Gradient Gibbs measures with disorder

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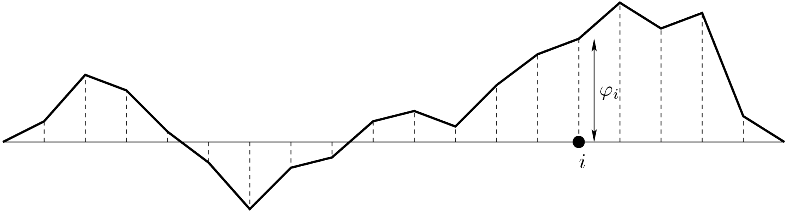
Partly based on joint works with Christof Külske

Outline

- Interface — transition region that separates different phases



- $\Lambda \subset \mathbb{Z}^d$ finite, $\partial\Lambda := \{x \notin \Lambda, \ ||x-y|| = 1 \text{ for some } y \in \Lambda\}$
- Height Variables (configurations) $\phi_x \in \mathbb{R}, x \in \Lambda$
- Boundary condition ψ, such that

$$\phi_x = \psi_x, \quad \text{when } \ x \in \partial\Lambda.$$

- tilt $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ and tilted boundary condition $\psi_x^u = x \cdot u, x \in \partial\Lambda$.
- Gradients $\nabla\phi$: $\eta_b = \nabla\phi_b = \phi_x - \phi_y$ for $b = (x,y), ||x-y|| = 1$

- The finite volume Gibbs measure on $\mathbb{R}^\Lambda$

$$\nu_\Lambda^\psi(\phi) := \frac{1}{Z_\Lambda^\psi} \exp\Big(-\beta \sum_{\substack{i,j\in\Lambda\cup\partial\Lambda \\ |i-j|=1}} V(\phi_i - \phi_j)\Big) \prod_{i\in\Lambda} d\phi_i,$$

  where $\phi_i = \psi_i$ for $i \in \partial\Lambda$.
- $V : \mathbb{R} \to \mathbb{R}^+$, $V \in C^2(\mathbb{R})$, satisfies:
  - symmetry: $V(x) = V(-x), x \in \mathbb{R}$
  - $V(x) \geq Ax^2 + B, A > 0, B \in \mathbb{R},$ for large $x \in \mathbb{R}$.
- Finite volume surface tension (free energy) $\sigma_\Lambda(u)$: macroscopic energy of a surface with tilt $u \in \mathbb{R}^d$.

$$\sigma_\Lambda(u) := \frac{1}{\beta|\Lambda|} \log Z_\Lambda^{\psi^u}.$$

For GFF

- If $V(s) = s^2$, then ν_Λ^ψ is a Gaussian measure, called the Gaussian Free Field (GFF).
- If $x, y \in \Lambda_n$

$$\text{cov}_{\nu_{\Lambda_n}^0}(\phi_x, \phi_y) = G_{\Lambda_n}(x, y),$$

  where $G_{\Lambda_n}(x, y)$ is the Green's function, that is, the expected number of visits to y of a simple random walk started from x killed when it exits Λ_n.
- GFF appears in many physical systems, and two-dimensional GFF has close connections to Schramm-Loewner Evolution (SLE).

Questions (for general potentials V):

- Existence and (strict) convexity of infinite volume surface tension

$$\sigma(u) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \sigma_\Lambda(u).$$

- Existence of shift-invariant infinite volume Gibbs measure

$$\nu := \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu_\Lambda^\psi$$

- Uniqueness of shift-invariant Gibbs measure under additional assumptions on the measure.
- Quantitative results for ν: decay of covariances with respect to ϕ, central limit theorem (CLT) results, large deviations (LDP) results.

