Planar maps, circle patterns, conformal point processes and two dimensional gravity

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joint work with Bertrand Eynard
(+ recent work with Séverin Charbonnier)

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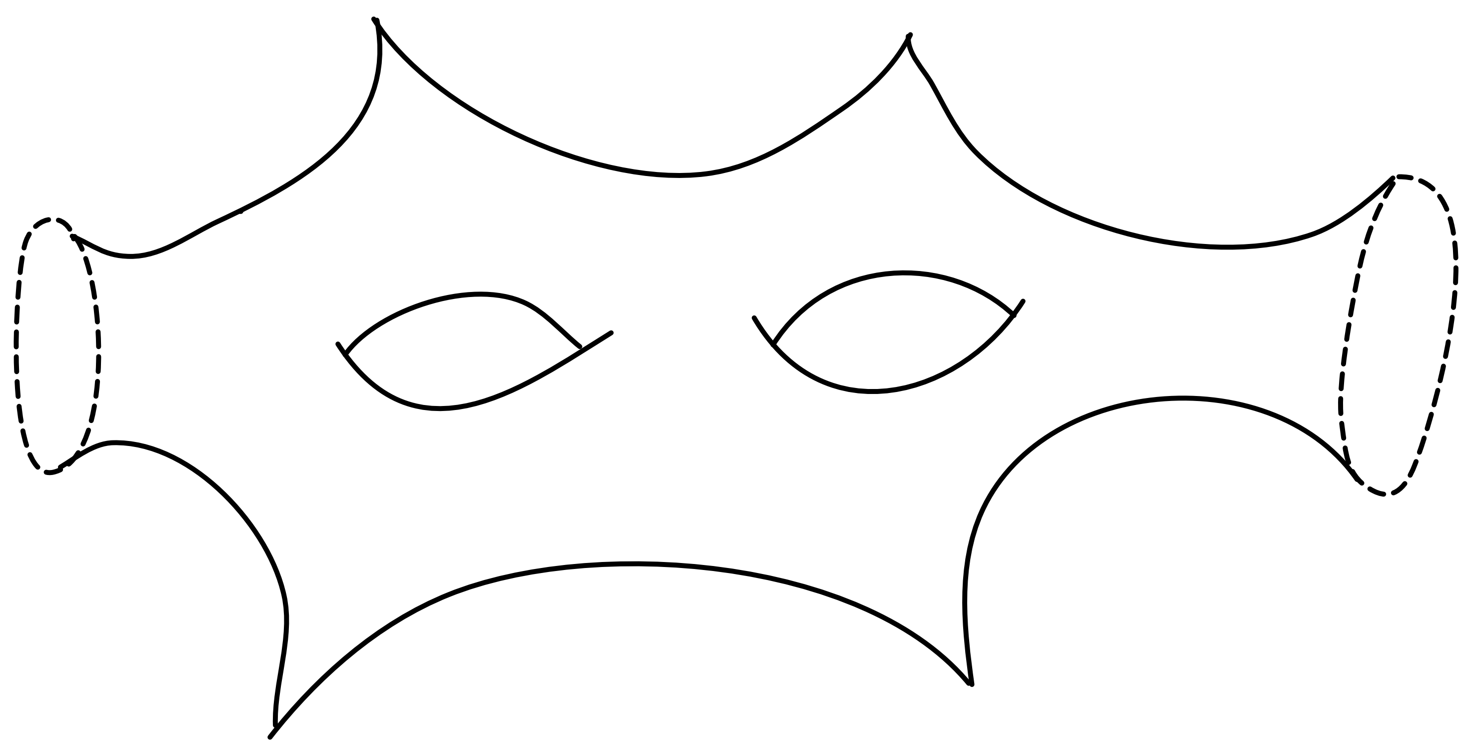
1. **Continuum and discrete 2D gravity: what remains to be understood?**
2. Circle packings and circle patterns
3. Delaunay circle patterns and planar maps
4. A measure over planar triangulations
5. Spanning 3-trees representation
6. Kähler geometry over triangulation space and 3D hyperbolic geometry
7. Discretized Faddev-Popov operator and Polyakov's 2D gravity
8. Local uniform bounds and the continuum limit

1. a: Continuum formulations of 2D quantum gravity

Quantization = Feynman path integral over 2D Riemannian metrics (+ matter fields)

$$\int \mathcal{D}[g_{ab}] \exp\left(-\mu_0 \int d^2x \sqrt{|g|}\right) \int \mathcal{D}[X]\, e^{-S[X,g]}$$



For physicists, this is a fairly well understood theory

Polyakov functional integral, use the conformal gauge *A. Polyakov 1981*

$$g_{ab} = \hat{g}_{ab}\, e^{\phi}$$

The Faddeev-Popov ghost systems leads to the effective action for the remaining conformal factor.

$$\int \mathcal{D}[g_{ab}] = \int \mathcal{D}_{\hat{g}e^{\phi}}[\phi]\ \det(\nabla^{FP}_{\hat{g}}) = \int \mathcal{D}_{\hat{g}}[\varphi]\ e^{-S_L[\varphi]}$$

The effective theory is the Liouville theory (from conformal anomaly consistency condition), a CFT and integrable quantum theory

$$S_L[\varphi] = \frac{1}{2\pi}\int \sqrt{\hat{g}}\left(\frac{1}{2}(\hat{\nabla}\varphi)^2 + \frac{Q}{2}\hat{R}\,\varphi + \mu_R\, e^{\gamma\varphi}\right)$$

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} \qquad\qquad c_L = 1 + 6Q^2 = 26 - c_M$$

