

THE Z -INVARIANT MASSIVE LAPLACIAN ON ISORADIAL GRAPHS

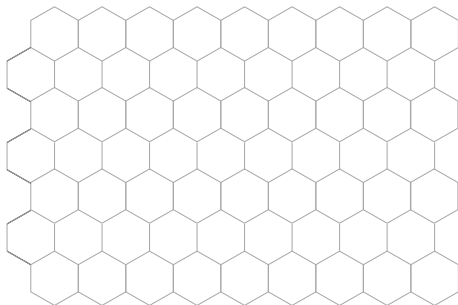
Béatrice de Tilière
University Paris 6

j.w. with Cédric Boutillier (Paris 6), Kilian Raschel (Tours)

Random Interface and Integrable Probability
GGI, Florence, June 25, 2015

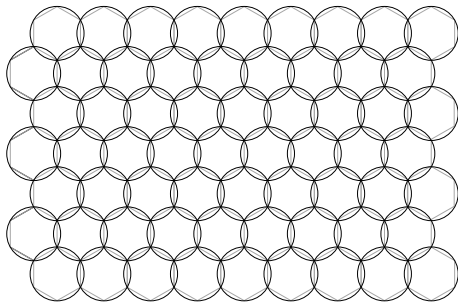
ISORADIAL GRAPHS

- ▶ A graph G is **isoradial** if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



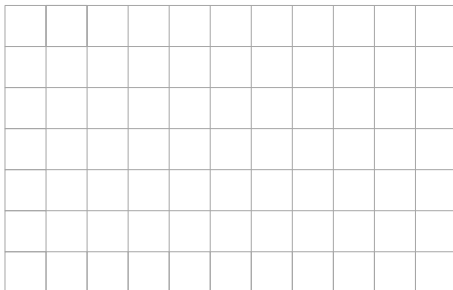
ISORADIAL GRAPHS

- ▶ A graph G is **isoradial** if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



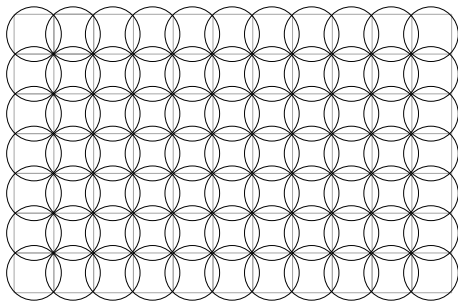
ISORADIAL GRAPHS

- ▶ A graph G is **isoradial** if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



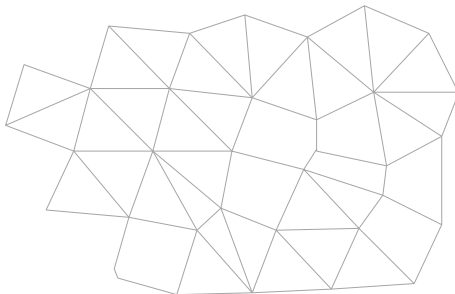
ISORADIAL GRAPHS

- ▶ A graph G is **isoradial** if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



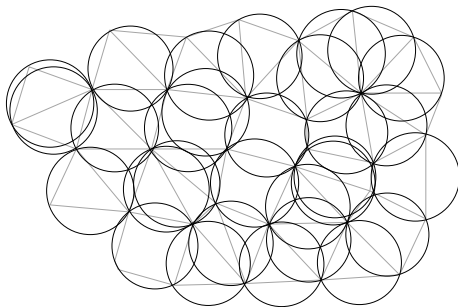
ISORADIAL GRAPHS

- ▶ A graph G is **isoradial** if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



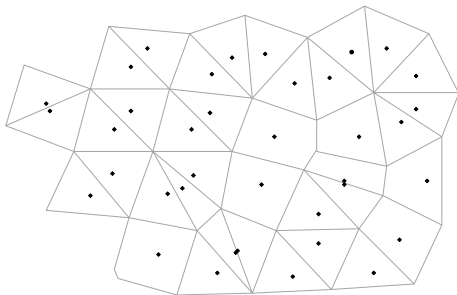
ISORADIAL GRAPHS

- ▶ A graph G is **isoradial** if it can be embedded in the plane in such a way that all (inner) faces are inscribed in a circle of radius 1, and such that the center of the circles are in the interior of the faces (Duffin-Mercat-Kenyon).



