Asymptotics of representations of classical Lie groups

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Plan

- Asymptotic representation theory $\to$ random matrices
- Asymptotic representation theory $\to$ lozenge tilings; domino tilings
- General theorem
- Our tools
- Further applications.

Representations of $U(N)$

- Let $U(N)$ denote the group of all $N \times N$ unitary matrices.
- A *signature* of length N is a N-tuple of integers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.
  For example, $\lambda = (5, 3, 3, 1, -2, -2)$ is a signature of length 6.
- It is known that all irreducible representations of $U(N)$ are parameterized by signatures (= highest weights). Let π^λ be an irreducible representation of $U(N)$ corresponding to λ.
- The character of π^λ is the Schur function

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det_{i,j=1,\ldots,N}\left(x_i^{\lambda_j + N - j}\right)}{\prod_{1 \leq i < j \leq N}(x_i - x_j)}$$

Tensor product

- Let λ and μ be signatures of length N. We consider the decomposition of the (Kronecker) tensor product $\pi^\lambda \otimes \pi^\mu$ into irreducible components

$$\pi^\lambda \otimes \pi^\mu = \bigoplus_\eta c_\eta^{\lambda,\mu} \pi^\eta,$$

  where η runs over signatures of length N.
- The decomposition is given by the classical Littlewood-Richardson rule. However, for large N it is hard to "extract information" this rule.

Finite level

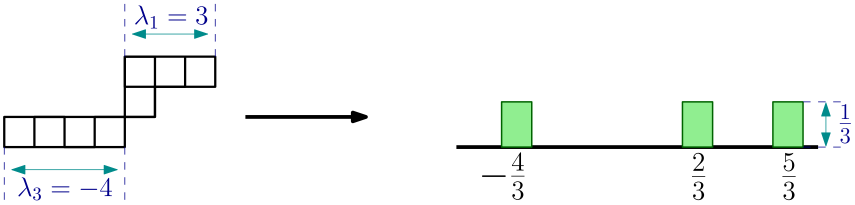
- Let A and B be two Hermitian matrices with known eigenvalues. What can we say about the eigenvalues of $A+B$?
- For which triples of signatures (λ, μ, η) the Littlewood-Richardson coefficient $c_{\eta}^{\lambda,\mu}$ is positive ?
- The two questions above are intimately related. The final answer to both of them was found by Knutson-Tao (1998).
- One can say that we study the *asymptotic* versions of these questions.

Measures related to signatures

It will be convenient for us to encode a representation π^λ and a signature λ by the *counting measure* $m[\lambda]$:

$$m[\lambda] := \frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{\lambda_i + N - i}{N}\right).$$

For the signature $(3, 1, -4)$ we have



Decomposition into irreducibles

- Given a finite-dimensional representation π of $U(N)$ we can decompose it into irreducible components:

$$\pi = \bigoplus_{\lambda} c_\lambda \pi^\lambda,$$

where non-negative integers c_λ are multiplicities.

- This decomposition can be identified with a probability measure ρ^π on signatures of length N such that

$$\rho^\pi(\lambda) := \frac{c_\lambda \dim(\pi^\lambda)}{\dim(\pi)}.$$

Probability measure on the real line

$$\rho^{\pi}(\lambda) := \frac{c_{\lambda} \dim(\pi^{\lambda})}{\dim(\pi)}.$$

- The pushforward of ρ^{π} with respect to the map $\lambda \to m[\lambda]$ is a **random probability measure** on $\mathbb{R}$; we denote this measure by $m[\pi]$.
- **Example.** Let $\pi = \pi^{(3,2)} \oplus \pi^{(3,1)}$. It is known that $\dim(\pi^{(3,2)}) = 2$, $\dim(\pi^{(3,1)}) = 3$; therefore, $m[\pi]$ is the random probability measure which takes the value $\frac{1}{2}(\delta(2) + \delta(1))$ with probability $2/5$, and $\frac{1}{2}(\delta(2) + \delta(1/2))$ with probability $3/5$.