Outline

Known results for potentials V with

$$0 < C_1 \leq V'' \leq C_2 :$$

- Existence and strict convexity of the surface tension for $d \geq 1$.
- Gibbs measures ν do not exist for $d = 1, 2$.
- We can consider the distribution of the $\nabla\phi$-field under the Gibbs measure ν. We call this measure the $\nabla\phi$-Gibbs measure μ.
- $\nabla\phi$-Gibbs measures μ exist for $d \geq 1$.
- (Funaki-Spohn: CMP 1997) For every $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ there exists a **unique shift-invariant ergodic** $\nabla\phi$- Gibbs measure μ with $E_\mu[\phi_{e_k} - \phi_0] = u_k$, for all $k = 1, \ldots, d$.
- Decay of covariance results, CLT results, LDP results
- Important properties for proofs: shift-invariance, ergodicity and extremality of the infinite volume Gibbs measures

Bolthausen, Brydges, Deuschel, Funaki, Giacomin, Ioffe, Naddaf, Olla, Sheffield, Spencer, Spohn, Velenik, Yau

Outline

For

$$0 < C_1 \le V'' \le C_2 :$$

- Brascamp-Lieb Inequality: for all $x \in \Lambda$ and for all $i \in \Lambda$

$$\frac{1}{C_2}\text{var}_{\tilde{\nu}_\Lambda^\psi}(\phi_i) \le \text{var}_{\nu_\Lambda^\psi}(\phi_i) \le \frac{1}{C_1}\text{var}_{\tilde{\nu}_\Lambda^\psi}(\phi_i),$$

  $\tilde{\nu}_\Lambda^\psi$ is the Gaussian Free Field with potential $\tilde{V}(s) = s^2$.
- More generally, for any real convex function F bounded below, we have

$$\mathbb{E}_{\nu_\Lambda^\psi}(F(v \cdot (\phi - \mu(\phi))) \le \frac{1}{C_1}\mathbb{E}_{\tilde{\nu}_\Lambda^\psi}(F(\phi)),\ \forall v \in \mathbb{R}^{|\Lambda|}.$$

Techniques: Strictly Convex Potentials (cont.)

- Random Walk Representation Deuschel-Giacomin-Ioffe (PTRF-2000): Representation of Covariance Matrix in terms of the Green function of a particular random walk.
  - GFF: If $x, y \in \Lambda$

    $$\text{cov}_{\nu_\Lambda^0}(\phi_x, \phi_y) = G_\Lambda(x, y),$$

    where $G_\Lambda(x, y)$ is the Green's function, that is, the expected number of visits to y of a simple random walk started from x killed when it exits Λ.
  - General $0 < C_1 \leq V'' \leq C_2$:
    $0 \leq \text{cov}_{\nu_\Lambda^\psi}(\phi_x, \phi_y) \leq \frac{C}{]|x-y|[^{d-2}}$, $|\text{cov}_{\mu_\Lambda^\rho}(\nabla_i \phi_x, \nabla_j \phi_y)| \leq \frac{C}{]|x-y|[^{d-2+\delta}}$

Techniques: Strictly Convex Potentials (cont.)

- The dynamic: SDE satisfied by $(\phi_x)_{x\in\mathbb{Z}^d}$

$$d\phi_x(t) = -\frac{\partial H}{\partial \phi_x}(\phi(t))dt + \sqrt{2}dW_x(t), \quad x \in \mathbb{Z}^d,$$

where $W_t := \{W_x(t), x \in \mathbb{Z}^d\}$ is a family of independent 1-dim Brownian Motions.

Outline

Why look at the case with non-convex potential V?

- Probabilistic motivation: Universality class
- Physics motivation: For lattice spring models a realistic potential has to be non-convex to account for the phenomena of fracturing of a crystal under stress.
- The Cauchy-Born rule: When a crystal is subjected to a small linear displacement of its boundary, the atoms will follow this displacement.
- Friesecke-Theil: for the 2-dimensional mass-spring model, Cauchy-Born holds for a certain class of non-convex potentials. Generalization to d-dimensional mass-spring model by Conti, Dolzmann, Kirchheim and Müller.