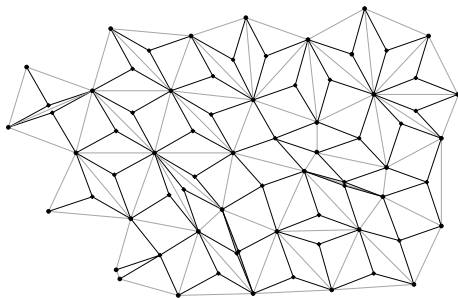
CORRESPONDING DIAMOND GRAPH, ANGLES

- ▶ Take the centers of the circumcircles (embedded dual vertices)



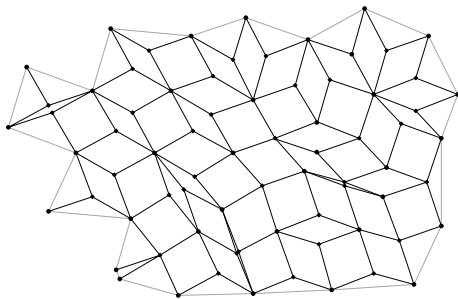
CORRESPONDING DIAMOND GRAPH, ANGLES

- ▶ Join them to the vertices of G of the face they correspond to.
⇒ Corresponding rhombus graph G^\diamond .



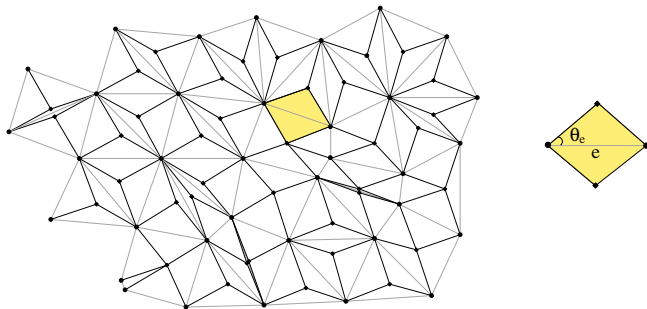
CORRESPONDING DIAMOND GRAPH, ANGLES

- ▶ Join them to the vertices of G of the face they correspond to.
⇒ Corresponding rhombus graph G^\diamond .



CORRESPONDING DIAMOND GRAPH, ANGLES

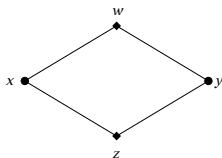
- ▶ To every edge e corresponds a rhombus and a half-angle θ_e .



DISCRETE COMPLEX ANALYSIS

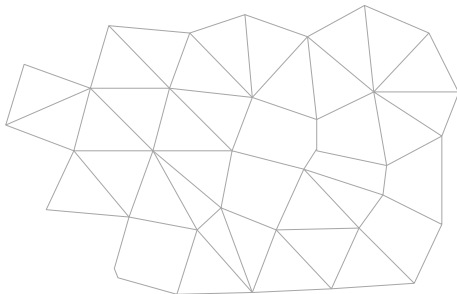
- ▶ Let f be a function defined on vertices of G and G^* .
- ▶ It is **discrete holomorphic** if, for every rhombus $xwyz$,

$$\frac{f(y) - f(x)}{y - x} = \frac{f(w) - f(z)}{w - z}.$$



Z-INVARIANT MODELS ON ISORADIAL GRAPHS

- ▶ Finite isoradial graph $G = (V, E)$.



- ▶ Set of configurations on G : $\mathcal{C}(G)$.

