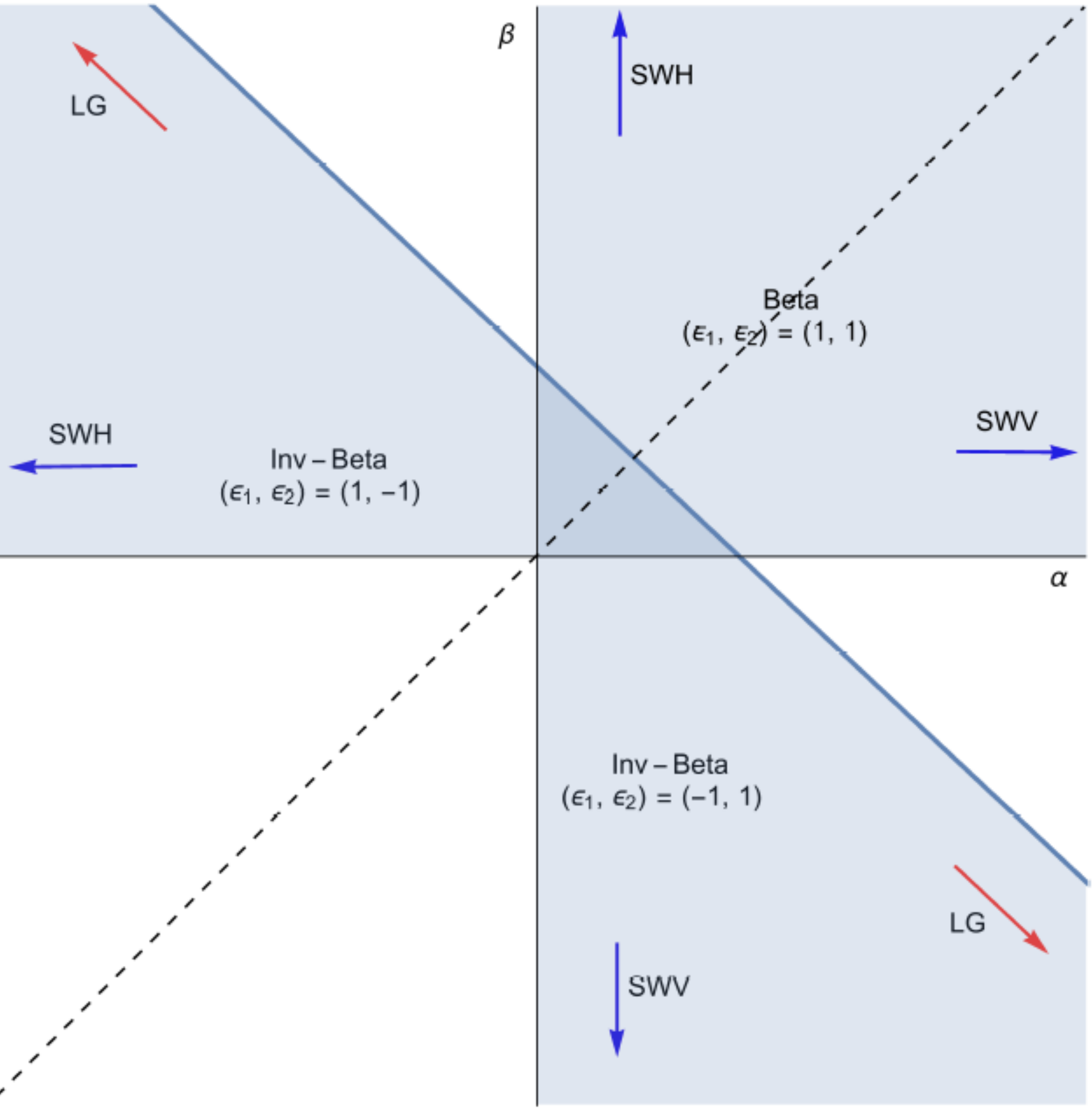
inverse-Beta distribution

$$\tilde{p}_{\gamma,\beta}(u) = \frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \qquad \gamma := 1 - (\alpha+\beta)$$

