Asymptotic behaviors in Schur processes

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(joint work with D. Betea, C. Boutillier, J. Bouttier, G. Chapuy and S. Corteel)

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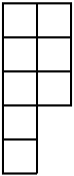
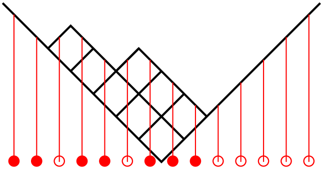
Outline

1. Schur Process/Models
2. Sampling algorithm (joint work with D. Betea, C. Boutillier, J. Bouttier, G. Chapuy and S. Corteel)
3. Asymptotics (joint work with D. Betea and C. Boutillier)
4. Symmetric Schur process

Partitions and Maya diagrams

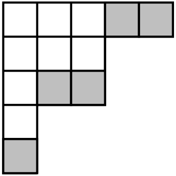
$$\lambda = (2, 2, 2, 1, 1)$$



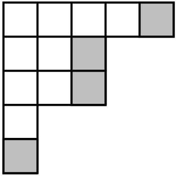


Interlacing

- interlacing $\lambda \succ \mu : \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots$
- dual interlacing $\lambda \succ' \mu$ means $\lambda' \succ \mu'$



$(5,3,3,1,1) \succ (3,3,1,1)$
horizontal strip



$(5,3,3,1,1) \succ' (4,2,2,1)$
vertical strip

- $w = (w_1, w_2, \ldots, w_n) \in \{\prec, \succ, \prec', \succ'\}^n$: w-interlaced sequences of partitions $\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$ means $\lambda(i-1) w_i \lambda(i)$, $\forall i$

Schur process (Okounkov–Reshetikhin [2003])

For a word $w = (w_1, w_2, \ldots, w_n) \in \{\prec, \succ, \prec', \succ'\}^n$, the *Schur process* of word w with parameters $Z = (z_1, \ldots, z_n)$ is a measure on the set of w-interlaced sequences of partitions $\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \emptyset)$ given by

$$Prob(\Lambda) \propto \prod_{i=1}^{n} z_i^{||\lambda(i)|-|\lambda(i-1)||}.$$

Remark 1.

$$s_{\lambda/\mu}(x_1) = x_1^{|\lambda|-|\mu|} \delta_{\lambda \succ \mu}.$$

Remark 2.

$$q^{vol} = q^{\sum_i |\lambda(i)|}$$

Reverse plane partitions

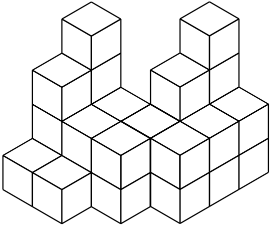
(m $\times n$)-boxed plane partitions

1	3	4
1	2	2
0	2	2
0	0	2

$$w = (\prec)^3(\succ)^4$$

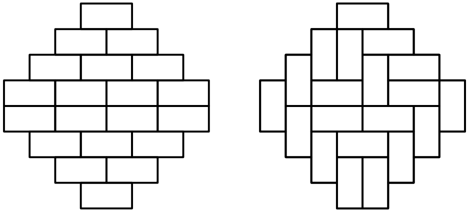
skew plane partitions

1	3	4		
1	2	2		
0	2	2	3	4
0	0	2	2	2



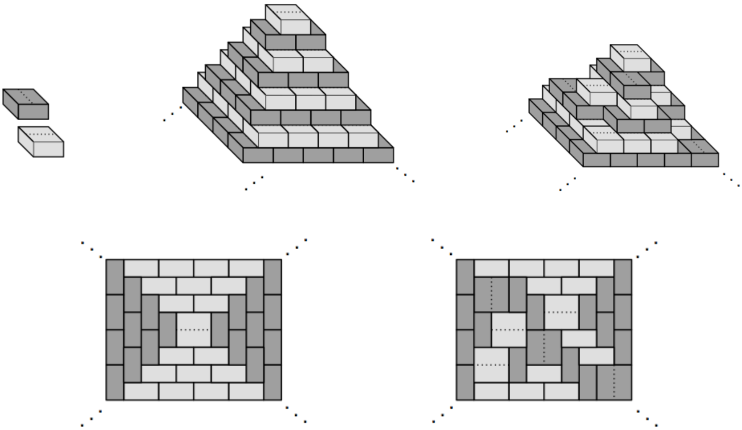
$$(\prec)^3(\succ)^2(\prec)^2(\succ)^2$$