Z-INVARIANT MODELS ON ISORADIAL GRAPHS

- ▶ **Parameters:** positive weight function on edges/vertices

w depends on angles $(\theta_e)_{e \in E}$

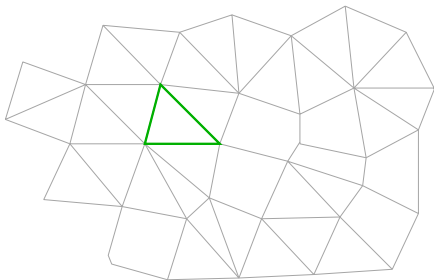
- ▶ **Boltzmann probability measure** on configurations:

$$\forall C \in \mathcal{C}(G), \quad \mathbb{P}(C) = \frac{e^{-\mathcal{E}_w(C)}}{Z(G, w)},$$

where $Z(G, w) = \sum_{C \in \mathcal{C}(G)} e^{-\mathcal{E}_w(C)}$ is the **partition function**.

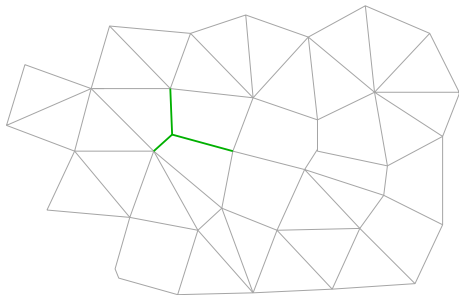
Z-INVARIANT MODELS ON ISORADIAL GRAPHS

- ▶ Star-triangle transformation preserves isoradiality.



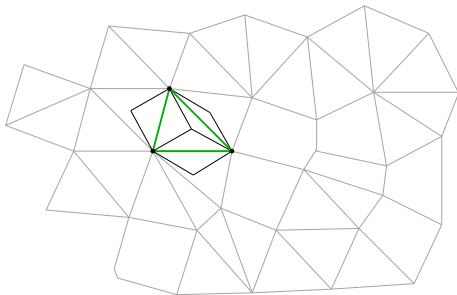
Z-INVARIANT MODELS ON ISORADIAL GRAPHS

- ▶ Star-triangle transformation preserves isoradiality.



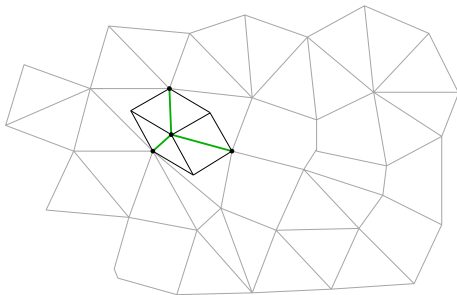
Z-INVARIANT MODELS ON ISORADIAL GRAPHS

- ▶ Star-triangle transformation preserves isoradiality.

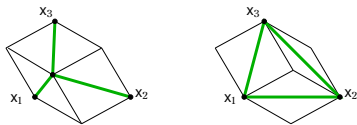


Z-INVARIANT MODELS ON ISORADIAL GRAPHS

- ▶ Star-triangle transformation preserves isoradiality.



Z-INVARIANT MODELS ON ISORADIAL GRAPHS



- ▶ Decompose the partition function according to the possible configurations outside of the star/triangle.
- ▶ The model is **Z-invariant** (Baxter) if \exists constant \mathcal{C} , s.t. for all outer configuration $\mathbf{C}(x_1, x_2, x_3)$:

$$Z(G_Y, w, \mathbf{C}(x_1, x_2, x_3)) = \mathcal{C} Z(G_\Delta, w, \mathbf{C}(x_1, x_2, x_3)).$$

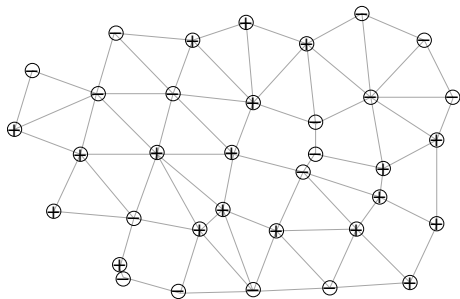
(Yang-Baxter equations)

\Rightarrow Transfer matrices commute (Onsager, 1944).

\Rightarrow Probabilities are not affected by $Y - \Delta$ transformations.