Tensor product in terms of characters

One can write the decomposition of tensor product in terms of Schur functions

$$s_\lambda(x_1,\ldots,x_N)s_\mu(x_1,\ldots,x_N)=\sum_\eta c_\eta^{\lambda,\mu}s_\eta(x_1,\ldots,x_N)$$

The explicit formula for the random measure on signatures

$$m[\pi^\lambda\otimes\pi^\mu](\eta)=c_\eta^{\lambda,\mu}\frac{s_\eta(1,1,\ldots,1)}{s_\lambda(1,1,\ldots,1)s_\mu(1,1,\ldots,1)}$$

How the decomposition of the tensor product looks for large N ?

- Assume that two sequences of signatures $\lambda = \lambda(N)$ and $\mu = \mu(N)$ satisfy

$$m[\lambda] \xrightarrow[N\to\infty]{} m_1, \qquad m[\mu] \xrightarrow[N\to\infty]{} m_2, \qquad \text{weak convergence},$$

  where m_1 and m_2 are probability measures. For example, $\lambda_1 = \cdots = \lambda_{[N/2]} = N$, $\lambda_{[N/2]+1} = \cdots = \lambda_N = 0$, or $\lambda_i = N - i$, for $i = 1, 2, \ldots, N$.
- We are interested in the **asymptotic behaviour** of the decomposition of the tensor product into irreducibles, i.e., we are interested in the asymptotic behaviour of the random probability measure $m[\pi^\lambda \otimes \pi^\mu]$.

Law of Large Numbers for tensor products

Theorem (Bufetov - Gorin, 2013, to appear in *Geometric And Functional Analysis*)

Under assumptions above, we have

$$\lim_{N\to\infty} m[\pi^\lambda \otimes \pi^\mu] = m_1 \otimes m_2, \qquad \textit{weak convergence; in probability,}$$

where $m_1 \otimes m_2$ is a deterministic measure on $\mathbb{R}$.

We also prove a similar result for symplectic and orthogonal groups.

We call $m_1 \otimes m_2$ the *quantized free convolution* of measures m_1 and m_2.

Random matrices

- Let A be a $N \times N$ Hermitian matrix with eigenvalues $\{a_i\}_{i=1}^{N}$. Let
$$m[A] := \frac{1}{N} \sum_{i=1}^{N} \delta(a_i)$$
be the *empirical* measure of A.
- For each $N = 1, 2, \ldots$ take two sets of real numbers $a(N) = \{a_i(N)\}_{i=1}^{N}$ and $b(N) = \{b_i(N)\}_{i=1}^{N}$.
- Let $\mathcal{A}(N)$ be the uniformly ($=$ Haar distributed) random $N \times N$ Hermitian matrix with fixed eigenvalues $a(N)$ and let $\mathcal{B}(N)$ be the uniformly ($=$ Haar distributed) random $N \times N$ Hermitian matrix with fixed eigenvalues $b(N)$ such that $\mathcal{A}(N)$ and $\mathcal{B}(N)$ are independent.

Free convolution

Suppose that as $N \to \infty$ the empirical measures of $\mathcal{A}(N)$ and $\mathcal{B}(N)$ weakly converge to probability measures $\mathbf{m}^1$ and $\mathbf{m}^2$, respectively.

Theorem (Voiculescu, 1991)

The random empirical measure of the sum $\mathcal{A}(N) + \mathcal{B}(N)$ *converges (weak convergence; in probability) to a deterministic measure* $\mathbf{m}^1 \boxplus \mathbf{m}^2$ *which is the free convolution of* $\mathbf{m}^1$ *and* $\mathbf{m}^2$.

Let us now describe the convolutions $\mathbf{m}^1 \otimes \mathbf{m}^2$ and $\mathbf{m}^1 \boxplus \mathbf{m}^2$. One way to do this is through certain power series called R-transforms.