Aztec diamond



$$w = (\prec', \succ)^n$$

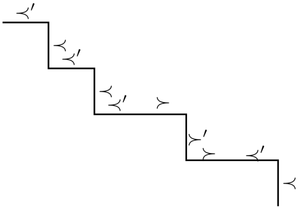
Pyramid partitions

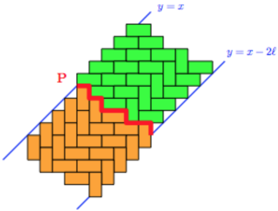


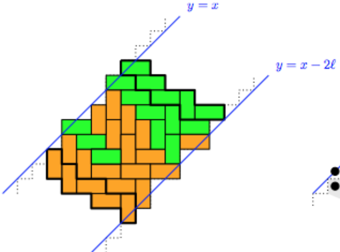
$$w = (\underbrace{\dots, \prec, \prec', \prec, \prec'}_{l}, \underbrace{\succ, \succ', \succ, \succ', \dots}_{l})$$

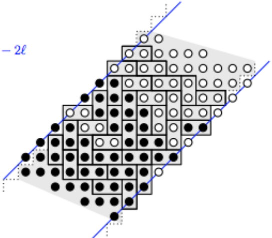
Steep tilings (Bouttier, Chapuy, Corteel [2014])

$w \in \{\prec, \succ, \prec', \succ'\}^{2l}$ $w_{2i} \in \{\prec, \succ\}$ and $w_{2i+1} \in \{\prec', \succ'\}$

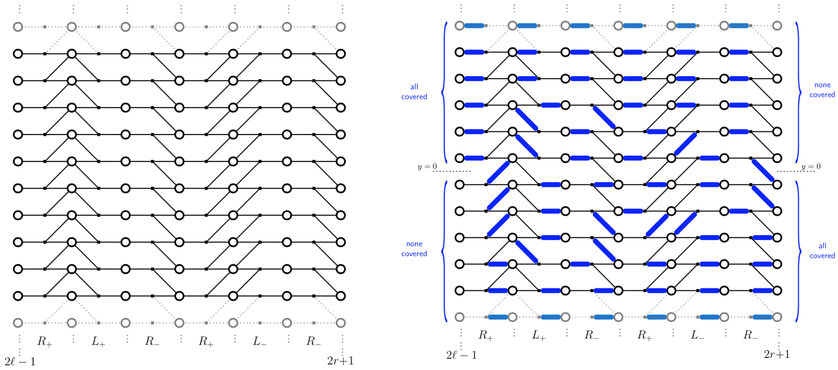








Rail yard graphs(Boutillier, Bouttier, Chapuy, Corteel and Ramassamy [2015])



$$R_+ = \prec', \; R_- = \succ', \; L_+ = \prec, \; L_- = \succ$$

Algorithm properties:

- generalizes RSK and shuffling algorithm
- exact
- entropy optimal
- based on bijections that are easy to implement
- polynomial time complexity
- uses samples from geometric and Bernoulli variables
- extends to one-sided free Schur process (symmetric Schur process) and also to Schur process for infinite words

Literature

- RSK: Gessel, Krattenthaler, Pak–Postnikov, Fomin
- shuffling algorithm: Elkies–Kuperberg–Larsen–Propp
- sampling of Schur processes: Borodin, Borodin-Ferrari
- sampling of Macdonald process: Borodin–Petrov
- coupling from the past (MCMC): Propp–Wilson