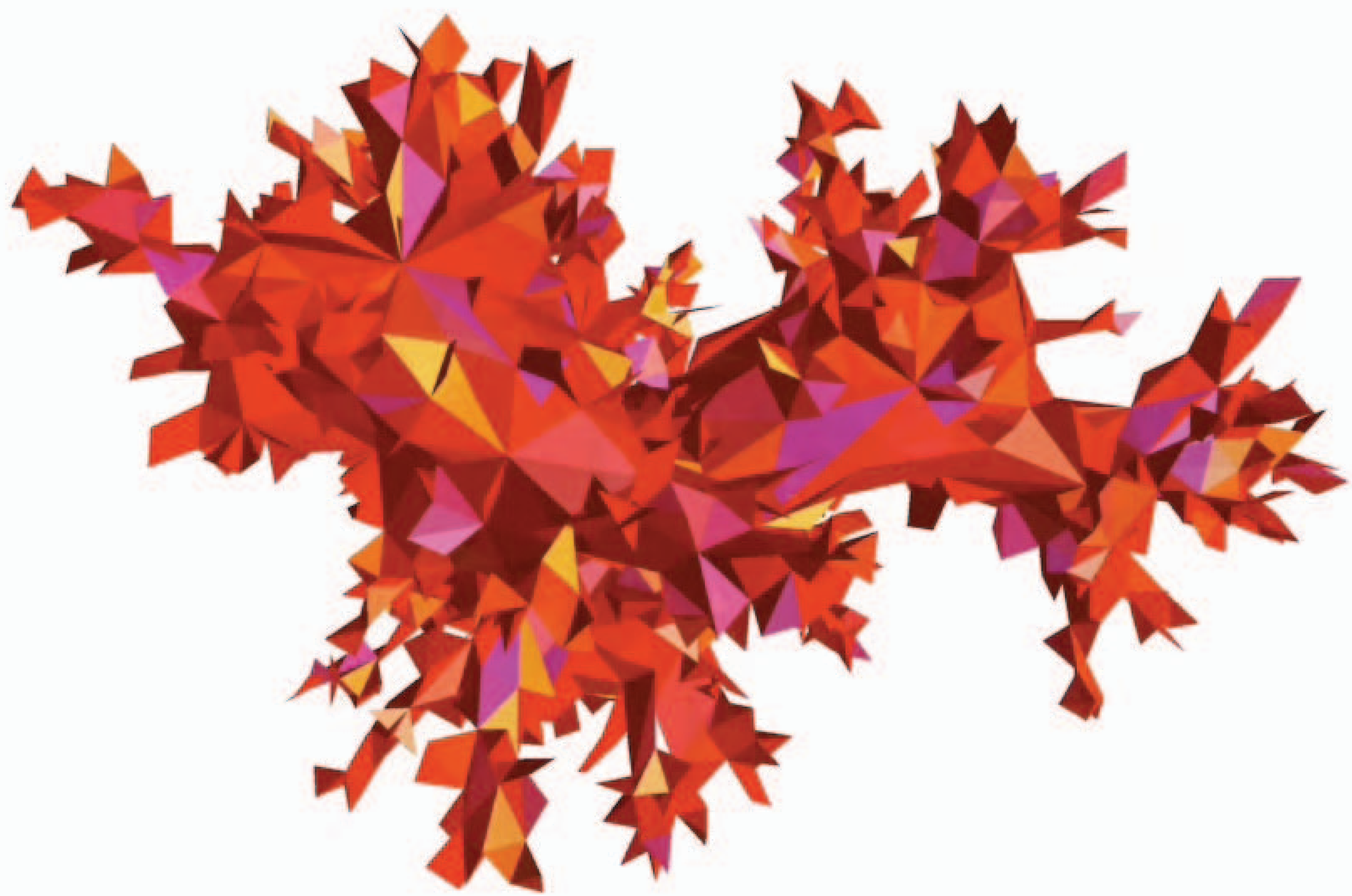
For pure gravity $\quad c_M = 0$

Real positive action for $\quad -\infty < c_M \leq 1$

1.b: Discretized formulations of 2D quantum gravity *

Random planar lattices (maps)

F. D., J. Fröhlich, V. Kazakov & A. Migdal (circa 1984-85)



$N \simeq 10^3$

⚠ not an isometric embedding!

Also fairly well understood

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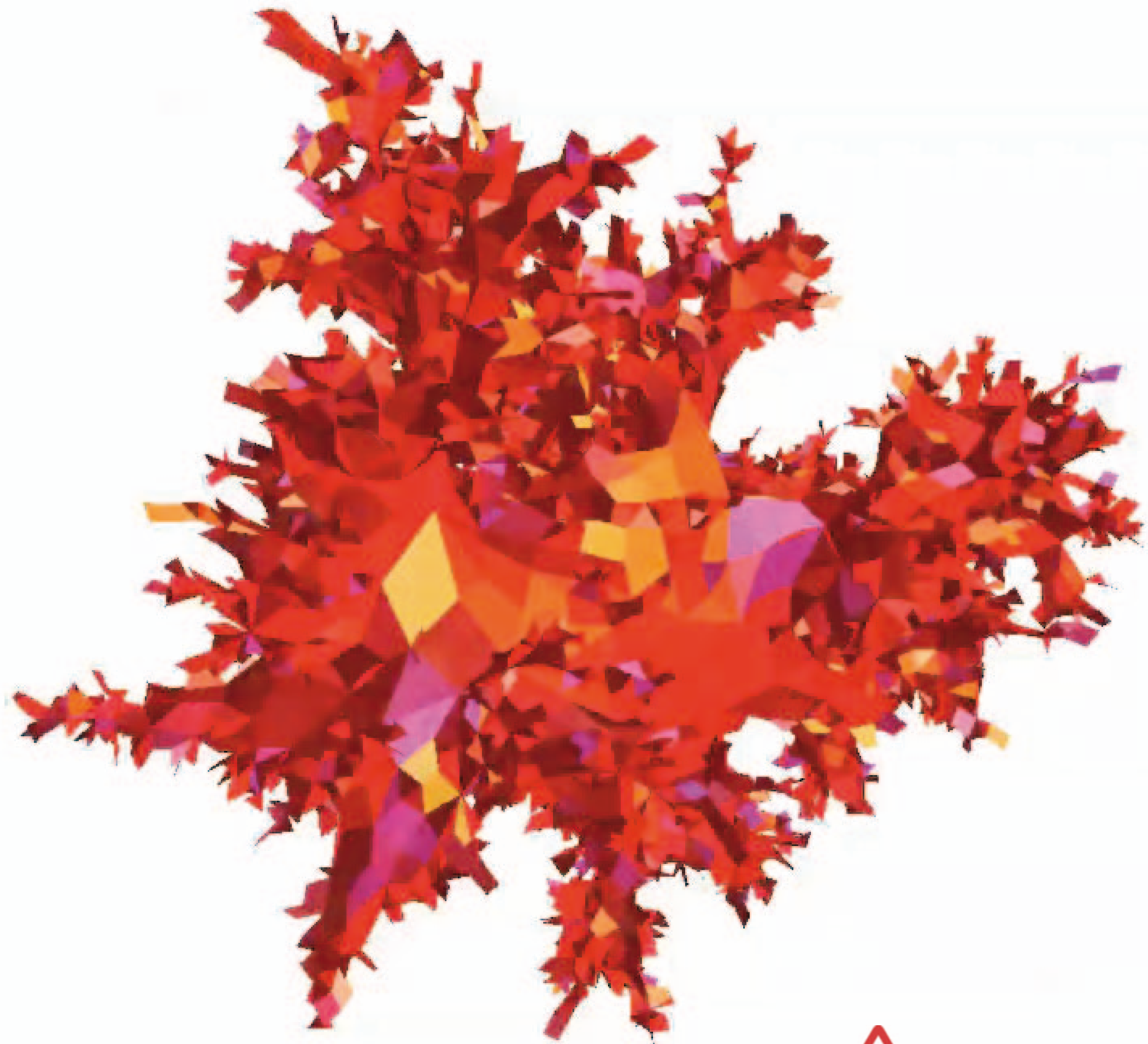