T. Thiery, PLD(2015)

Integrable directed polymer models on square lattice

T. Thiery, PLD(2015)



What do they have in common ?

$$\overline{u^{n_1} v^{n_2}} = (\epsilon_1)^{n_1} (\epsilon_2)^{n_2} \frac{(\alpha)_{n_1} (\beta)_{n_2}}{(\alpha+\beta)_{n_1+n_2}}$$

$$(\epsilon_1, \epsilon_2) \in \{-1, 1\}^2$$

$$(a)_n = a(a+1)..(a+n-1)$$

$$\gamma := 1 - (\alpha + \beta)$$

$$\psi_t(x_1 \dots, x_n) = \overline{Z_t(x_1) \dots Z_t(x_n)}$$

$$Z_{t+1}(x) = u_{t+1,x} Z_t(x) + v_{t+1,x} Z_t(x-1)$$

=> Transfer matrix $\Lambda_\mu \psi_\mu = T_n \psi_\mu$

contains all integer moments $\overline{u^{n_1} v^{n_2}}$

$n_1 + n_2 = n$

$$\psi_t(x_1 \dots, x_n) = \overline{Z_t(x_1) \dots Z_t(x_n)}$$

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$n_1 + n_2 = n$

Bethe Ansatz

$$\psi_\mu(x_1, \dots, x_n) = \tilde{\psi}_\mu(x_1, \dots, x_n) \text{ if } x_1 \leq \dots \leq x_n$$

$$\tilde{\psi}_\mu(x_1, \dots, x_n) = \sum_{\sigma \in S_n} A_\sigma \prod_{i=1}^{n} z_{\sigma(i)}^{x_i}$$

$$\tilde{\psi}_\mu(x_i, x_i - 1) = \mathsf{a}\tilde{\psi}_\mu(x_i, x_i) + \mathsf{b}\tilde{\psi}_\mu(x_i - 1, x_i) + \mathsf{c}\tilde{\psi}_\mu(x_i - 1, x_i - 1)$$

$$A_\sigma = \epsilon(\sigma) \prod_{1 \leq i < j \leq n} \frac{\mathsf{c} + \mathsf{b} z_{\sigma(i)} + \mathsf{a} z_{\sigma(i)} z_{\sigma(j)} - z_{\sigma(j)}}{\mathsf{c} + \mathsf{b} z_i + \mathsf{a} z_i z_j - z_j}$$

$$\mathsf{a} = \frac{\overline{u^2} - (\overline{u})^2}{(\overline{u})(\overline{v})} \quad \mathsf{b} = \frac{2\overline{uv} - (\overline{u})(\overline{v})}{(\overline{u})(\overline{v})} \quad \mathsf{c} = \frac{\overline{v^2} - (\overline{v})^2}{(\overline{u})(\overline{v})}$$

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$$\tilde\psi_\mu(x_1,\ldots,x_n)=\sum_{\sigma\in S_n}A_\sigma\prod_{i=1}^n z_{\sigma(i)}^{x_i}$$

$$\tilde\psi_\mu(x_i,x_i-1)=\mathsf{a}\tilde\psi_\mu(x_i,x_i)+\mathsf{b}\tilde\psi_\mu(x_i-1,x_i)+\mathsf{c}\tilde\psi_\mu(x_i-1,x_i-1)$$

$$A_\sigma=\epsilon(\sigma)\prod_{1\le i<j\le n}\frac{\mathsf{c}+\mathsf{b}z_{\sigma(i)}+\mathsf{a}z_{\sigma(i)}z_{\sigma(j)}-z_{\sigma(j)}}{\mathsf{c}+\mathsf{b}z_i+\mathsf{a}z_iz_j-z_j}$$

$$\mathsf{a}=\frac{\overline{u^2}-(\overline{u})^2}{(\overline{u})(\overline{v})}\quad \mathsf{b}=\frac{2\overline{uv}-(\overline{u})(\overline{v})}{(\overline{u})(\overline{v})}\quad \mathsf{c}=\frac{\overline{v^2}-(\overline{v})^2}{(\overline{u})(\overline{v})}$$