RSK (Robinson–Schensted–Knuth) correspondence

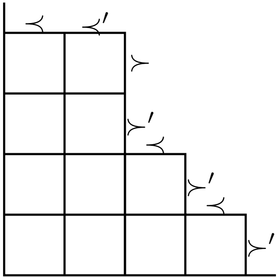
$m \times n$ non-negative integer matrices $\leftrightarrow$ $m \times n$ boxed plane partitions

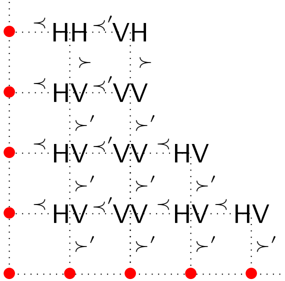
- $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
- $P = \begin{matrix} 1 & 1 & 2 & 2 \\ 2 & 3 & & \\ 3 & & & \end{matrix}$, $Q = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{matrix}$
- $\pi = \begin{matrix} 4 & 3 & 3 \\ 4 & 2 & 2 \\ 2 & 1 & 1 \end{matrix}$

Algorithm

Ex. $w = (\prec, \prec', \succ, \succ', \prec, \succ', \prec, \succ')$

- shape: path of horizontal ($w_i \in \{\prec, \prec'\}$) and vertical ($w_i \in \{\succ, \succ'\}$) segments
- : type: HH $(\prec, \succ)$, HV $(\prec, \succ')$, VH $(\prec', \succ)$, VV $(\prec', \succ')$

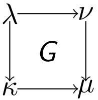




Cauchy identity

$$\sum_{\nu} s_{\nu/\lambda}(x) s_{\nu/\mu}(y) = \frac{1}{1-xy} \sum_{\kappa} s_{\lambda/\kappa}(y) s_{\mu/\kappa}(x)$$

$$\begin{array}{ccc} \lambda & & \\ \curlyvee & G & \\ \kappa & \prec & \mu \end{array} \longrightarrow \begin{array}{ccc} \lambda & \prec & \nu \\ & & \curlyvee \\ & & \mu \end{array}$$

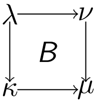


- sample $G \sim Geom(xy)$
- $\nu_i = \begin{cases} \max(\lambda_1, \mu_1) + G & \text{if } i = 1, \\ \max(\lambda_i, \mu_i) + \min(\lambda_{i-1}, \mu_{i-1}) - \kappa_{i-1} & \text{if } i > 1 \end{cases}$

Dual Cauchy identity

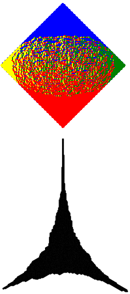
$$\sum_{\nu} s_{\nu/\lambda}(x) s_{\nu'/\mu'}(y) = (1+xy) \sum_{\kappa} s_{\lambda'/\kappa'}(y) s_{\mu/\kappa}(x)$$

$$\begin{array}{ccc} \lambda & & \\ \curlyvee & B & \\ \kappa & \prec & \mu \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \lambda & \prec & \nu \\ & & \curlyvee \\ & & \mu \end{array}$$

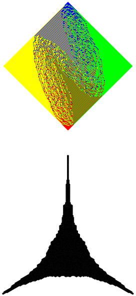


- sample $B \sim Bernoulli(\frac{xy}{1+xy})$
- for $i = 1 \ldots \max(\ell(\lambda), \ell(\mu)) + 1$
  if $\lambda_i \leq \mu_i < \lambda_{i-1}$ then $\nu_i = \max(\lambda_i, \mu_i) + B$
  else $\nu_i = \max(\lambda_i, \mu_i)$
  if $\mu_{i+1} < \lambda_i \leq \mu_i$ then $B = \min(\lambda_i, \mu_i) - \kappa_i$