Probabilities should only depend on the *local* geometry of the graph

EXAMPLE: THE Z-INVARIANT ISING MODEL (BAXTER)




$$\forall \sigma \in \{-1, 1\}^V, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{\exp\left(\sum_{e=xy \in E} J(\theta_e) \sigma_x \sigma_y\right)}{Z_{\text{Ising}}(\mathbf{G}, J)},$$

THEOREM (BAXTER)

The Ising model is Z -invariant iff

$$\forall \mathbf{e} \in \mathbf{E}, J(\theta_{\mathbf{e}}) = \frac{1}{2} \log \left(\frac{1 + \operatorname{sn}\left(\frac{2K}{\pi} \theta_{\mathbf{e}} | k\right)}{\operatorname{cn}\left(\frac{2K}{\pi} \theta_{\mathbf{e}} | k\right)} \right), \quad k \in [0, 1).$$

- $K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2 \sin^2 \tau}} d\tau$: *complete elliptic integral of the first kind.*
- $\operatorname{sn}, \operatorname{cn}$: *Jacobi elliptic functions.*
- ▶ If $k = 0$: $\forall \mathbf{e} \in \mathbf{E}, J(\theta_{\mathbf{e}}) = \frac{1}{2} \log \left(\frac{1 + \sin \theta_{\mathbf{e}}}{\cos \theta_{\mathbf{e}}} \right)$.
 - ▶ The model is critical (Li, Duminil-Copin-Cimasoni, Lis), conformally invariant (Chelkak - Smirnov).
 - ▶ Local expressions for probabilities of the corresponding dimer model (Boutillier-dT).
- ▶ $k \neq 0$:  (Boutillier-dT-Raschel).

THE LAPLACIAN [...] ON CRITICAL PLANAR GRAPHS (KENYON)

- ▶ Infinite isoradial graph G .
- ▶ **Conductances**: $\rho = (\tan(\theta_e))_{e \in E}$.
- ▶ Let Δ be the **discrete Laplacian** on G represented by the matrix Δ :

$$\forall x, y \in V, \quad \Delta(x, y) = \begin{cases} \rho(\theta_{xy}) & \text{if } x \sim y \\ -\sum_{y \sim x} \rho(\theta_{xy}) & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

- ▶ The Laplacian Δ is an operator from \mathbb{C}^V to \mathbb{C}^V

$$\forall f \in \mathbb{C}^V, \quad (\Delta f)(x) = \sum_{y \in V} \Delta(x, y) f(y) = \sum_{y \sim x} \rho(\theta_{xy}) (f(y) - f(x)).$$

- ▶ The restriction to G of a discrete holomorphic function is discrete harmonic.

THE LAPLACIAN [...] ON CRITICAL PLANAR GRAPHS (KENYON)

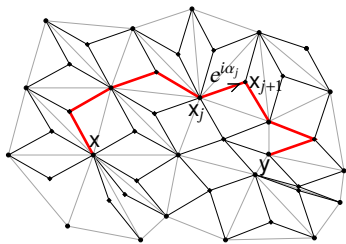
- ▶ The **Green function** G is the inverse of the Laplacian: $\Delta G = \text{Id}$.
- ▶ **Discrete exponential function** (Mercat):

$\text{Exp} : V^\diamond \times V^\diamond \times \mathbb{C} \rightarrow \mathbb{C}$. Let $x, y \in V^\diamond$.

Path in E^\diamond : $x = x_1, \dots, x_n = y$,

$$\text{Exp}_{x_j, x_{j+1}}(\lambda) = \frac{(\lambda + e^{i\alpha_j})}{(\lambda - e^{i\alpha_j})}$$

$$\text{Exp}_{x, y}(\lambda) = \prod_{j=1}^{n-1} \text{Exp}_{x_j, x_{j+1}}(\lambda).$$



THEOREM (KENYON)

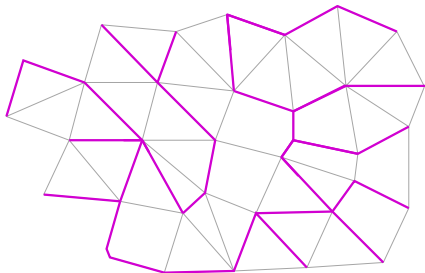
The Green function has the following explicit expression:

$$\forall x, y \in V, \quad G(x, y) = -\frac{1}{8\pi^2 i} \oint_{\gamma} \text{Exp}_{x, y}(\lambda) \log(\lambda) d\lambda,$$

where γ is a contour in \mathbb{C} containing all the poles of $\text{Exp}_{x, y}$.