Description of convolutions: formulas

Let $c_k(\mathbf{m})$ be the kth moment of $\mathbf{m}$

$$S_{\mathbf{m}}(z) := z + c_1(\mathbf{m})z^2 + c_2(\mathbf{m})z^3 + \dots,$$

$$R^{free}_{\mathbf{m}}(z) := \frac{1}{S^{(-1)}_{\mathbf{m}}(z)} - \frac{1}{z}$$

$$R^{quantized}_{\mathbf{m}}(z) := \frac{1}{S^{(-1)}_{\mathbf{m}}(z)} - \frac{1}{1 - e^{-z}}$$

We have

$$R^{free}_{\mathbf{m}_1 \boxplus \mathbf{m}_2}(z) = R^{free}_{\mathbf{m}_1}(z) + R^{free}_{\mathbf{m}_2}(z)$$

$$R^{quantized}_{\mathbf{m}_1 \otimes \mathbf{m}_2}(z) = R^{quantized}_{\mathbf{m}_1}(z) + R^{quantized}_{\mathbf{m}_2}(z)$$

Degeneration: Semiclassical limit

There is a **degeneration** of the tensor product of representations of unitary groups to the summation of Hermitian matrices.
On the level of formulas for R-transforms this degeneration can be seen as follows.
Given a probability measure $\mathbf{m}$ let $\mathbf{m} \star L$ be a probability measure such that

$$(\mathbf{m} \star L)(A) := \mathbf{m}\left(\frac{A}{L}\right), \qquad \text{for any measurable } A \subset \mathbb{R}$$

Then we have

$$\lim_{L \to \infty} \frac{R^{quantized}_{m \star L}\left(\frac{z}{L}\right)}{L} = R^{free}_{\mathbf{m}}(z).$$

Related results

- In our situation coordinates of signatures λ and μ grow linearly in N. The situation when this growth is superlinear was considered by Biane (1995), and Collins-Sniady (2007). The resulting operation on measures is the conventional free convolution. This regime of growth is related to the degeneration discussed above.
- In the case of the symmetric group similar results were obtained by Biane (1998).

CLT for tensor products

$p_k := \int x^k dm[\pi^\lambda \otimes \pi^\mu]$.

Theorem (Bufetov-Gorin,2015)

As $N \to \infty$, *we have*

$$\lim_{N\to\infty} \operatorname{cov}(p_k, p_s)$$

$$= \frac{1}{(2\pi i)^2} \oint_{|z|=\epsilon} \oint_{|w|=\epsilon/2} \left(\frac{1}{z} + 1 + (1+z) H'_{\mathbf{m}_1}(1+z) \right)^k$$

$$\times \left(\frac{1}{w} + 1 + (1+w) H'_{\mathbf{m}_2}(1+w) \right)^s Q^{\otimes}_{\mathbf{m}_1,\mathbf{m}_2}(1+z, 1+w) dz dw,$$

More formulas...

$$H_{\mathbf{m}}(u) := \int_0^{\ln(u)} R_{\mathbf{m}}(t)dt + \ln\left(\frac{\ln(u)}{u-1}\right),$$

For two probability measures $\mathbf{m}_1$ and $\mathbf{m}_2$:

$$\begin{aligned}
&Q^{\otimes}_{\mathbf{m}_1,\mathbf{m}_2}(x,y)\\
&:= \partial_x\partial_y \left(\log\left(1-(x-1)(y-1)\frac{xH'_{\mathbf{m}_1}(x)-yH'_{\mathbf{m}_1}(y)}{x-y}\right)\right.\\
&+\log\left(1-(x-1)(y-1)\frac{xH'_{\mathbf{m}_2}(x)-yH'_{\mathbf{m}_2}(y)}{x-y}\right)-\log\Big(1\\
&-(x-1)(y-1)\frac{x(H'_{\mathbf{m}_1}(x)+H'_{\mathbf{m}_2}(x))-y(H'_{\mathbf{m}_1}(y)+H'_{\mathbf{m}_2}(y))}{x-y}\Big)\\
&\left. -\log(x-y)\right).
\end{aligned}$$