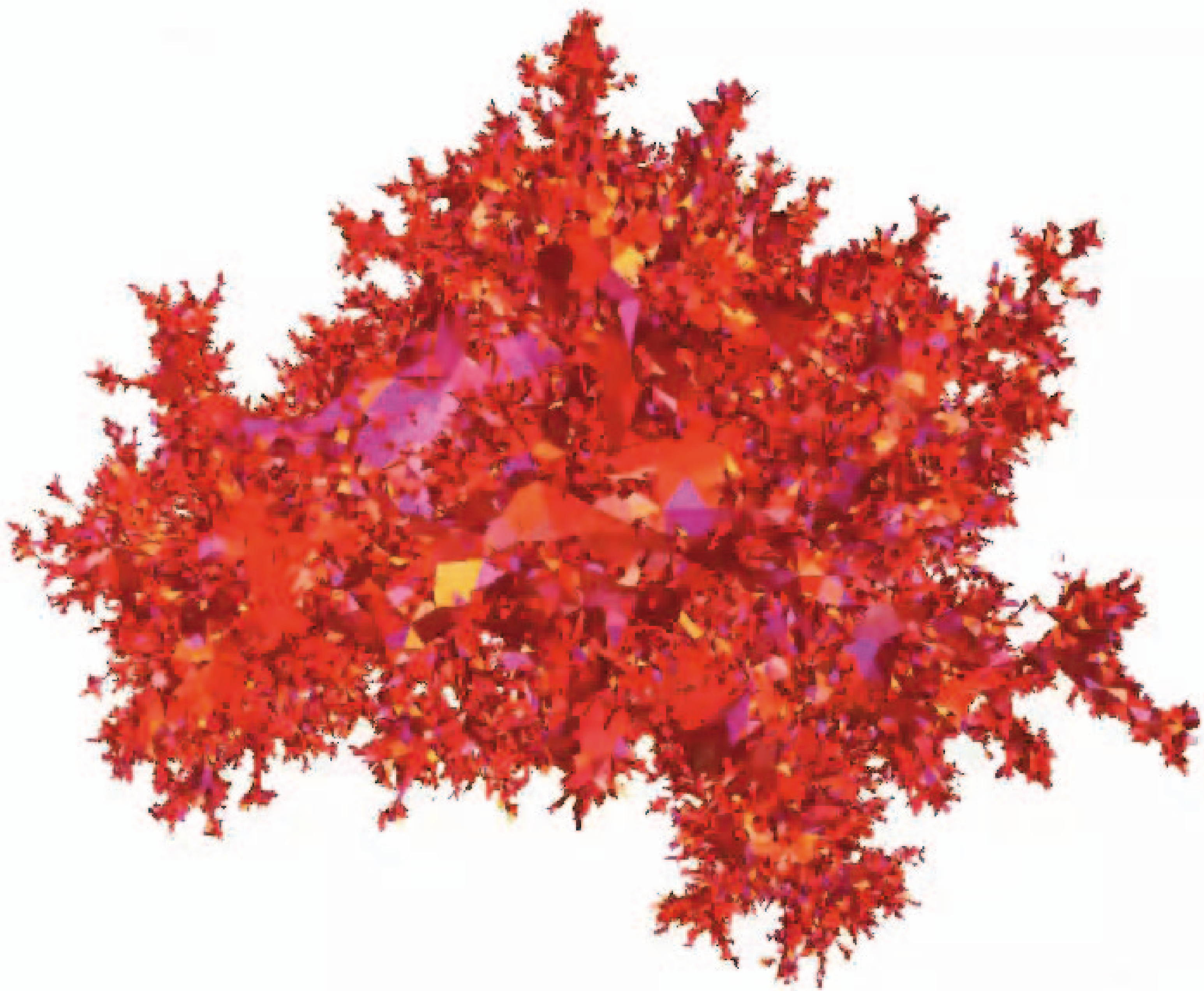
$N \simeq 10^4$

⚠ not an isometric embedding!

Also fairly well understood

1.b: Discretized formulations of 2D quantum gravity *

Random planar lattices (maps) F. D., J. Fröhlich, V. Kazakov & A. Migdal (circa 1984-85)



$N \simeq 10^5$

⚠ not an isometric embedding!

Also fairly well understood

Discrete 2d gravity

Continuous 2d gravity

recursion relations
integrability

Topological gravity

Random matrices

QFT
CFT
integrability

combinatorics

Liouville theory &
conformal field theories

Planar maps
(diagrams)

conformal invariance
probabilities

KPZ relations

Cori-Vauquelin-
Schaeffer-....
bijections
combinatorics
probabilities

Gaussian free field, SLE

Brownian maps &
well labelled trees

Still lacking: a constructive & geometrical link between the discrete and the continuous formulations