RELATION TO STATISTICAL MECHANICS

- ▶ Spanning trees of G



- ▶ Boltzmann probability measure:

$$\forall T \in \mathcal{T}(G), \quad \mathbb{P}_{\text{tree}}(T) = \frac{\prod_{e \in T} \rho(\theta_e)}{Z_{\text{tree}}(G, \rho)}.$$

RELATION TO STATISTICAL MECHANICS

THEOREM (KIRCHHOFF)

$$Z_{\text{tree}}(\mathbf{G}, \rho) = \det \Delta^{(r)},$$

where $\Delta^{(r)}$ is the matrix Δ from which the line and column corresponding to the vertex r are removed.

THEOREM (BURTON - PEMANTLE)

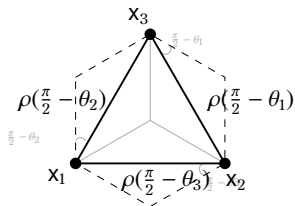
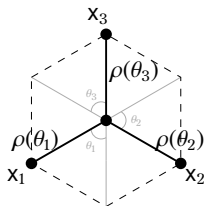
For every subset of edges $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ of \mathbf{G} :

$$\mathbb{P}_{\text{tree}}(\mathbf{e}_1, \dots, \mathbf{e}_k) = \det[(H(\mathbf{e}_i, \mathbf{e}_j))_{1 \leq i, j \leq k}],$$

where H is the *transfer impedance* matrix. Coefficients are differences of Green functions.

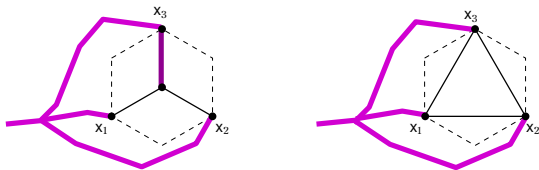
► Kenyon's results yield local formulas for \mathbb{P}_{tree} and for the free energy when the graph is infinite.

Z-INVARIANCE FOR SPANNING TREES



Decompose $Z_{\text{tree}}(\mathbf{G}, \rho)$ according to the possible configurations outside of the $Y - \Delta$.

Z-INVARIANCE FOR SPANNING TREES



Example: x_1, x_2, x_3 are connected to r

	C_Y	C_Δ
$\{x_1, x_2, x_3\}$	$\sum_{\ell=1}^3 \rho(\theta_\ell)$	1
$\{x_i, x_j\}$	$\rho(\theta_k) (\sum_{\ell \neq k} \rho(\theta_\ell))$	$\sum_{\ell \neq k} \rho(\frac{\pi}{2} - \theta_\ell)$
$\{x_i\}$	$\prod_{\ell=1}^3 \rho(\theta_\ell)$	$\sum_{\ell=1}^3 \prod_{\ell' \neq \ell} \rho(\frac{\pi}{2} - \theta_{\ell'})$
$\{\emptyset\}$	0	0

REMARK

The spanning tree model with conductances $\rho = (\tan(\theta_e))_{e \in E}$ is Z-invariant [Kenelly].

AWAY FROM THE CRITICAL POINT ? MASSIVE LAPLACIAN

- ▶ Let $k \in [0, 1)$ (the **elliptic modulus**), $k' = \sqrt{1 - k^2}$, $\bar{\theta}_e = \frac{2K}{\pi}\theta_e$.
- ▶ Define conductances and masses on G :

$$\forall e \in E, \rho(\theta_e) = \text{sc}(\bar{\theta}_e | k)$$

$$\forall x \in V, m^2(x) = \sum_{j=1}^n A(\bar{\theta}_j | k) - \frac{2}{k'}(K - E) - \sum_{j=1}^n \rho(\bar{\theta}_j | k).$$

- ▶ $E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \tau} d\tau$: **complete elliptic int. of the second kind**.
- ▶ $E(u|k) = \int_0^u \text{dn}^2(v|k) dv$: **Jacobi epsilon function**.
- ▶ $A(u|k) = -\frac{i}{k'} E(iu|k')$.