Restriction of π^λ

Let λ be a signature of length N. Let us restrict π^λ to $U(N-1)$:

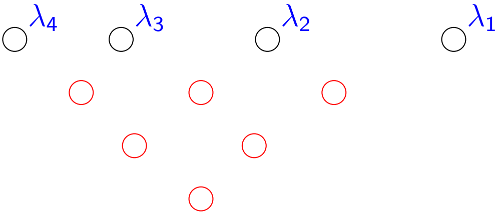
$$\pi^\lambda\big|_{U(N-1)} = \bigoplus_{\mu \prec \lambda} \pi^\mu,$$

where $\mu \prec \lambda$ means that they *interlace*:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N.$$

Gelfand-Tsetlin arrays

Restricting π^λ to $U(M)$, for $M < N$, we obtain the following picture:



For large N we consider uniformly random Gelfand-Tsetlin arrays with fixed upper row λ. What is the behavior of the signature on level $[\alpha N]$, $0 < \alpha < 1$. ?

Given a signature λ of length N let

$$\pi_{\lambda,M} := \pi^{\lambda}\Big|_{U(M)}.$$

Theorem (Bufetov-Gorin, 2013)

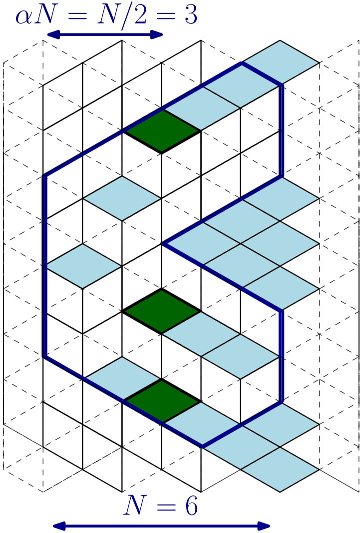
Assume that a sequence of signatures $\lambda = \lambda(N)$ *satisfies*

$$m[\lambda] \xrightarrow[N\to\infty]{} \mathbf{m}, \qquad \textit{weak convergence.}$$

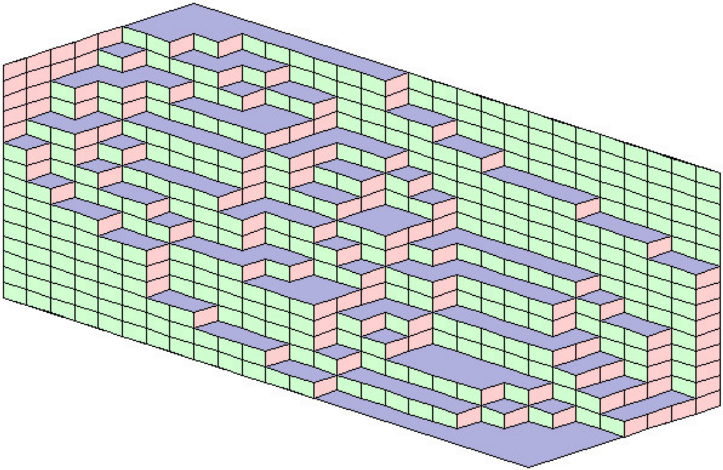
Let $M = M(N) = [\alpha N]$, $0 < \alpha < 1$. *Then, as* $N \to \infty$, *the random measure* $m[\pi_{\lambda,M}]$ *converges (in the sense of moments; in probability) to a deterministic measure* $m^{pr}_{\alpha,\mathbf{m}}$. *The measure* $m^{pr}_{\alpha,\mathbf{m}}$ *is determined by*

$$R^{quantized}_{m^{pr}_{\alpha,\mathbf{m}}}(z) = \frac{1}{\alpha} R^{quantized}_{\mathbf{m}}(z).$$