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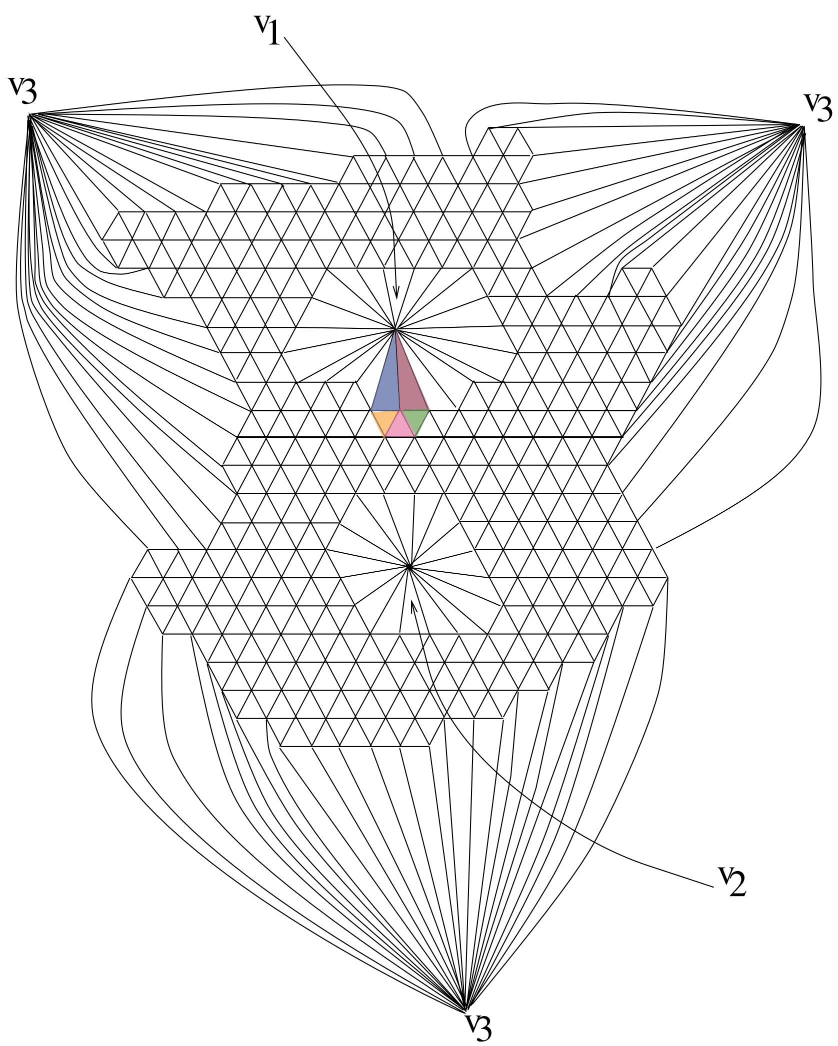
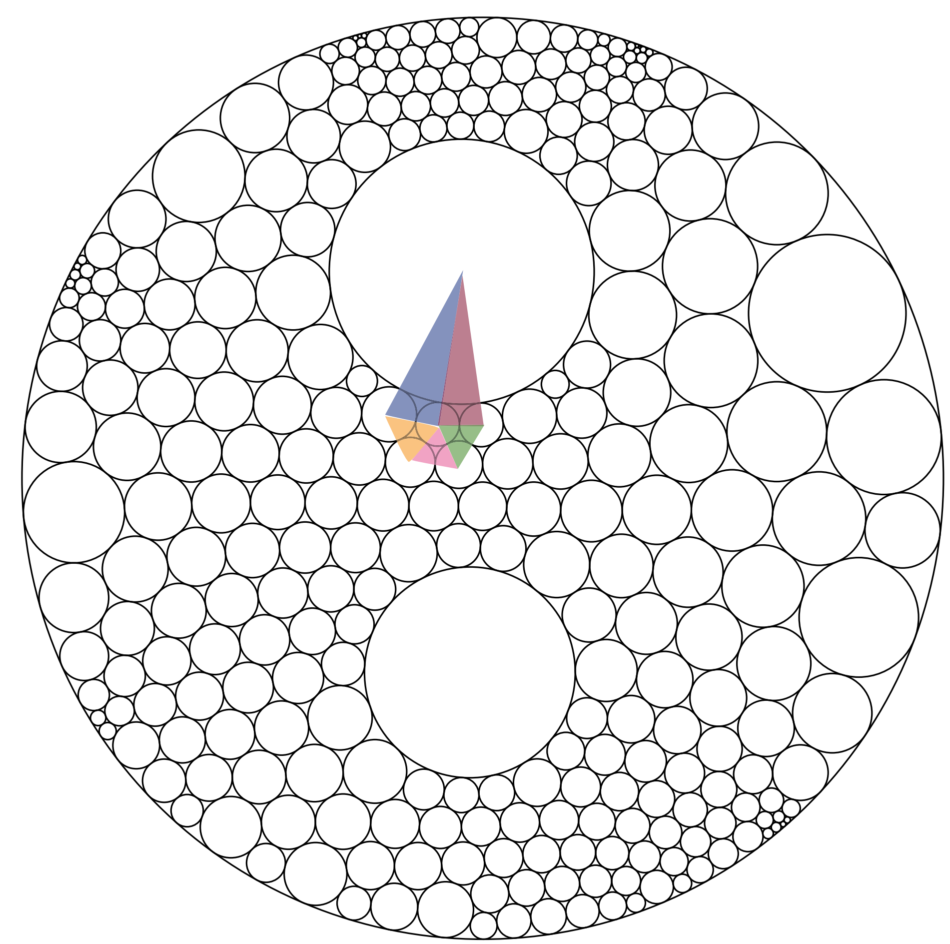
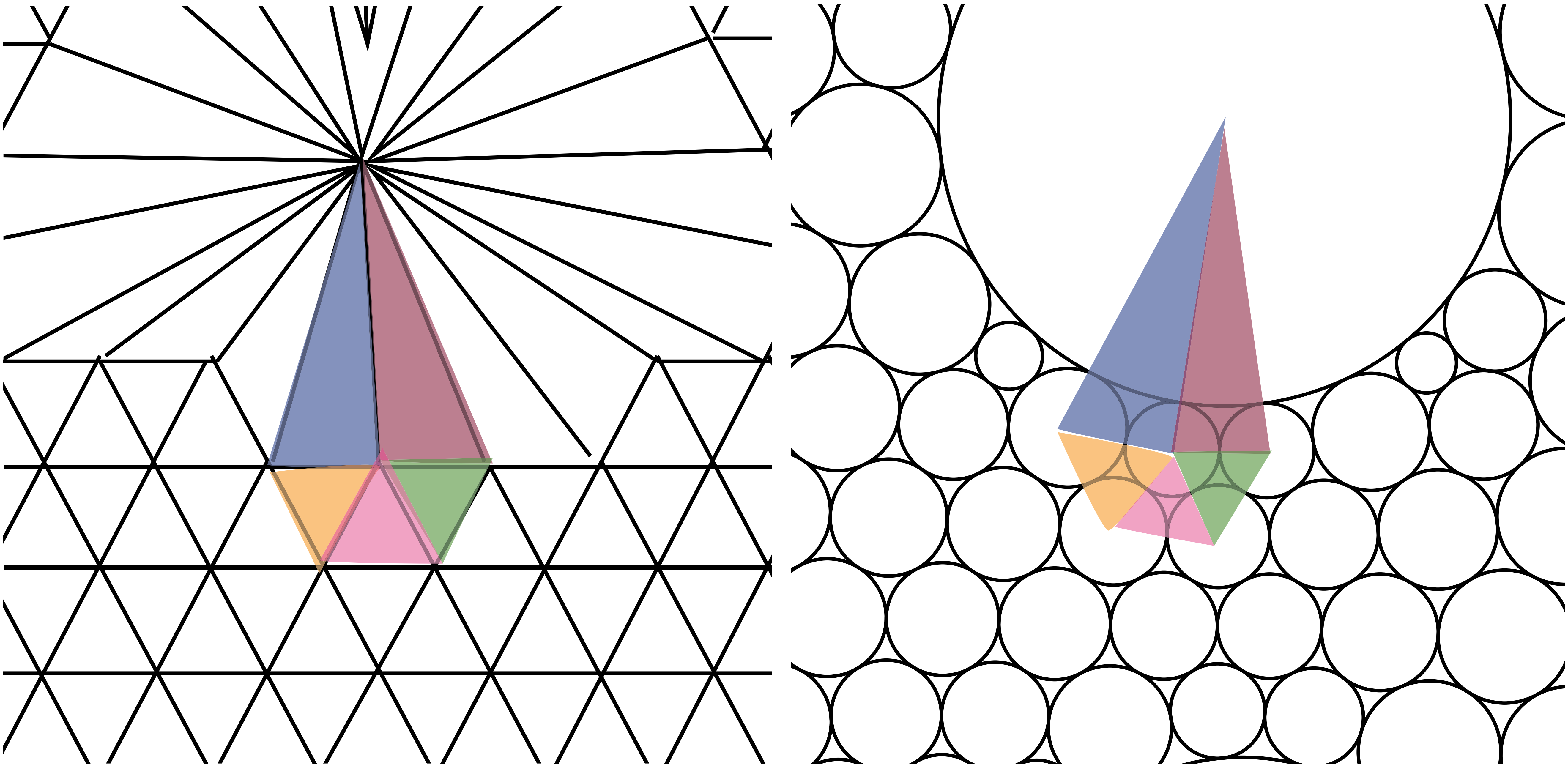
2.a: Circle packings
The Koebe-Andreev-Thurston theorem