FAMILY OF MASSIVE LAPLACIANS

- ▶ The massive Laplacian $\Delta^{m(k)}$ on G is represented by the matrix :

$$\forall x, y \in V, \quad \Delta^{m(k)}(x, y) = \begin{cases} \rho(\theta_{xy}) & \text{if } x \sim y \\ -m^2(x) - \sum_{y \sim x} \rho(\theta_{xy}) & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

- ▶ The massive Laplacian $\Delta^{m(k)}$ is the operator:

$$\forall f \in \mathbb{C}^V, \quad (\Delta^{m(k)}f)(x) = \sum_{y \sim x} \rho(\theta_{xy})(f(y) - f(x)) - m^2(x)f(x).$$

- ▶ The massive Green function $G^{m(k)}$ is the inverse of the massive Laplacian: $\Delta^{m(k)}G^{m(k)} = \text{Id}$.

THE DISCRETE MASSIVE EXPONENTIAL FUNCTION

► Let $\mathbb{T}(k) = \mathbb{C}/(4K\mathbb{Z} + i4K'\mathbb{Z})$.

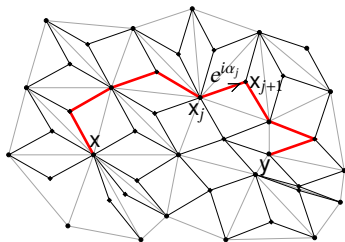
$\text{Exp}(\cdot | k) : V^\diamond \times V^\diamond \times \mathbb{T}(k) \rightarrow \mathbb{C}$.

Let $x, y \in V^\diamond$.

Path in E^\diamond : $x = x_1, \dots, x_n = y$,

$$\text{Exp}_{x_j, x_{j+1}}(u|k) = -i \sqrt{k'} \text{sc}(u \bar{\alpha}_j), \quad u_{\bar{\alpha}_j} = \frac{u - \bar{\alpha}_j}{2}.$$

$$\text{Exp}_{x,y}(u|k) = \prod_{j=1}^{n-1} \text{Exp}_{x_j, x_{j+1}}(u|k).$$



LEMMA

The discrete massive exponential function is well defined, i.e., independent of the choice of the path from x to y .

PROPOSITION

For every $u \in \mathbb{T}(k)$, for every $y \in V$, the function $\text{Exp}_{(\cdot, y)}(u|k) \in \mathbb{C}^V$ is massive harmonic: $\Delta^m \text{Exp}_{(\cdot, y)}(u|k) = 0$.

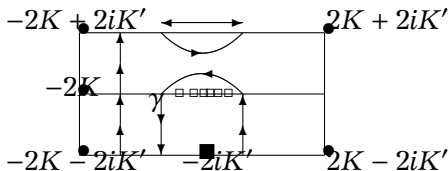
LOCAL EXPRESSION FOR THE MASSIVE GREEN FUNCTION

THEOREM

For every pair of vertices x, y of G ,

$$G^{m(k)}(x, y) = -\frac{k'}{4i\pi} \oint_{\gamma_{x,y}} H(u|k) \text{Exp}_{x,y}(u|k) du,$$

where $\gamma_{x,y}$ is the following contour, $H(u|k) = \frac{u}{4K} + \frac{K'}{\pi} Z(u/2|k)$ and Z is *Jacobi zeta function*.



Torus $\mathbb{T}(k)$, contour of $\gamma_{x,y}$. White squares are poles of $\text{Exp}_{x,y}(\cdot|k)$, the black square is the pole of H .