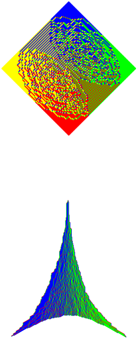
Algorithm



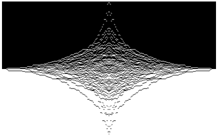
plane partition



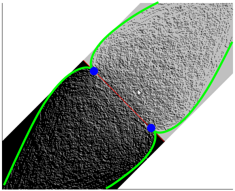
symmetric plane partition



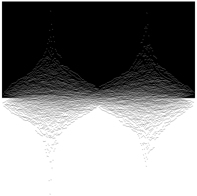
plane overpartition



symmetric pyramid partition



finite width pyramid partiton



steep tilings

Correlation functions (Okounkov-Reshetikhin 2003)

Let $X = \{(n_j, k_j)\}$ with $|X| = m$. The correlation function has the form

$$\rho(X) = Prob(k_j \in \lambda(n_j), \forall j) = \det\left[(K(n_i, k_i; k_j, n_j))\right]_{i,j=1}^{m}$$

where $K(n_i, x; n_j, y)$ is the coefficient of $z^x w^{-y}$ in the formal power series expansion of

$$\frac{\sqrt{zw}}{z-w}\frac{F(n_i, z)}{F(n_j, w)}$$

in the region $|z| > |w|$ if $n_i \geq n_j$ and $|z| < |w|$ if $n_i < n_j$ where

$$F(n_i, z) = \frac{\prod\limits_{j:j\leq n_i, w_j=\prec'} (1+z_j z) \prod\limits_{j:j>n_i, w_j=\succ} (1-z_j z^{-1})}{\prod\limits_{j:j\leq n_i, w_j=\prec} (1-z_j z) \prod\limits_{j:j>n_i, w_j=\succ'} (1+z_j z^{-1})}.$$

Fock space formalism

Vector space:

$$V = \bigoplus_{\lambda \text{ partition}} v_\lambda = \bigoplus_{\lambda} e_{\lambda_1 - 1/2} \wedge e_{\lambda_2 - 3/2} \wedge e_{\lambda_3 - 5/2} \wedge \dots,$$

bosonic operators $\Gamma'_{\pm}(x)$, $\Gamma'_{\pm}(x)$ and fermionic operators ψ and ψ^* plus their commutation relations

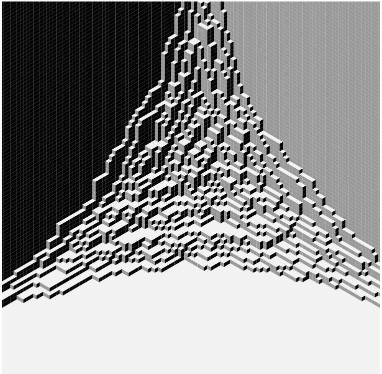
$$\Gamma_-(x) v_\mu = \sum_\lambda s_{\lambda/\mu}(x) v_\lambda, \quad \Gamma'_-(x) v_\mu = \sum_\lambda s_{\lambda'/\mu'}(x) v_\lambda,$$

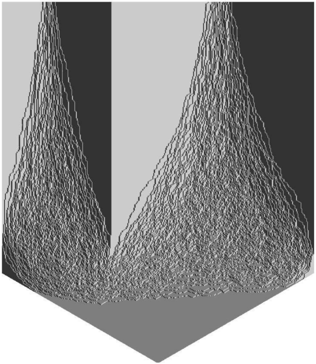
$$\Gamma_+(x) v_\lambda = \sum_\mu s_{\lambda/\mu}(x) v_\mu, \quad \Gamma'_+(x) v_\lambda = \sum_\mu s_{\lambda'/\mu'}(x) v_\mu.$$

$$\psi_i \psi_i^* v_\lambda = \begin{cases} v_\lambda & \text{if } i \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

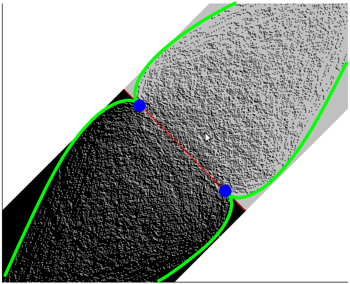
Limit shape- plane partitions

- uniform distribution on partitions of volume n, when $n \to \infty$: Cerf-Kenyon [01] and Okounkov-Reshetikhin [2003]
- skew plane partions Okounkov-Reshetikhin [2005] and [2006]
- behavior in the bulk and on the boundary