Results for non-convex potentials

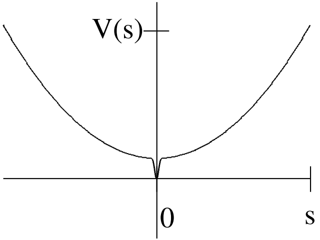
- Funaki-Spohn: The surface tension $\sigma(u)$ is convex as a function of $u \in \mathbb{R}^d$.
- Existence of infinite volume $\nabla\phi$-Gibbs measure μ with expected tilt $E_\mu[\phi_{e_k} - \phi_0] = u_k, k = 1, 2, \ldots d$.
- Hariya (2014): Brascamp-Lieb inequality in $d = 1$.
- Brascamp-Lieb inequality for $d \geq 2$ and 0-boundary condition holds for a class of potentials at all temperatures

$$e^{-V(s)} = \sum_{i=1}^{n} p_i e^{-k_i \frac{s^2}{2}}, \sum_i p_i = 1.$$

- Conjecture: Brascamp-Lieb holds for $\psi \equiv 0$ for all V with $V(x) \geq Ax^2 + B, A > 0, B \in \mathbb{R}$, and $V'' \leq C_2$.

- For the potential

$$e^{-V(s)} = pe^{-k_1 \frac{s^2}{2}} + (1-p)e^{-k_2 \frac{s^2}{2}}, \ \beta = 1, k_1 << k_2, \ p = \left(\frac{k_1}{k_2}\right)^{1/4}$$



- Biskup-Kotecký: (PTRF 2007) Existence of several $\nabla\phi$-Gibbs measures with expected tilt $E_\mu[\phi_{e_k} - \phi_0] = 0, k = 1, 2, \ldots d$, but with different variances.

Results (cont)

- Cotar-Deuschel-Müller (CMP 2009)/ Cotar-Deuschel (AIHP 2012):
  Let

  $$V = V_0 + g,\ C_1 \leq V_0'' \leq C_2,\ g'' < 0.$$

  If

  $$C_0 \leq g'' < 0 \ \text{ and } \quad \sqrt{\beta}||g''||_{L^1(\mathbb{R})} \ \text{ small}(C_1, C_2).$$

  then we prove uniqueness of $\nabla\phi$-Gibbs measures μ such that $E_\mu\left[\phi_{e_k} - \phi_0\right] = u_k$ for all $k = 1, 2, \ldots, d$. Our results includes the Biskup-Kotecký model, but for different range of choices of p, k_1 and k_2.
- Adams-Kotecký-Müller (in preparation): Strict convexity of the surface tension for small tilt u and large β.

Outline

$(\Omega, \mathcal{F}, \mathbb{P})$ the probability space of the disorder, $\mathbb{E}$ the expectation w.r.t $\mathbb{P}$, $\mathbb{V}$ the variance w.r.t. $\mathbb{P}$ and $\mathbb{C}$ov the covariance w.r.t $\mathbb{P}$.

- The Hamiltonian (random external field)

$$H_\Lambda^\psi[\xi](\phi) := \frac{1}{2} \sum_{\substack{x,y \in \Lambda \cup \partial\Lambda \\ |x-y|=1}} V(\phi_x - \phi_y) + \sum_{x \in \Lambda} \xi_x \phi_x,$$

  χ is the set of η_b, with $b = (x, y)$ bonds,
- $(\xi_x)_{x \in \mathbb{Z}^d}$ are assumed to be *i.i.d.* real-valued random variables, with *finite non-zero second moments*.
- $V \in C^2(\mathbb{R})$ is an even function such that there exist $0 < C_1 < C_2$ with

$$C_1 \leq V''(s) \leq C_2 \text{ for all } s \in \mathbb{R}.$$

- The finite volume Gibbs measure on $\mathbb{R}^\Lambda$

$$\nu_\Lambda^\psi[\xi](\phi) := \frac{1}{Z_\Lambda^\psi[\xi]} \exp(-\beta H_\Lambda^\psi[\xi](\phi)) \prod_{x \in \Lambda} d\phi_x,$$

  where $\phi_x = \psi_x$ for $x \in \partial\Lambda$.