A. Povolotsky (2013) zero-range process

quantum binomial formula

$$BA=\mathsf{a}A^2+\mathsf{b}AB+\mathsf{c}B^2$$

$$(\overline{u}A+\overline{v}B)^m=\sum_{n_1=0}^m\rho^{n_1}\phi_{q,\mu,\nu}(n_1|m)A^{n_1}B^{m-n_1}$$

condition for integrability

$$(\overline{u}A+\overline{v}B)^m=\sum_{n_1=0}^m\overline{u^{n_1}v^{m-n_1}}C_m^{n_1}A^{n_1}B^{m-n_1}$$

$$\boxed{\overline{u^{n_1}v^{n_2}}=\frac{(\frac{\nu}{\mu};q)_{n_1}(\mu;q)_{n_2}}{(\nu;q)_{n_1+n_2}}\frac{(q;q)_{n_1+n_2}}{(q;q)_{n_1}(q;q)_{n_2}}\frac{1}{C_{n_1+n_2}^{n_1}}}$$

T. Thiery, PLD(2015)

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$$

Inverse-Beta polymer: main results via Bethe Ansatz

T. Thiery, PLD(2015)

- Laplace-transform representations $\overline{e^{-uZ_t(x)}} = \text{Det}\,(I + K_{tx})$

$$K^{BA}_{t,x}(z,z') = \int_{2a+\tilde{a}+i\mathbb{R}} dw \frac{1}{4\pi(w-z')} \frac{1}{\sin(\pi(w-z))} u^{w-z} \left(\frac{\Gamma(\gamma+a-w)}{\Gamma(\gamma+a-z)}\right)^{1+x} \left(\frac{\Gamma(z-a)}{\Gamma(w-a)}\right)^{1-x+t} \left(\frac{\Gamma(w-a+\beta)}{\Gamma(z-a+\beta)}\right)^{t}$$

$K^{BA}_{t,x} : L^2(a+\tilde{a}+i\mathbb{R}) \to L^2(a+\tilde{a}+i\mathbb{R})$ $0 < a < \min(1,\gamma)$ and $0 < \tilde{a} < \gamma - a$.

conjecture

$$\overline{e^{-uZ_t(x)}} = \frac{1}{J!}\int_{(iR)^J} \prod_{j=1}^{J} \frac{dw_j}{2i\pi} \prod_{j\neq k=1}^{J} \frac{1}{\Gamma(w_j-w_k)} \left(\prod_{j=1}^{J} u^{w_j-a}\Gamma[a-w_j]^J \left(\frac{\Gamma(\gamma+a-w_j)}{\Gamma(\gamma)}\right)^I \left(\frac{\Gamma(w_j-a+\beta)}{\Gamma(\beta)}\right)^{I+J-2}\right)$$

contains log-Gamma polymer $\beta\to\infty$

- large time

$$\lim_{t\to\infty} Prob\left(\frac{\log Z_t((1/2+\varphi)t)+tc_\varphi}{\lambda_\varphi} < 2^{\frac{2}{3}} z\right) = F_2(z)$$

optimal angle

$$\varphi* = -\frac{1}{2}\frac{\psi'(\beta+\gamma/2)}{\psi'(\gamma/2)} < 0$$

$$c_{\varphi*} = \psi(\gamma/2) - \psi(\beta+\gamma/2)$$

$$\lambda_{\varphi*} = \left(\frac{t}{8}(\psi''(\beta+\gamma/2)-\psi''(\gamma/2))\right)^{1/3}$$