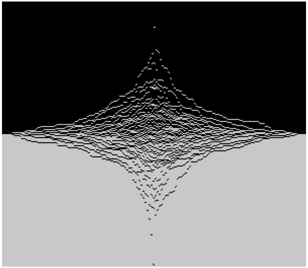




Pyramid partitions



finite width pyramid partiton



pyramid partition

Scaling

The correlation kernel is given by

$$\frac{1}{(2\pi)^2}\iint \frac{J(z;k,n)}{J(w;k,n)}\frac{1}{z-w}\frac{1}{z^l w^{-l}}dzdw$$

When $q=e^{-\epsilon}$, $\epsilon n = a$, $\epsilon k = x$, $\epsilon l = y$, when $\epsilon \to 0+$ asymptotics is determined by

$$\begin{aligned} S(z;x,y) \;=\; & -\mathrm{dilog}(e^{-a}z)+\mathrm{dilog}(z)-\mathrm{dilog}(-z)+\mathrm{dilog}(-e^{-a}z) \\ & -\mathrm{dilog}(-e^{-a}z^{-1})+\mathrm{dilog}(-e^{-x}z^{-1}) \\ & -\mathrm{dilog}(e^{-x}z^{-1})+\mathrm{dilog}(e^{-a}z^{-1})-2y\log z \end{aligned}$$

where

$$\mathrm{dilog}(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}, \quad |z|<1,$$

analytically continued to $z \in \mathbb{C}\backslash[1,\infty)$.

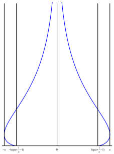
Frozen boundary

Determined by the double crtical points of $S(z;x,y)$:

$$f(z,X) = Y, f'(z,X) = 0,$$

where $A = e^a$, $X = e^x$, Y=e^{2y} and

$$f(z,X) = \frac{(z+1)(z+1/X)(z-1/A)(z-A)}{(z+A)(z+1/A)(z-1/X)(z-1)}.$$

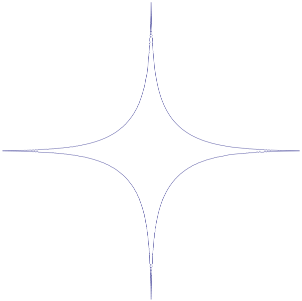


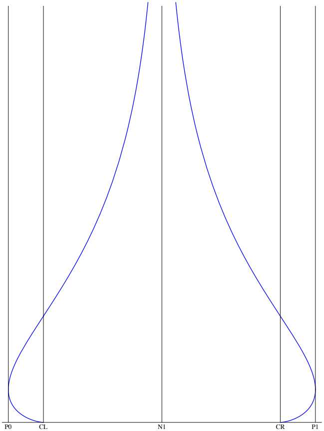
cusps at $(\pm \log(A + 1/A - 1), 0)$.

In the case of unbounded pyramid partitions

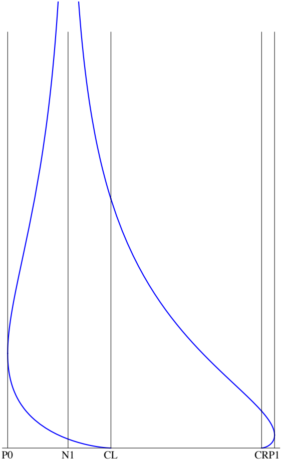
$$f(z) = \frac{(z+1)(z+1/X)}{(z-1)(z-1/X)}$$

and the frozen boundary is the boundary of the amoeba of the polynomial $-1+z+w+zw$, which is expected from the limit shape of the strict plane partitions (will be explained later)