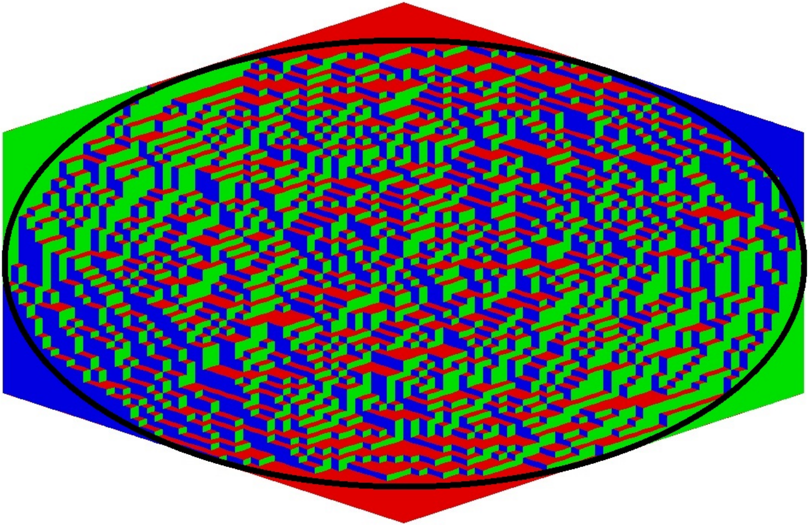
Projections and random tilings



Projections and random tilings



Projections and random tilings



Projection and random tilings: limit shapes

- The theorem for projection implies the limit shape phenomenon for uniform random lozenge tilings of certain polygons. The existence of the limit shape is known (Cohn-Kenyon-Propp (2001), Kenyon-Okounkov-Sheffield (2006)).
- However, our theorem directly links the computation of the limit shape (for these polygons) with the operation of the free projection from free probability.

$p_k^{(\alpha N)} := \int_{\mathbb{R}} x^k dm[\pi_{\lambda,[\alpha N]}]$. $S_{\mathbf{m}}(z) := \int_{\mathbb{R}} \frac{z}{1-zx} d\mathbf{m}(x)$.

Theorem (Petrov'12, Bufetov-Gorin'15)

Assume that a sequence of signatures $\lambda = \lambda(N)$ *satisfies* $m[\lambda] \xrightarrow[N\to\infty]{} \mathbf{m}$. *Then*

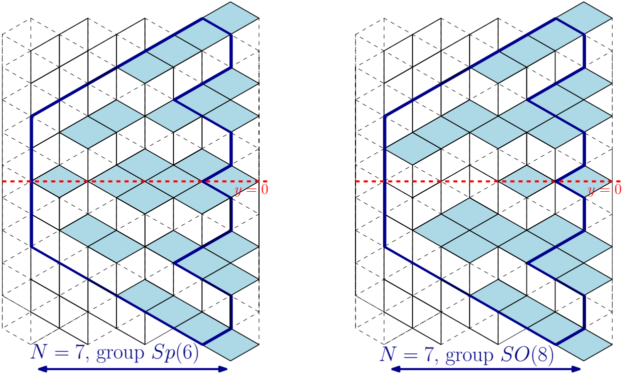
$$\lim_{N\to\infty} \mathrm{cov}(p_k^{(\alpha_1 N)}, p_s^{(\alpha_2 N)})$$

$$= \frac{1}{2\pi \mathbf{i}^2} \oint_{|z|=\epsilon} \oint_{|w|=\epsilon/2} \left(\frac{1}{z} + \frac{1-\alpha_1}{\exp(-S_{\mathbf{m}}(z)) - 1} \right)^k$$

$$\times \left(\frac{1}{w} + \frac{1-\alpha_2}{\exp(-S_{\mathbf{m}}(w)) - 1} \right)^s \frac{1}{(z-w)^2} dzdw,$$

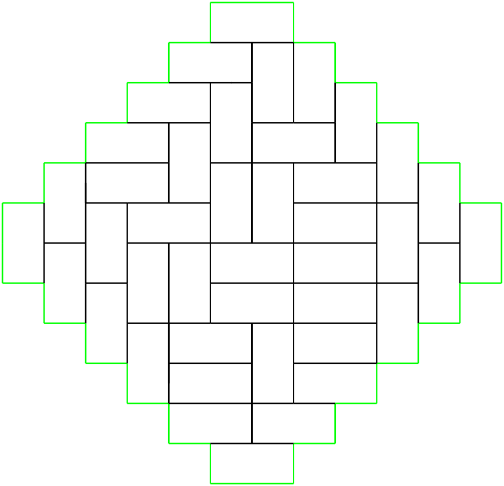
The covariance can be written in terms of a *Gaussian Free Field* (an idea of such a description first push forward by Kenyon).