- zero temperature limit

$\gamma = \epsilon\gamma'$ and $\beta = \epsilon\beta'$ with $\epsilon \to 0$

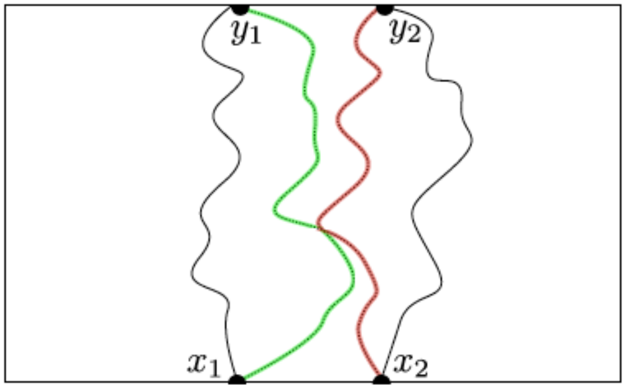
exponential-Bernoulli bond energies

$\beta'/\gamma' \to \infty$ exponential site energies

recovers Johansson’s 2000 formula

Probability that 2 directed polymers in same disorder do not cross

Andrea de Luca, PLD, arXiv 1505.04802



non-crossing probability

Karlin Mc Gregor

$$p_\eta(x_1, x_2; y_1, y_2|t) \equiv 1 - \frac{Z_\eta(x_2; y_1|t) Z_\eta(x_1; y_2|t)}{Z_\eta(x_1; y_1|t) Z_\eta(x_2; y_2|t)}$$

$$p_\eta(t) \equiv \lim_{\epsilon \to 0} \frac{p_\eta(-\epsilon, \epsilon| - \epsilon, \epsilon; t)}{4\epsilon^2} = \partial_x \partial_y \ln Z_\eta(x; y|t)|_{\substack{x=0 \\ y=0}}$$

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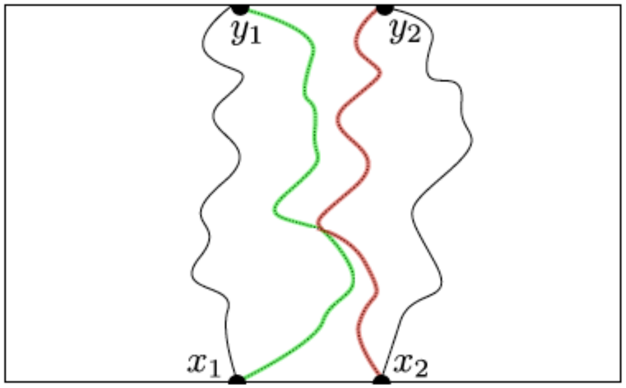

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Moments of the non-crossing probability

$$\overline{p_\eta(t)^m} = \lim_{n \to 0} \Theta_{n,m}(t)$$

$$\Theta_{n,m}(t) \equiv \lim_{\epsilon \to 0} \overline{[(2\epsilon)^{-2} Z^{(2)}_\eta(\epsilon)]^m [Z_\eta(0; 0|t)]^{n-2m}}$$

$$Z^{(2)}_\eta(\epsilon) = Z_\eta(\epsilon; \epsilon|t) Z_\eta(-\epsilon; -\epsilon|t) - Z_\eta(-\epsilon; \epsilon|t) Z_\eta(\epsilon; -\epsilon|t)$$

Probability that 2 directed polymers in same disorder do not cross

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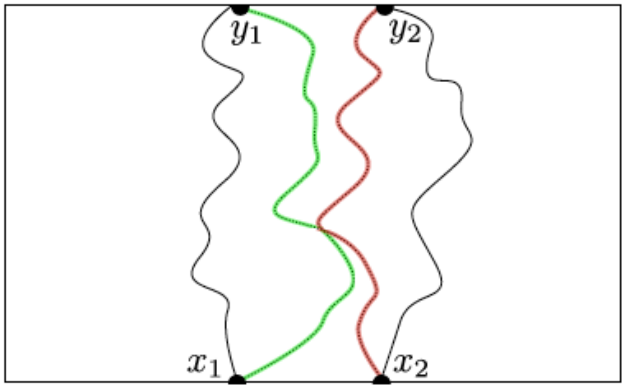

non-crossing probability

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$$Z_\eta^{(2)}(\epsilon) = Z_\eta(\epsilon;\epsilon|t)Z_\eta(-\epsilon;-\epsilon|t) - Z_\eta(-\epsilon;\epsilon|t)Z_\eta(\epsilon;-\epsilon|t)$$