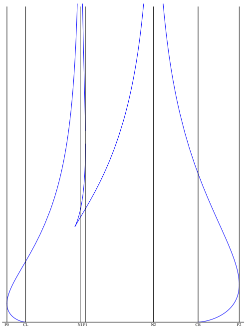


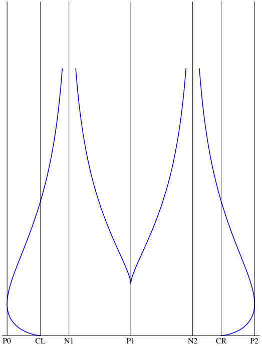


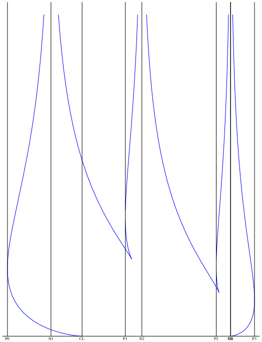
$$\Big(\underbrace{\ldots,\prec,\prec',\prec,\prec'}_{l},\underbrace{\succ,\succ',\succ,\succ',\ldots}_{l}\Big)$$



$$\Big(\underbrace{\ldots,\prec,\prec',\prec,\prec'}_{l},\underbrace{\succ,\succ',\succ,\succ',\ldots}_{m}\Big)$$

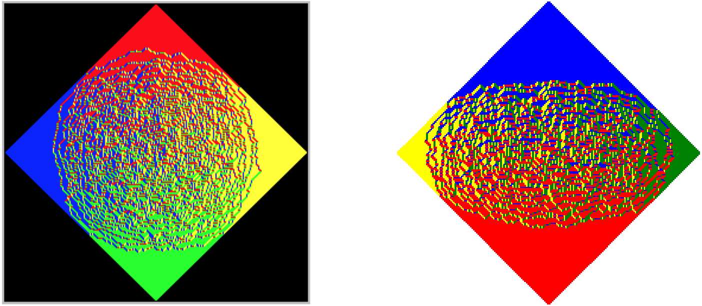






- generic point on the boundary: Airy
- horizontal cusps and vertical cusps(if any): cusp Airy process (OR[06])
- other cusps: Pearcy process
- turning points: GUE minor process (OR[06]), (Johansson–Nordenstam [2010])

Arctic Circle Theorem (Jockusch, Propp, Schor [1998])



$q^{vol} = q^{\# flips}$ left: $q = 1$, right: $q < 1$ (Chhita–Young [2013])

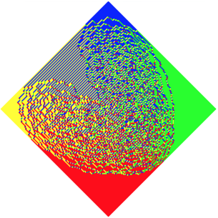
Aztec diamond with periodic weights

- Similar results: Mkrtchyan [2013] -plane partitions with periodic weights
- periodic z_{odd} (adding vertical strips) weights

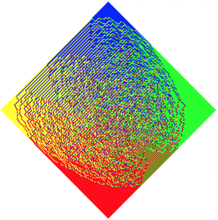
$$(z_1, z_3, z_5 \dots) = (a_1, a_2, \dots a_k, a_1, a_2, \dots a_k, \dots)$$

- periodic weights z_{even} (removing horizontal strips) weights

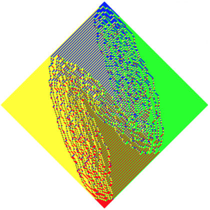
$$(z_2, z_4, z_6 \dots) = (b_1, b_2, \dots b_l, b_1, b_2, \dots b_l, \dots)$$



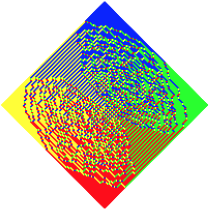
$(a_1, a_2) = (4, 1/4)$
$b_1 = 1$



$(a_1, a_2, a_3) = (8, 1, 1/8)$
$(b_1, b_2) = (3, 1/3)$



$(a_1, a_2) = (48, 1)$
$(b_1, b_2) = (16, 1/8)$



$(a_1, a_2) = (30, 1/30)$
$(b_1, b_2) = (30, 1/30)$

The asymptotics is determined by

$$S(z;x,y) = \frac{x}{k}\log\left(\prod_{i=1}^{k}(1+a_i z)\right) + \left(1-\frac{x}{l}\right)\log\left(\prod_{i=1}^{l}\left(1-\frac{b_i}{z}\right)\right) - y\log z$$