IDEA OF THE PROOF, CONSEQUENCES

Idea of the proof (Kenyon)

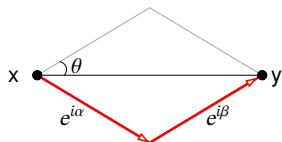
- ▶ Show that $\forall x, y \in V$, $\Delta^{m(k)} G^{m(k)}(x, y) = \delta(x, y)$.
- ▶ If $x \neq y$, deform the contours into a common contour and use the fact that massive exponential functions are massive harmonic.
- ▶ If $x = y$, explicit residue computation. Use the jump of the function H on the torus $\mathbb{T}(k)$.

Consequences

- ▶ Locality of the formula.
- ▶ Asymptotics of $G^{m(k)}(x, y)$, when $|x - y| \rightarrow \infty$.
- ▶ Explicit computations.

EXAMPLE OF COMPUTATION

If $x \sim y$ in G , then

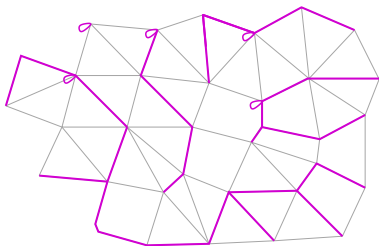


$$\text{Exp}_{x,y}(u) = -(k')^2 \text{sc}(u_{\bar{\alpha}}) \text{sc}(u_{\bar{\beta}}).$$

$$\begin{aligned} G^{m(k)}(x, y) &= \frac{(k')^2}{4i\pi} \oint_{\gamma} H(u) \text{sc}(u_{\bar{\alpha}}) \text{sc}(u_{\bar{\beta}}) du \\ &= \frac{(k')^2}{4i\pi} \oint_{\gamma} H(u) \text{sc}\left(\frac{u}{2}\right) \text{sc}\left(\frac{u - 2\bar{\theta}}{2}\right) du, \quad (\text{change of variable}) \\ &= \frac{H(2K + 2\bar{\theta}) - H(2K)}{\text{sc}(\bar{\theta})} - \frac{K'k'}{\pi \text{dn}(\bar{\theta})}, \quad (\text{residues } 2K, 2K + 2\bar{\theta}, 2iK') \\ &= \frac{H(2\bar{\theta})}{\text{sc}(\bar{\theta})} - \frac{K' \text{dn}(\bar{\theta})}{\pi}, \quad (\text{addition formula for } H). \end{aligned}$$

LOCAL FORMULA FOR ROOTED SPANNING FORESTS

- ▶ Rooted spanning forests



- ▶ Boltzmann probability measure:

$$\forall F \in \mathcal{F}(\mathbf{G}), \quad \mathbb{P}_{\text{forest}}(F) = \frac{\prod_{T \in \mathcal{F}, T \text{ rooted in } x} (\prod_{e \in T} \rho(\theta_e)) m^2(x)}{Z_{\text{forest}}(\mathbf{G}, \rho, m^2)}.$$

- ▶ Explicit expression for probability measure on spanning forests of an infinite isoradial graph, periodic or not.