Lieb-Liniger with general symmetry (beyond bosons)

$$H_n = -\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - 2\bar{c}\sum_{1\le i<j\le n}\delta(x_i-x_j)$$

$$\Theta_{n,m}(t) = \lim_{\epsilon\to 0}(2\epsilon)^{-2m}\langle\Psi_m(\epsilon)|e^{-tH_n}|\Psi_m(\epsilon)\rangle = \sum_\mu \frac{|\mathcal{D}_m\psi_\mu(\mathbf{x})|^2}{||\mu||^2}e^{-tE_\mu}$$

$$|\Psi_m(\epsilon)\rangle = 2^{-m/2}(\otimes_{j=1}^m|\epsilon,-\epsilon\rangle - |-\epsilon,\epsilon\rangle)\otimes|0\dots 0\rangle$$

$$\mathcal{D}_1 = 2^{-1/2}(\partial_{x_1}-\partial_{x_2})\big|_{\mathbf{x}=0}$$

Nested Bethe ansatz

$\{\mu_1, \ldots, \mu_n\}$ $E_\mu = \sum_{j=1}^n \mu_j^2$

$$\psi_\mu(\mathbf{x}) = \sum_{P,Q \in \mathcal{S}_n} \vartheta_Q(\mathbf{x}) A_Q^P \exp\Big[i \sum_{j=1}^n x_{Q_j} \mu_{P_j}\Big]$$

$x_{Q_1} \le x_{Q_2} \ldots \le x_{Q_n}$

A_Q^P inside irreducible representation of S_n

2-row Young diagram $\xi = (n-m, m)$

example $n = 8$ and $m = 3$

$$(5,3) \equiv \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 9 \\ \hline 2 & 4 & 6 \\ \cline{1-3}\end{array}$$

antisymmetry $\mathcal{D}_{m=3}$

$x_1 \leftrightarrow x_2 \,, x_3 \leftrightarrow x_4 \,, x_5 \leftrightarrow x_6$

C-N Yang PRL 19,1312 (1967)

$$\prod_{\substack{b=1 \\ b \neq a}}^m \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^n \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2} \,,$$

$$\prod_{\substack{k=1 \\ k \neq j}}^n \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^m \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}$$

$\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$

auxiliary spin chain

auxiliary rapidities λ_a $a = 1, ..m$

=> large L: strings again !
but not all strings contribute !

$$\mu_j^a = k_j + \frac{ic}{2}(m_j + 1 - 2a) + \delta_j^a$$

$$\Theta_{n,1}(t) = \sum_{n_s=1}^{n} \frac{n!\bar{c}^n}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,..m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j e^{-A_2 t}}{m_j} \Phi(\mathbf{k},\mathbf{m}) \Lambda_{n,1}(\mathbf{k},\mathbf{m})$$

LL conserved charges $A_p = \sum_{j=1}^{n} \mu_j^p$

$$\Lambda_{n,1} = \frac{1}{n(n-1)} \left(nA_2 - A_1^2 + \frac{n^2(n^2-1)}{12} \bar{c}^2 \right):$$

$$\Theta_{n,0}(t) = \mathcal{Z}_n(t) \equiv \mathcal{Z}_n(\mathbf{x} = \mathbf{0}; \mathbf{0}|t)$$

$$\Lambda_{n,1} \to \Lambda_{n,0} \equiv 1$$

$$\Theta_{n,1}(t) = \sum_{n_s=1}^{n} \frac{n!\bar{c}^n}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,..m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j e^{-A_2 t}}{m_j} \Phi(\mathbf{k},\mathbf{m})\Lambda_{n,1}(\mathbf{k},\mathbf{m})$$