Using this we get

- arctic curve
- cusps
- location of point where arctic curve touches the boundary
- behavior at special points (work in progress)

Symmetric Schur process

$w \in \{\prec, \succ, \prec', \succ'\}^n$: *right-free w-interlaced* sequence of partitions, i.e. $\Lambda = (\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n))$ such that $\lambda(i-1) w_i \lambda(i)$

Right-free Schur process

$$Prob(\Lambda) \propto \prod_{i=1}^{n} z_i^{||\lambda(i)|-|\lambda(i-1)||}$$

Symmetric Schur process

$$(\emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n-1), \lambda(n) = \lambda, \lambda(n-1), \ldots, \lambda(1), \lambda(0) = \emptyset)$$

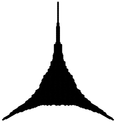
$$\prod_{i=1}^{2n+1} t_i^{||\lambda(i)|-|\lambda(i-1)||}.$$

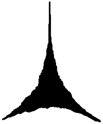
where $t_i t_{2n-i+1} = z_i$, for $i = 1, \ldots, n$.

Examples

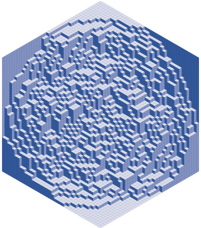
Symmetric plane partitions:

1	2	4
0	2	2
0	0	1





Similar result: uniform plane partitions that fit in $n \times n \times n$ box, when $n \to \infty$: Cohn-Larsen-Propp [98], symmetric: Panova [2014]

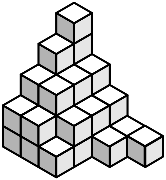


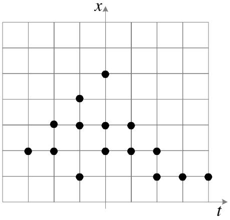
Plane overpartitions

- plane overpartition:

4	$\bar{4}$	$\bar{3}$	2	2
3	3	$\bar{3}$	$\bar{2}$	
$\bar{3}$	$\bar{1}$			
1				

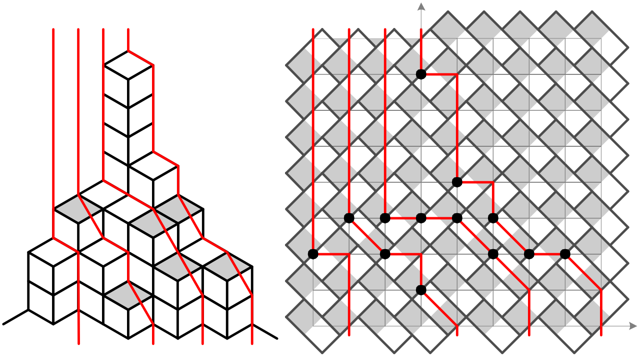
- half pyramid partition: $\emptyset \prec (1) \prec' (2) \prec (2,2) \prec' (3,3,1) \prec (5,3,1) \prec' (5,4,1) \prec (5,4,1,1) \prec' (5,4,2,1)$
- strict plane partition:





Domino tilings

- plane overpartitions $\longleftrightarrow$ steep domino tilings with one-side free boundary



Symmetric Schur process

$\mathfrak{M}_q$ is a probability measure on the set of plane overpartitions defined by

$$\mathfrak{M}_q(\pi) \propto q^{|\pi|}$$

Shifted MacMahon's formula

$$\sum_{\substack{\pi \text{ is a plane} \\ \text{overpartition}}} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n$$