Z-INVARIANCE FOR ROOTED SPANNING FORESTS

	C_Y	C_Δ
$\{x_1, x_2, x_3\}$	$m^2(x_0) + \sum_{\ell=1}^3 \rho(\theta_\ell)$	1
$\{x_i, x_j\}$	$\rho(\theta_k) [\sum_{\ell \neq k} \rho(\theta_\ell)] + m^2(x_0) \rho(\theta_k) +$ $m^2(x_k) [\sum_{\ell=1}^3 \rho(\theta_\ell) + m^2(x_0)]$	$\sum_{\ell \neq k} \rho(K - \theta_\ell) + m'^2(x_k)$
$\{x_i\}$	$\prod_{\ell=1}^3 \rho(\theta_\ell) + m^2(x_0) \prod_{\ell \neq i} \rho(\theta_\ell) +$ $\sum_{\ell \neq i} m^2(x_\ell) \rho(\theta_{\overline{\{i, \ell\}}}) [\sum_{\ell' \in \{i, \ell\}} \rho(\theta_{\ell'})] +$ $m^2(x_0) [m^2(x_k) \rho(\theta_j) + m^2(x_j) \rho(\theta_k)] +$ $[\prod_{\ell \neq i} m^2(x_\ell)] [\sum_{\ell=1}^3 \rho(\theta_\ell) + m^2(x_0)]$	$\sum_{\ell=1}^3 \prod_{\ell' \neq \ell} \rho(K - \theta_{\ell'}) +$ $\sum_{\ell \neq i} m'^2(x_\ell) [\sum_{\ell' \in \{i, \ell\}} \rho(K - \theta_{\ell'})] + \prod_{\ell \neq i} m'^2(x_\ell)$
$\{\emptyset\}$	$[\sum_{i=0}^3 m^2(x_i)] [\prod_{i=1}^3 \rho(\theta_i)] + m^2(x_0) \sum_{i=1}^3 m^2(x_i) \prod_{\ell \neq i} \rho(\theta_\ell) +$ $\sum_{i=1}^3 [\prod_{\ell \neq i} m^2(x_\ell)] \rho(\theta_i) [\sum_{\ell \neq i} \rho(\theta_\ell)] +$ $m^2(x_0) \sum_{i=1}^3 [\prod_{\ell \neq i} m^2(x_\ell)] \rho(\theta_i) +$ $[\prod_{i=1}^3 m^2(x_k)] [\sum_{i=1}^3 \rho(\theta_i) + m^2(x_0)]$	$[\sum_{i=1}^3 m'^2(x_i)] [\prod_{i=1}^3 \rho(K - \theta_\ell)] +$ $\sum_{i=1}^3 [\prod_{\ell \neq i} m'^2(x_\ell)] [\sum_{i \neq \ell} \rho(K - \theta_\ell)] +$ $\prod_{i=1}^3 m'^2(x_k)$

Z-INVARIANCE FOR ROOTED SPANNING FORESTS

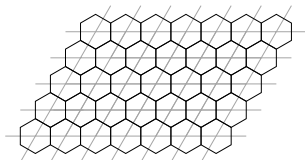
THEOREM

For every $k \in [0, 1)$, the rooted spanning forest model with weights ρ, m^2 , is Z-invariant.

- ▶ When $k = 0$, $\rho(\theta_e) = \tan(\theta_e)$, $m^2(x) = 0$: one recovers the “critical” case.

WHEN THE GRAPH G IS \mathbb{Z}^2 -PERIODIC

Exhaustion by toroidal graphs G : $G_n = G/n\mathbb{Z}^2$.



The free energy is:

$$f(k) = - \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z_{\text{forest}}(G_n, \rho, m^2).$$

THEOREM

The free energy is equal to

$$f(k) = |V_1| \int_0^K 4H'(2\theta) \log \text{sc}(\theta) d\theta + \sum_{e \in E_1} \int_0^{\theta_e} \frac{2H(2\theta) \text{sc}'(\theta)}{\text{sc}(\theta)} d\theta,$$

When $k = 0$, one recovers Kenyon's result.

SECOND ORDER PHASE TRANSITION

PROPOSITION

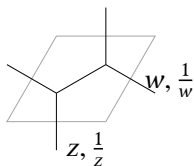
When $k \rightarrow 0$,

$$f(k) = f(0) - k^2 \log(k)|V_1| + O(k^2).$$

where $f(0)$ is the free energy of spanning trees.

SPECTRAL CURVE

- ▶ Fundamental domain: G_1 .



- ▶ $\Delta^{m(k)}(z, w)$: massive Laplacian matrix of G_1 , with weights $z, \frac{1}{z}, w, \frac{1}{w}$.
- ▶ Characteristic polynomial: $P_{\Delta^{m(k)}}(z, w) = \det \Delta^{m(k)}(z, w)$.
- ▶ Spectral curve of the massive Laplacian:

$$C_{\Delta^{m(k)}} = \{(z, w) \in \mathbb{C}^2 : P_{\Delta^{m(k)}}(z, w) = 0\}$$

THEOREM

- ▶ For every $k \in (0, 1)$, $C_{\Delta^{m(k)}}$ is a Harnack curve of genus 1.
- ▶ Every Harnack curve of genus 1 with $(z, w) \leftrightarrow (z^{-1}, w^{-1})$ symmetry arises for such a massive Laplacian.