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Introduce GGE partition function $\mathcal{Z}_n^{\boldsymbol{\beta}}(t)$ $\qquad e^{-A_2 t}$ => $e^{-A_2 t + \sum_{p \geq 1} \beta_p A_p}$

$$A_p \to \partial_p \equiv \partial_{\beta_p} \qquad \Theta_{n,m}(t) = \Lambda_{n,m}(\{\partial_p\})[\mathcal{Z}_n^{\boldsymbol{\beta}}(t)]$$

$$\lim_{n\to 0} \partial_{i_1} \dots \partial_{i_k} \frac{\mathcal{Z}_n^{\boldsymbol{\beta}}(t) - 1}{n} \bigg|_{\boldsymbol{\beta}=0} = -\int_0^\infty \frac{du}{u} \partial_{i_1} \dots \partial_{i_k} \operatorname{Det}(1 + \Pi_0 \mathcal{K}_u^{\boldsymbol{\beta}} \Pi_0)$$

$$\Theta_{n,0}(t) = \mathcal{Z}_n(t) \equiv \mathcal{Z}_n(\mathbf{x}=\mathbf{0};\mathbf{0}|t) \qquad \Lambda_{n,1} \to \Lambda_{n,0} \equiv 1$$

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We calculated the $\Lambda_{n,m}(A_p)$ from the Borodin-Corwin “conjecture”

BC arXiv11114408

$$\Theta_{n,m}(t) = \frac{1}{2^m} \int \frac{dz_1}{2\pi} \cdots \int \frac{dz_n}{2\pi} e^{-t\sum_{k=1}^{n} z_k^2} \times \left(\prod_{1\leq k<j\leq n} f(z_{kj})\right)\left(\prod_{q=1}^{m} h(z_{2q-1,2q})\right)$$

$h(u) = u(u - ic)$

$f(u) \equiv u/(u - ic)$

$z_{kj} = z_k - z_j$ imaginary part C_j for z_j satisfying $C_{j+1} > C_j + \bar{c}$

Results: - first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

$$\overline{p_\eta(t)} = \lim_{n\to 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

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$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

$$\simeq -\frac{\bar{c}^2 t}{12} + \overline{\chi_2}(\bar{c}^2 t)^{1/3}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right) h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$

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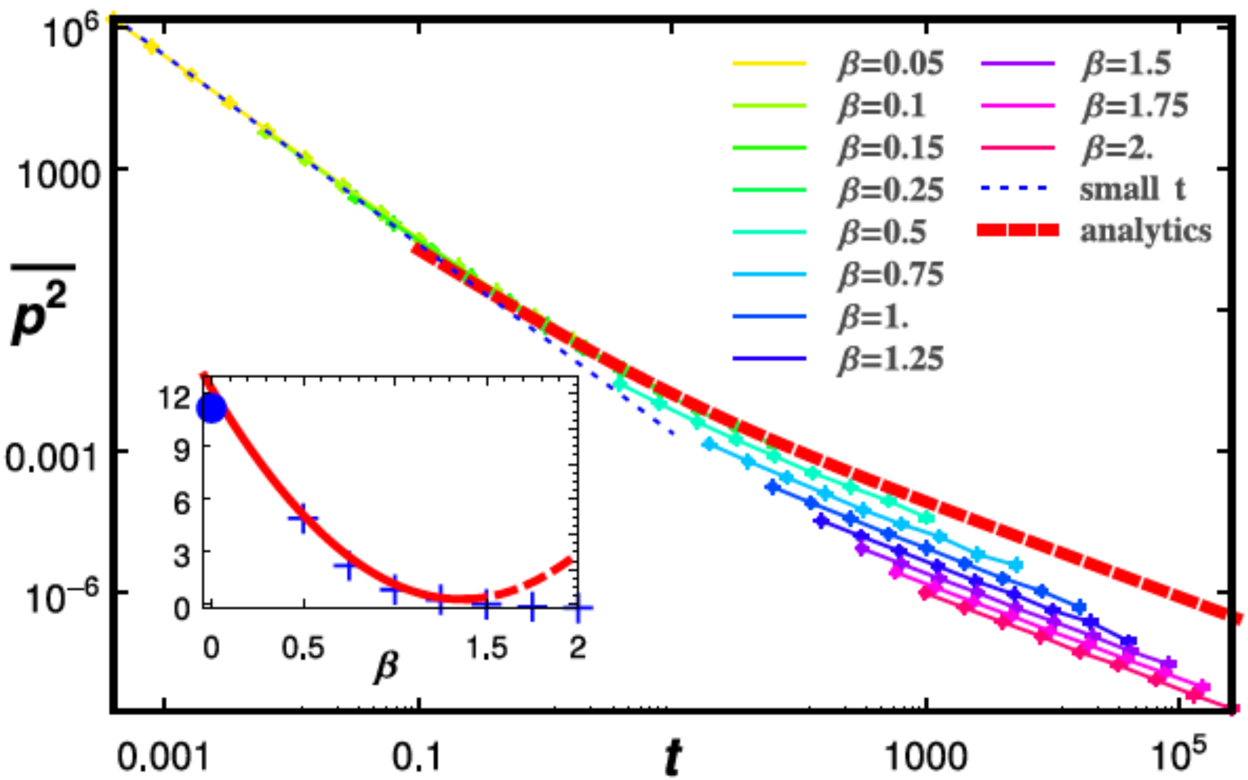