Theorem

The correlation function has the form

$$\rho(X) = \mathrm{Pf}(M_X)$$

where M_X *is a skew-symmetric* $2n \times 2n$ *matrix*

$$M_X(i,j) = \begin{cases} K_{x_i,x_j}(t_i,t_j) & 1 \le i < j \le n, \\ (-1)^{x_{j'}} K_{x_i,-x_j'}(t_i,t_{j'}) & 1 \le i \le n < j \le 2n, \\ (-1)^{x_{i'}+x_{j'}} K_{-x_{i'},-x_{j'}}(t_{i'},t_{j'}) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$ *and* $K_{x,y}(t_i,t_j)$ *is the coefficient of* $z^x w^y$ *in the formal power series expansion of*

$$\frac{z-w}{2(z+w)} J_q(z,t_i) J_q(w,t_j)$$

in the region $|z| > |w|$ *if* $t_i \ge t_j$ *and* $|z| < |w|$ *if* $t_i < t_j$.

Here $J_q(z,t)$ is given with

$$J_q(z,t) = \begin{cases} \dfrac{(q^{1/2}z^{-1};q)_\infty(-q^{t+1/2}z;q)_\infty}{(-q^{1/2}z^{-1};q)_\infty(q^{t+1/2}z;q)_\infty} & t \geq 0, \\[2ex] \dfrac{(-q^{1/2}z;q)_\infty(q^{-t+1/2}z^{-1};q)_\infty}{(q^{1/2}z;q)_\infty(-q^{-t+1/2}z^{-1};q)_\infty} & t < 0, \end{cases}$$

where

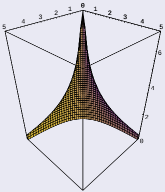
$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-q^n z)$$

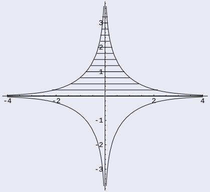
is the quantum dilogarithm function.

Asymptotics for $\mathfrak{M}_q$

Theorem (Asymptotics)

Limit shape (half amoeba of $-1 + z + w + zw$):





Bulk: Determinantal kernel

$$K(i,j) = \frac{1}{2\pi i} \int_{\gamma^{\pm}_{\tau,\chi}} \left(\frac{1-z}{1+z} \right)^{\Delta t_{ij}} \frac{1}{z^{\Delta x_{ij}+1}} dz.$$

Edge-Floor: Pfaffian kernel similar to the one in the bulk.
Edge-Walls: Airy kernel.

Height fluctuations $H = h - E(h)$- Gaussian free field

Theorem

Height fluctuations converge to the Gaussian free field on the first quadrant Q under the push forward given by $z : \mathcal{D} \to Q$.

Higher moments

Let $rx_i \to \chi_i$ and $rt_i \to \tau_i$ when $r \to 0+$ then

$$\lim_{r \to 0+} E[H(x_1, t_1) \cdots H(x_n, t_n)] = \begin{cases} \sum_\sigma \prod_{i=1}^{n/2} G(z_{\sigma(2i-1)}, z_{\sigma(2i)}) & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases},$$

where the sum is taken over all pairings of $\{1, 2, \ldots, n\}$.

The shifted Schur process

specializations: $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \ldots, \rho_T^-)$ sequences of strict partitions: $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^T)$ and $\mu = (\mu^1, \mu^2, \ldots, \mu^{T-1})$

$$W(\lambda, \mu) = \prod_{i=0}^{T} P_{\lambda^i/\mu^i}(\rho_i^-) Q_{\lambda^{i+1}/\mu^i}(\rho_i^+).$$

ρ_0^- and ρ_T^+ are trivial specializations

Fock space formalism

$$V = \bigoplus_{\lambda \text{ strict}} v_\lambda = \bigoplus_{\lambda \text{ strict}} e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_l},$$

$$\Gamma^-(x) v_\mu = \sum_{\lambda \text{ strict}} Q_{\lambda/\mu}(x) v_\lambda, \qquad \Gamma^+(x) v_\lambda = \sum_{\mu \text{ strict}} P_{\lambda/\mu}(x) v_\mu.$$

$$\psi_i \psi_i^* v_\lambda = \begin{cases} v_\lambda/2 & \text{if } i \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Symmetric Schur process

Thank you.