- For $v \in \mathbb{Z}^d$, we define the shift operators τ_v:
  - For the bonds by $(\tau_v \eta)(b) := \eta(b - v)$ for b bond and $\eta \in \chi$
  - For the disorder by $(\tau_v \xi)(y) := \xi(y - v)$ for $y \in \mathbb{Z}^d$ and $\xi \in \mathbb{R}^{\mathbb{Z}^d}$.
- A measurable map $\xi \to \mu[\xi]$ is called a shift-covariant random gradient Gibbs measure if $\mu[\xi]$ is a $\nabla\phi-$ Gibbs measure for $\mathbb{P}$-almost every ξ, and if

$$\int \mu[\tau_v \xi](\mathrm{d}\eta) F(\eta) = \int \mu[\xi](\mathrm{d}\eta) F(\tau_v \eta),$$

for all $v \in \mathbb{Z}^d$ and for all $F \in C_b(\chi)$, where χ is the set of gradients.

Outline

Model B

- For each $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $|x-y|=1$, we define the measurable map $V^{\omega}_{(x,y)}(s) : (\omega, s) \in \Omega \times \mathbb{R} \to \mathbb{R}$.
- $V^{\omega}_{(x,y)}$ are random variables with *uniformly-bounded finite second moments* and jointly *stationary* distribution.
- For some given $0 < C^{\omega}_{1,(x,y)} < C^{\omega}_{2,(x,y)}, \omega \in \Omega$, with $0 < \inf_{(x,y)} \mathbb{E}\left(C^{\omega}_{1,(x,y)}\right) < \sup_{(x,y)} \mathbb{E}\left(C^{\omega}_{2,(x,y)}\right) < \infty$, $V^{\omega}_{(x,y)}$ obey for $\mathbb{P}$-almost every $\omega \in \Omega$ the following bounds, uniformly in the bonds (x,y)

$$C^{\omega}_{1,(x,y)} \leq (V^{\omega}_{(x,y)})''(s) \leq C^{\omega}_{2,(x,y)} \text{ for all } s \in \mathbb{R}.$$

- For each fixed $\omega \in \Omega$ and for each bond (x,y), $V^{\omega}_{(x,y)} \in C^2(\mathbb{R})$ is an even function.

- The Hamiltonian for each fixed $\omega \in \Omega$ (random potentials)

$$H_{\Lambda}^{\psi}[\omega](\phi) := \frac{1}{2} \sum_{x,y \in \Lambda \cup \partial\Lambda, |x-y|=1} V_{(x,y)}^{\omega}(\phi_x - \phi_y)$$

- Let $\omega \in \Omega$ be fixed. We will denote by $\mu[\tau_v \omega]$ the infinite-volume gradient Gibbs measure with given Hamiltonian $\bar{H}[\omega](\eta) := \left(H_{\Lambda}^{\rho}[\omega](\tau_v \eta)\right)_{\Lambda \subset \mathbb{Z}^d}$. This means that we shift the field of disorded potentials on bonds from $V_{(x,y)}^{\omega}$ to $V_{(x+v,y+v)}^{\omega}$.
- Questions of interest: Disorder-relevance, universality

Outline

Results for gradients with disorder

- For model A, van Enter-Külske (AAP-2007): For $d = 2$, there exists no shift-covariant gradient Gibbs measure $\mu[\xi]$ with $\mathbb{E}\left|\int \mu[\xi](d\eta)V'(\eta(b))\right| < \infty$ for all bonds b.
- For model A, Cotar-Külske (AAP-2010): For $d = 3, 4$, there exists no shift-covariant Gibbs measure.
- Cotar-Külske (PTRF-to appear): **(Model A)** Let $d \geq 3$, $\xi(0)$ with symmetric distribution and $u \in \mathbb{R}^d$. Assume $0 < C_1 \leq V'' \leq C_2$. Then there exists exactly one shift-covariant random gradient Gibbs measure $\xi \to \mu[\xi]$ with $\mathbb{E}\left(\int \mu[\xi]\right)$ ergodic and such that

$$\mathbb{E}\left(\int \mu[\xi](\mathrm{d}\eta)\eta_b\right) = \langle u, y_b - x_b \rangle \text{ for all } b = (x_b, y_b).$$