There is a bijection between simple triangulations and circle packings, modulo SL(2,C) Möbius transformations



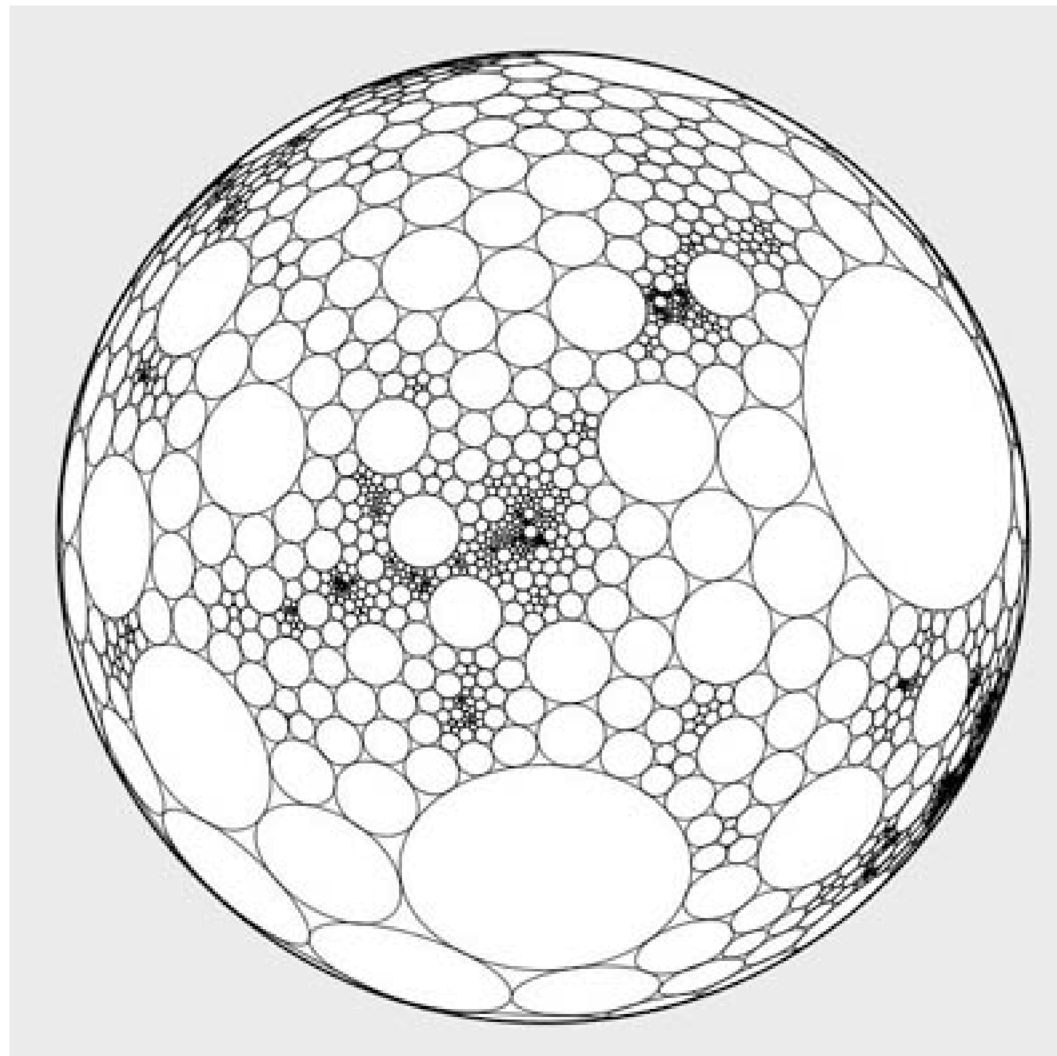


Illustrations borrowed to Schramm & Mishenko

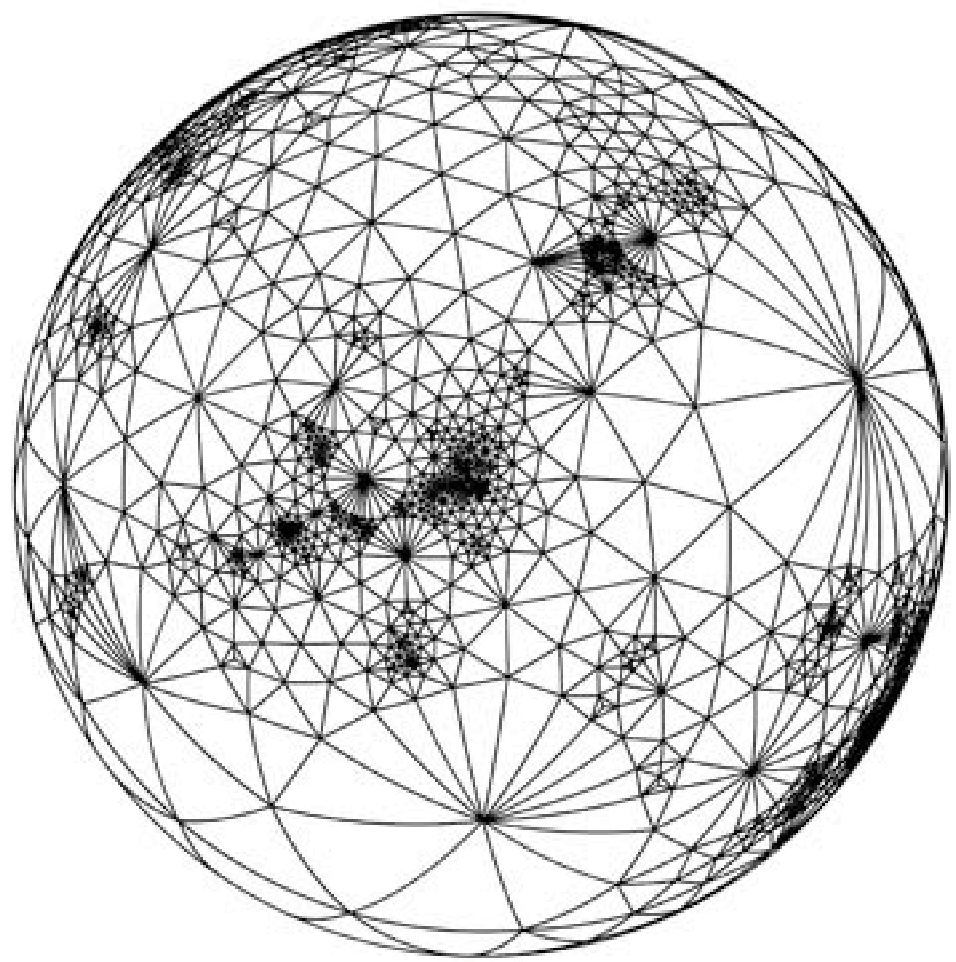


Planar triangulation $\quad N \simeq 10^3$

Circle packing embedding

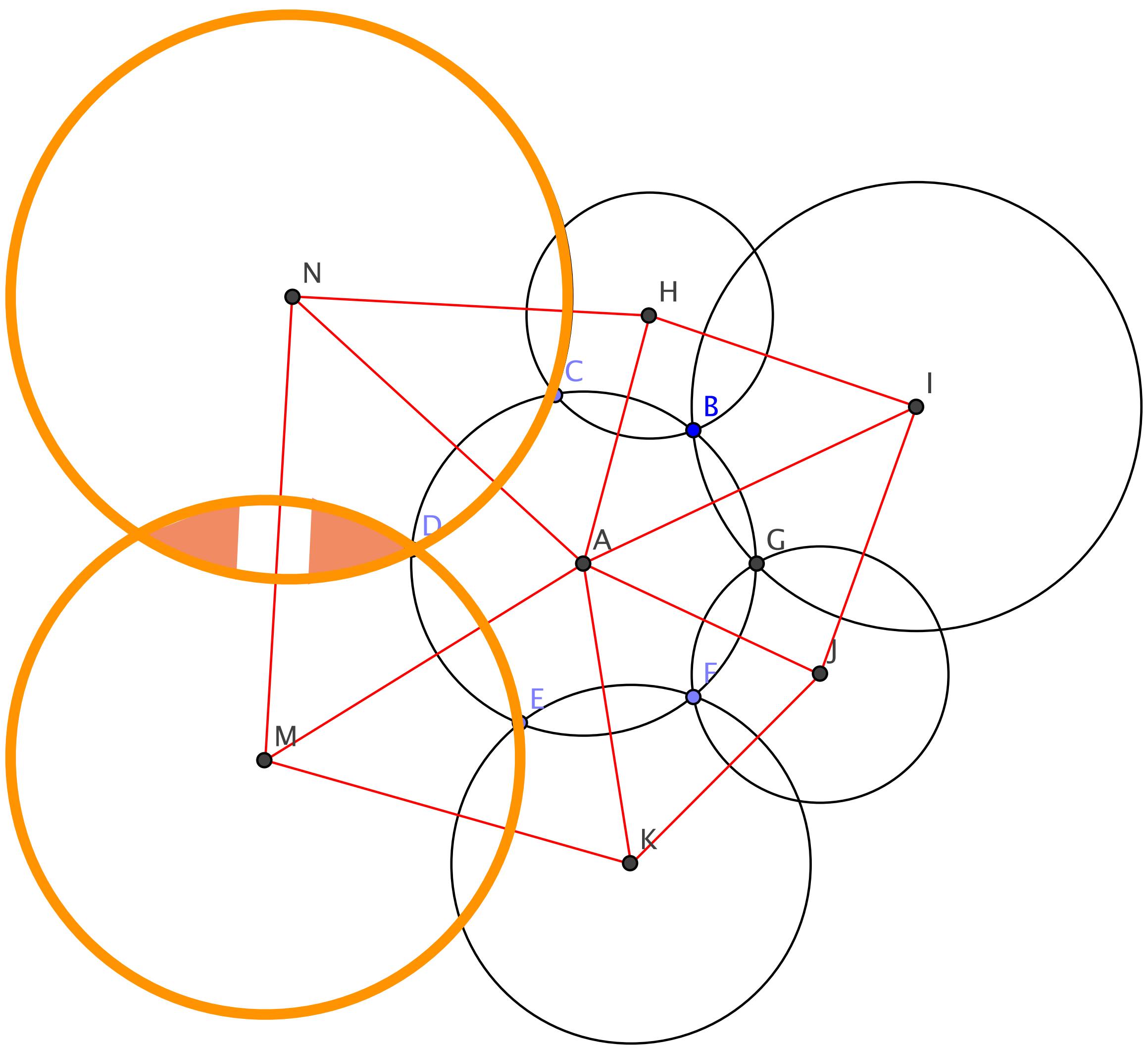