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- higher moments

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\overline{\chi_2}\bar{c}^{8/3}}{9t^{5/3}}$$ + complicated

=> conjecture 1

$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

Results: - first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

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$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\overline{\chi_2}\bar{c}^{8/3}}{9t^{5/3}}$$ + complicated

=> conjecture 1

$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

- conjecture 2

$$\overline{\ln p_\eta(t)} \sim -a(\bar{c}^2 t)^{1/3} \qquad a = \overline{\chi_2} - \overline{\chi_2'} \approx 1.9043$$

=> p(t) (sub-)exponentially small for typical environments

and $p_\eta(t) \sim \bar{c}^2$ for a fraction $\sim 1/(\bar{c}^2 t)$ of environments

Results:

- first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

$$\overline{p_\eta(t)} = \lim_{n\to 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right) h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$

- higher moments

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\overline{\chi_2}\bar{c}^{8/3}}{9t^{5/3}}$$ + complicated

=> conjecture 1

$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

- conjecture 2 $\overline{\ln p_\eta(t)} \sim -a(\bar{c}^2 t)^{1/3}$ $a = \overline{\chi_2} - \overline{\chi_2'} \approx 1.9043$

=> p(t) (sub-)exponentially small for typical environments

and $p_\eta(t) \sim \bar{c}^2$ for a fraction $\sim 1/(\bar{c}^2 t)$ of environments

compare with proba q(t) of single DP not crossing a hard-wall at 0 T. Gueudre, PLD 2012

$$\overline{\ln q_\eta(t)} = -(\overline{\chi_2} - \overline{\chi_4})(\bar{c}^2 t)^{1/3} \approx -1.49134(\bar{c}^2 t)^{1/3}$$

$$\overline{\ln p_\eta(t)} < \overline{\ln q_\eta(t)}$$

Perspectives/other works

- replica BA method

Airy process

stationary KPZ — Sasamoto Inamura — $t \to \infty$ $A_2(y)$

2 space points — $Prob(h(x_1,t),h(x_2,t))$ — Prohlac-Spohn (2011), Dotsenko (2013)

2 times — $Prob(h(0,t),h(0,t'))$ — Dotsenko (2013)

endpoint distribution of DP — Dotsenko (2012) — Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

q-TASEP — $q \to 1$ — avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

WASEP — Bose gas — moments as nested contour integrals

- sine-Gordon FT

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- Lattice directed polymers