- **(Model B)** Let $d \geq 1$ and $u \in \mathbb{R}^d$. Assume $0 < C_1 \leq (V^{\omega}_{(x,y)})'' \leq C_2$ for all ω. Then there exists exactly one shift-covariant random gradient Gibbs measure $\omega \to \mu[\omega]$ with $\mathbb{E}\left(\int \mu[\omega]\right)$ ergodic and such that

$$\mathbb{E}\left(\int \mu[\omega](d\eta)\eta_b\right) = \langle u, y_b - x_b \rangle \text{ for all } b = (x_b, y_b).$$

For our 2nd main result, we need

- Poincaré inequality assumption on the distribution γ of the disorder $\xi(0)$, (respectively of $V^{\omega}_{(0,e_1)}$): There exists $\lambda > 0$ such that for all smooth enough real-valued functions f on Ω, we have for the probability measure γ

$$\lambda \text{var}_{\gamma}(f) \leq \int |\nabla f|^2 \, \mathrm{d}\gamma, \tag{1}$$

  where $|\nabla f|$ is the Euclidean norm of the gradient of f smooth enough.

- Let

$$\partial_b F(\eta) := \frac{\partial F(\eta)}{\partial \eta_b}, \; ||\partial_b F||_\infty := \sup_{\eta \in \chi} |\partial_b F(\eta)| \text{ and }]|b|[= \max\{|x_b|, 1\}.$$

- Cotar-Külske (PTRF-to appear): Let $u \in \mathbb{R}^d$.
  (a) **(Model A)** Let $d \geq 3$. Assume that $(\xi(x))_{x\in\mathbb{Z}^d}$ are i.i.d with mean 0 and the distribution of $\xi(0)$ satisfies (1). Then for all $F, G \in C_b$

$$|\mathbb{C}\text{ov}\left(\mu[\xi](F(\eta)), \mu[\xi](G(\eta))\right)| \leq c \sum_{b,b'} \frac{||\partial_b F||_\infty ||\partial_{b'} G||_\infty}{]|b-b'|[^{d-2}},$$

  for some $c > 0$ which depends only on d, C_1, C_2 and on the number of terms b, b' in F and G.
  (b) **(Model B)** Let $d \geq 1$. Assume that $V^\omega_{(x,y)}$ are i.i.d., and they also satisfy (1) for $\mathbb{P}$-almost every ω and uniformly in the bonds (x, y). Then for all $F, G \in C^1_b$

$$|\mathbb{C}\text{ov}\left(\mu[\omega](F(\eta)), \mu[\omega](G(\eta))\right)| \leq c \sum_{b,b'} \frac{||\partial_b F||_\infty ||\partial_{b'} G||_\infty}{]|b-b'|[^{d}}.$$

- The independence assumption can be relaxed by using, for example, Marton (2013) and Caputo, Menz, Tetali (2014)

Outline

Conjecture for disordered non-convex potentials

- Consider for simplicity the corresponding disordered model

$$e^{-V_b(\eta_b)} := pe^{-k_1(\eta_b)^2+\omega_b}+(1-p)e^{-k_2(\eta_b)^2-\omega_b},\ (w_b)_b \text{ i.i.d. Bernoulli.}$$

Conjectures (work-in-progress):

  - uniqueness for low enough $d \leq d_c$ (shows disorder relevance);
  - uniqueness/non-uniqueness phase transition for high enough $d > d_c \geq 2$ (disorder relevance?).
  - Strict convexity for the surface tension when the gradient Gibbs measure is unique.

Adaptation of the Aizenman-Wehr (CMP-1990) argument.