Projections for *Sp* and *SO*



Bufetov-Gorin'13: limit shapes for these tilings; connection with free probability.

Domino tilings



Schur-generating functions

$\text{Sign}(N)$ — the set of all signatures of length N. Let $prob(\lambda)$ be a probability measure on Sign_N.

$$\Phi(x_1, \ldots, x_N) = \sum_{\lambda \in \text{Sign}(N)} prob(\lambda) \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1, 1, \ldots, 1)}.$$

We call $\Phi(x_1, \ldots, x_N)$ the *Schur-generating function* of $prob(\lambda)$. Our method allows to extract information about $prob(\lambda)$ from Φ. Moreover, it is enough to know the behavior of Φ in the neighborhood of the point $(1, 1, \ldots, 1)$.

$$1^{N-k} := \underbrace{(1, 1, \ldots, 1)}_{N-k}.$$

General conditions

- $$\lim_{N\to\infty} \frac{\partial_i \log \Phi(x_1, x_2, \ldots, x_k, 1^{N-k})}{N} = F(x_i),$$
- $$\lim_{N\to\infty} \partial_i \partial_j \log \Phi(x_1, x_2, \ldots, x_k, 1^{N-k}) = G(x_i, x_j), \qquad i \neq j$$
- $$\lim_{N\to\infty} \partial_{i_1} \partial_{i_2} \partial_{i_3} \log \Phi(x_1, \ldots, x_k, 1^{N-k}) = 0, \qquad i_1 < i_2 < i_3,$$

everywhere the convergence is uniform over an open complex neighborhood of $(x_1, \ldots, x_k) = (1^k)$.

General theorem

Theorem (Bufetov-Gorin'13, Bufetov-Gorin'15)

Under conditions above, the Law of Large Numbers and the Central Limit Theorem holds.
Limit shape can be expressed through $F(x)$*; the covariance can be expressed through* $F(x)$ *and* $G(x, y)$*.*

We need: relation of combinatorics to Schur functions + limit regime as in lozenge tilings.

This theorem covers:

- tensor products of irreducibles
- restrictions of irreducibles; corresponding models of tilings without and with additional symmetries.
- domino tilings of Aztec diamond.
- other probabilistic models, in particular, more general Schur processes (??).
- probabilistic models related to extreme characters of infinite-dimensional groups.
- Perelomov-Popov measures (on tilings).

Method of proof

Consider the differential operators:

$$D_k := \prod_{i<j} \frac{1}{x_i - x_j} \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^k \prod_{i<j} (x_i - x_j).$$

They act nicely on Schur-generating functions.

$$\Phi(x_1, \ldots, x_N) = \sum_{\lambda \in \mathsf{Sign}_N} prob(\lambda) \frac{s_\lambda(x_1, x_2, \ldots, x_N)}{s_\lambda(1, 1, \ldots, 1)}$$

Method of proof

$$\Phi(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Sign}_N} prob(\lambda) \frac{s_\lambda(x_1, x_2, \ldots, x_N)}{s_\lambda(1, 1, \ldots, 1)}$$

$$D_k \Phi(x_1, \ldots, x_N)\big|_{x_i=1} = \sum_{\lambda \in \mathrm{Sign}_N} prob(\lambda) \frac{s_\lambda(x_1, x_2, \ldots, x_N)}{s_\lambda(1, 1, \ldots, 1)} \times \left(\sum_{i=1}^{N} (\lambda_i + N - i)^k \right) \Bigg|_{x_i=1} = \mathbf{E}_{prob} \sum_{i=1}^{N} (\lambda_i + N - i)^k.$$

General conditions on Φ allow to compute the left-hand side; this gives us moments of our measure.