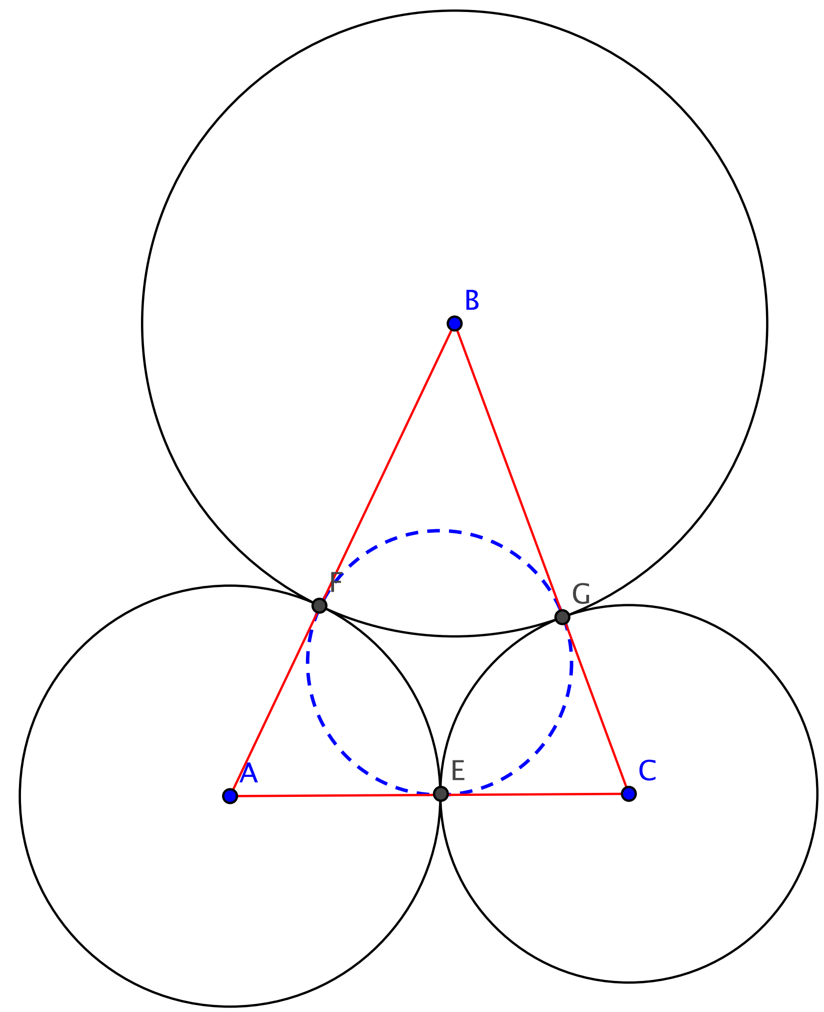
Dual spherical triangle embedding



2.b: A generalisation of circle packings: circle patterns

Circles meeting at common points.
The angles of intersection of the circles are given.
Find the planar pattern and the radii of the circles.





theorem of existence and unicity