- Gloria-Otto (AOP-2012)/ Ledoux (2001): Fix $n \in \mathbb{N}$ and let $a = (a_i)_{i=1}^n$ be independent random variables with uniformly-bounded finite second moments on $(\Omega, \mathcal{F}, \mathbb{P})$. Let X, Y be Borel measurable functions of $a \in \mathbb{R}^n$ (i.e. measurable w.r.t. the smallest σ-algebra on $\mathbb{R}^N$ for which all coordinate functions $\mathbb{R}^n \ni a \to a_i \in \mathbb{R}$ are Borel measurable). Then $|\mathrm{cov}\,(X, Y)| \leq$ $\max_{1\leq i\leq n} \mathrm{var}\,(a_i) \sum_{i=1}^n \left(\int \sup_{a_i} \left|\frac{\partial X}{\partial a_i}\right|^2 \, \mathrm{d}\mathbb{P}\right)^{1/2} \left(\int \sup_{a_i} \left|\frac{\partial Y}{\partial a_i}\right|^2 \, \mathrm{d}\mathbb{P}\right)^{1/2}$ where $\sup_{a_i} \left|\frac{\partial Z}{\partial a_i}\right|$ denotes the supremum of

$$\frac{\partial Z}{\partial a_i}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

of Z with respect to the variable a_i, for $Z = X, Y$.

The theorem below will allow us to pass from results for the annealed measure to results for the quenched measure.

- Komlos (1967): If $(\zeta_n)_{n\in\mathbb{N}}$ is a sequence of real-valued random variables with $\liminf_{n\to\infty}\mathbb{E}(|\zeta_n|) < \infty$, then there exists a subsequence $\{\theta_n\}_{n\in\mathbb{N}}$ of the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ and an integrable random variable θ such that for any arbitrary subsequence $\{\tilde{\theta}_n\}_{n\in\mathbb{N}}$ of the sequence $\{\theta_n\}$, we have almost surely that

$$\lim_{n\to\infty}\frac{\tilde{\theta}_1+\tilde{\theta}_2+\ldots+\tilde{\theta}_n}{n}=\theta.$$

We will first prove:

Theorem

Fix $u \in \mathbb{R}^d$. Let for all $\alpha \in \{1, 2, \dots, d\}$

$$E_\alpha := \{\eta \mid \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \eta(b_{x,\alpha}) = u_\alpha\},$$

along the sequence with $b_{x,\alpha} := (x + e_\alpha, x) \in \chi$.
Then there exists a unique shift-covariant random gradient Gibbs measure $\xi \to \mu[\xi]$ which satisfies for $\mathbb{P}$-almost every ξ

$$\mu[\xi](E_\alpha) = 1, \ \alpha \in \{1, 2, \dots, d\}.$$

Moreover, $\mu[\xi]$ satisfies the integrability condition

$$\mathbb{E} \int \mu[\xi](\mathrm{d}\eta)(\eta(b))^2 < \infty \text{ for all bonds } b \in \chi.$$

Ergodicity of the unique averaged measure:

- Let $\mathcal{F}_{inv}(\chi)$ the σ-algebra of shift-invariant events on χ. Let

$$\mu_{av} = \left(\int \mathbb{P}(d\xi)\mu[\xi] \right) (\,\mathrm{d}\eta).$$

  We need to show that for all $A \in \mathcal{F}_{inv}(\chi)$, we have $\mu_{av}(A) = 0$ or $\mu_{av}(A) = 1$. We will show that this holds by contradiction.

- Suppose that there exists $A \in \mathcal{F}_{inv}(\chi)$ such that $0 < \mu_{av}(A) < 1$. Then, for $\mathbb{P}$-almost all ξ we have $0 < \mu[\xi](A) < 1$. We define now for all ξ the *distinct* measures on χ

$$\mu_A[\xi](B) := \frac{\mu[\xi](B \cap A)}{\mu[\xi](A)} \quad \text{and} \quad \mu_{A^c}[\xi](B) := \frac{\mu[\xi](B \cap A^c)}{\mu[\xi](A^c)}, \quad \forall B \in \mathcal{T},$$

  where we denoted by $\mathcal{T} := \sigma(\{\eta_b : b \in \chi\})$ the smallest σ-algebra on χ generated by all the edges in χ.

THANK YOU!