Perelomov-Popov measures

For a signature λ we set

$$m_{PP}[\lambda] := \frac{1}{N}\sum_{i=1}^{N}\left(\prod_{j\neq i}\frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)}\right)\delta\left(\frac{\lambda_i + N - i}{N}\right).$$

This definition is inspired by the theorem of Perelomov and Popov (1968).

For any representation π we define the random probability measure $m_{PP}[\pi]$ as the pushforward of ρ^{π} with respect to the map $\lambda \to m_{PP}[\lambda]$.

Law of Large Numbers

Consider two sequences of signatures $\lambda = \lambda(N)$ and $\mu = \mu(N)$ which satisfy

$$m_{PP}[\lambda] \xrightarrow[N\to\infty]{} m_1, \qquad m_{PP}[\mu] \xrightarrow[N\to\infty]{} m_2, \qquad \text{weak convergence,}$$

where m_1 and m_2 are probability measures.
We are interested in the **asymptotic behaviour** of the random probability measure $m_{PP}[\pi^\lambda \otimes \pi^\mu]$.

Theorem (Bufetov-Gorin, 2013)

As $N \to \infty$, random measures $m_{PP}[\pi^{\lambda(N)} \otimes \pi^{\mu(N)}]$ converge in the sense of moments, in probability to a deterministic measure $m_1 \boxplus m_2$ which is the free convolution of m_1 and m_2.

Theorem (Bufetov-Gorin, 2013)

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Theorem (Bufetov-Gorin,2013)

For $0 < \alpha < 1$, as $N \to \infty$, random measures $m_{PP}[\pi_{\lambda(N),[\alpha N]}]$ converge in the sense of moments, in probability to a deterministic measure m^{pr}_{α,m_1}, where

$$R^{free}_{m^{pr}_{\alpha,\mathbf{m}_1}}(z) = \frac{1}{\alpha} R^{free}_{\mathbf{m}_1}(z).$$

This means that the Perelomov-Popov measure is more natural (!) than the uniform one from some point of view.

Universal enveloping algebra

- Let $\mathcal{U}(\mathfrak{gl}_N)$ denote the complexified universal enveloping algebra of $U(N)$. This algebra is spanned by generators E_{ij} subject to the relations

$$[E_{ij}, E_{kl}] = \delta_j^k E_{il} - \delta_i^l E_{kj}.$$

- Let $E(N) \in \mathcal{U}(\mathfrak{gl}_N) \otimes \mathrm{Mat}_{N\times N}$ denote the following $N \times N$ matrix, whose matrix elements belong to $\mathcal{U}(\mathfrak{gl}_N)$:

$$E(N) = \begin{pmatrix} E_{11} & E_{12} & \dots & E_{1N} \\ E_{21} & \ddots & & E_{2N} \\ \vdots & & & \vdots \\ E_{N1} & E_{N2} & \dots & E_{NN} \end{pmatrix}$$

Let $\mathcal{Z}(\mathfrak{gl}_N)$ denote the center of $\mathcal{U}(\mathfrak{gl}_N)$.

Theorem (Perelomov–Popov, 1968)

For $p = 0, 1, 2, \ldots$ consider the element

$$X_p = \mathrm{Trace}\,(E^p) = \sum_{i_1,\ldots,i_p=1}^{N} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_p i_1} \in \mathcal{U}(\mathfrak{gl}_N).$$

Then $X_p \in \mathcal{Z}(\mathfrak{gl}_N)$. Moreover, in the irreducible representation π^λ the element X_p acts as scalar $C_p[\lambda]$

$$C_p[\lambda] = \sum_{i=1}^{N} \left(\prod_{j \neq i} \frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)} \right) (\lambda_i + N - i)^p \,.$$