(Igor Rivin 1994, Ann. of Math.)

circle packing = circle patterns
with angles 0 or $\pi/2$ only

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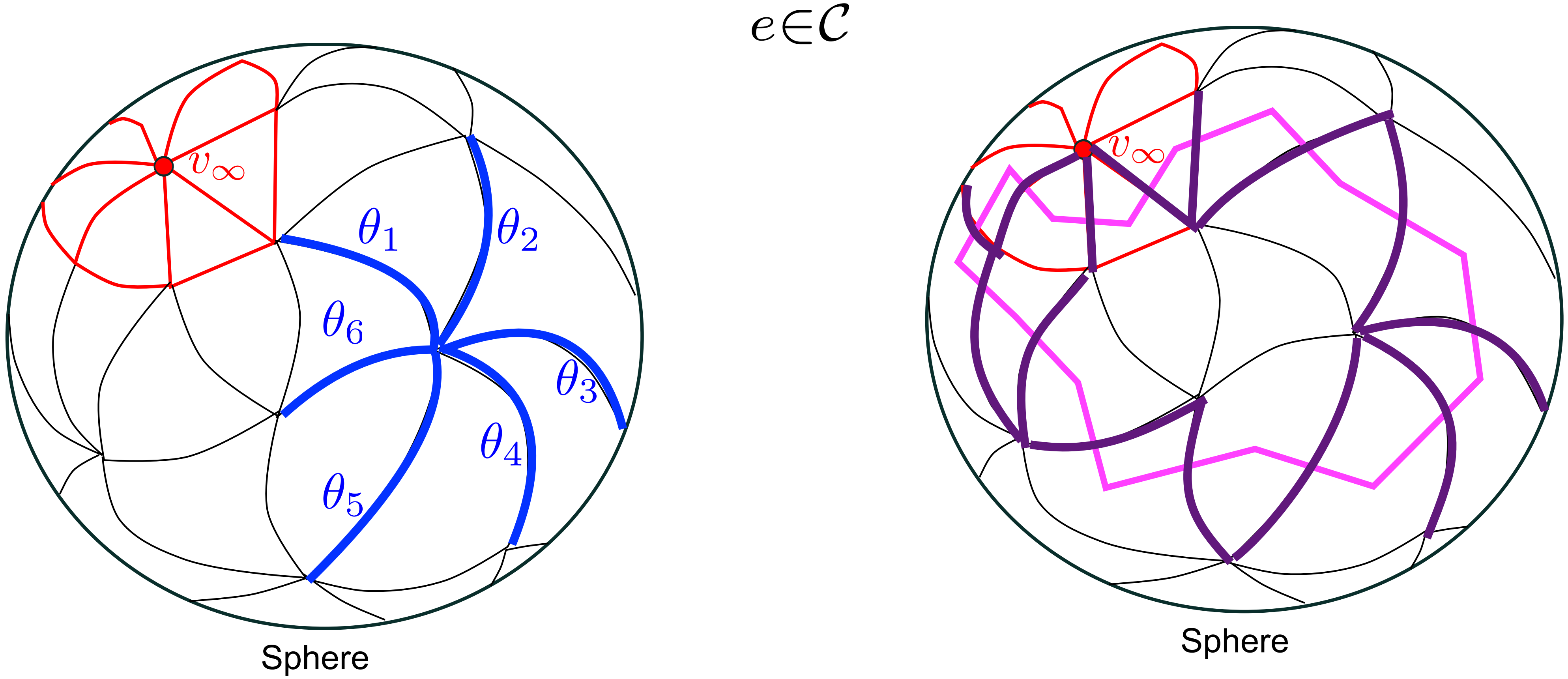
«Dressed» planar maps

$\mathcal{T} = (T, \{\theta\})$ T an abstract triangulation of the sphere, with $\{\theta\}$ angles θ_e attached to the edges e such that

1) the angles are positive $0 < \theta_e < \pi$

2) around each vertex $$\sum_{e \to v} \theta_e = 2\pi$$

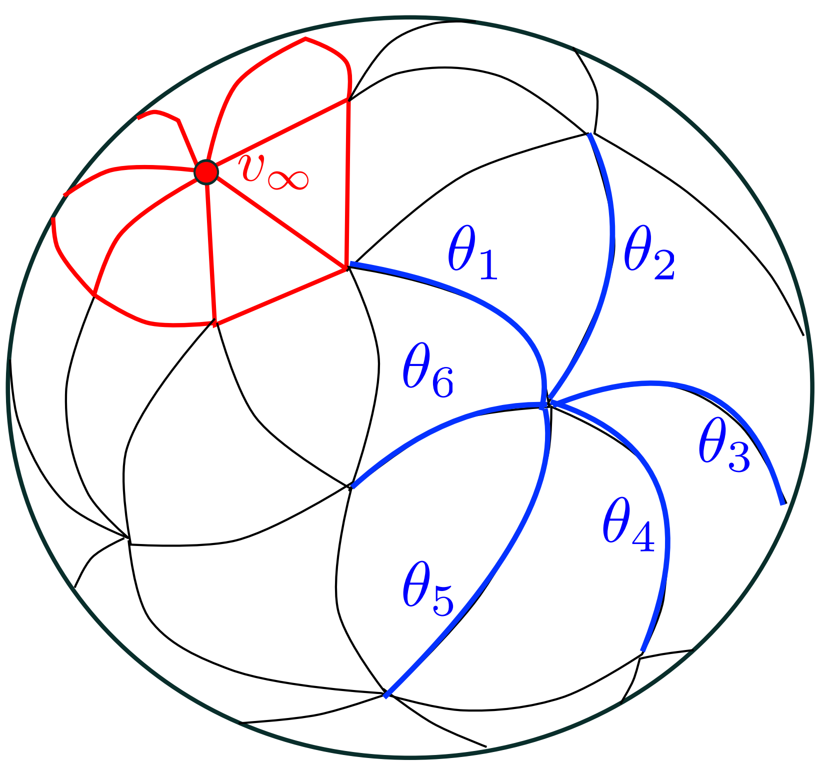
3) around each closed cycle $$\sum_{e \in \mathcal{C}} \theta_e \geq 2\pi$$



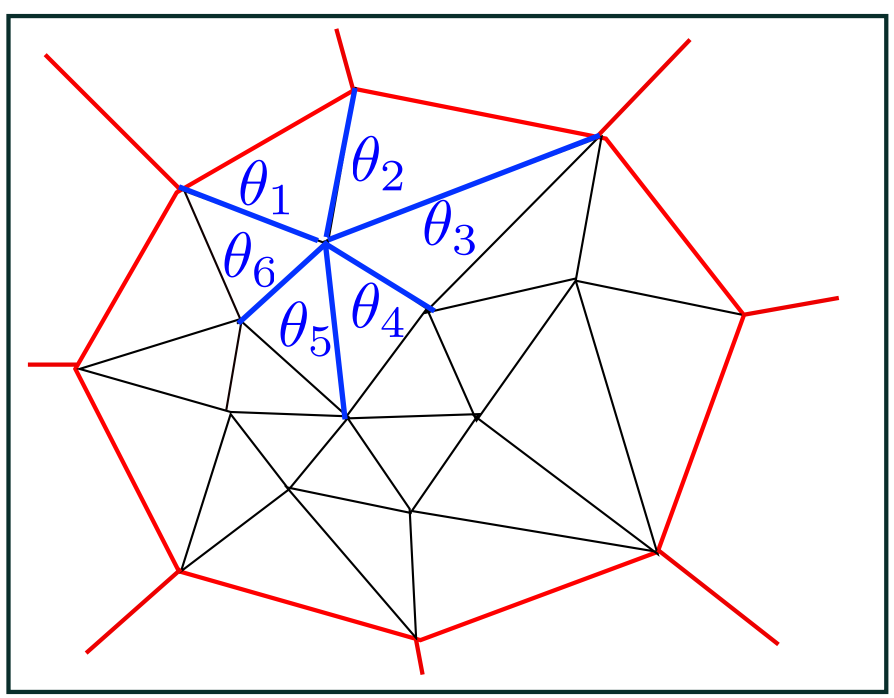
Delaunay triangulations and planar maps

There is a bijection* between such «dressed» planar maps and Delaunay triangulations in the plane, such that the circle intersection angles are $\theta_e^* = \pi - \theta_e$

* modulo global $\mathrm{SL}(2,\mathbb{C})$ transformations (as usual)

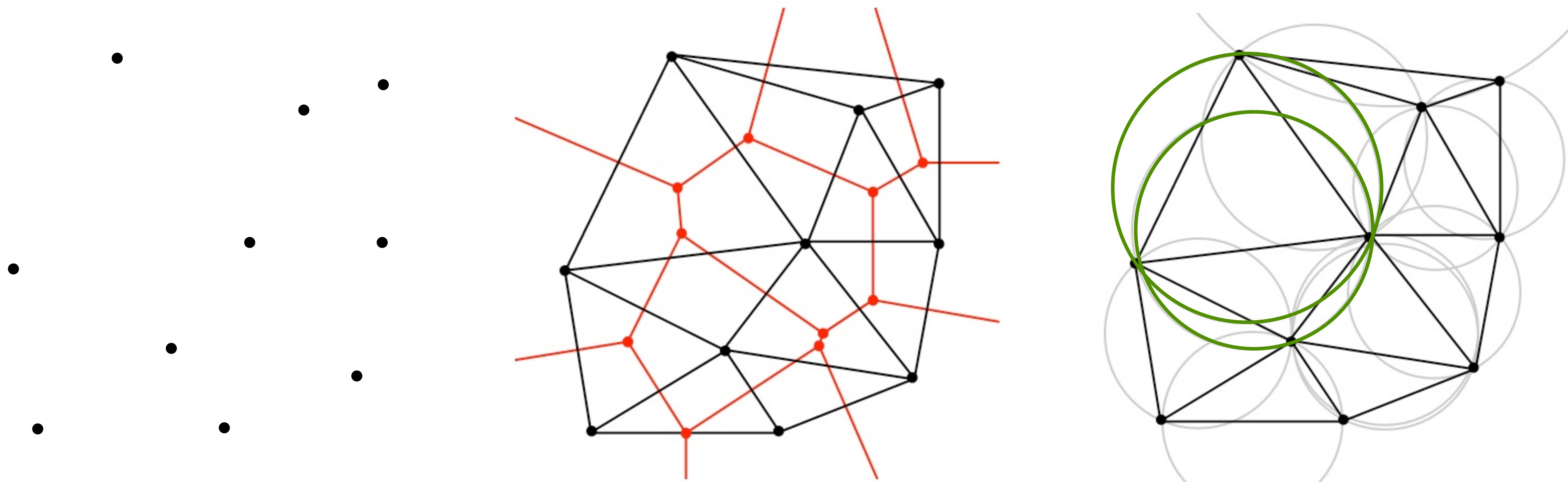


Sphere



Plane

Voronoï tessellation and Delaunay triangulation in the plane



From Wikimedia Commons

Delaunay condition: no vertex of a triangle must be inside the circumscribed circle to any another triangle ⇒ $\theta > 0$

Local flatness (p.l. manifold) ⇒ $$\sum_{e \to v} \theta_e = 2\pi$$

Last condition on cycles ⇒ no change of orientations or foldings (to be discussed later)

This excludes some triangulations but it is conjectured that one keeps generic ones (universality)

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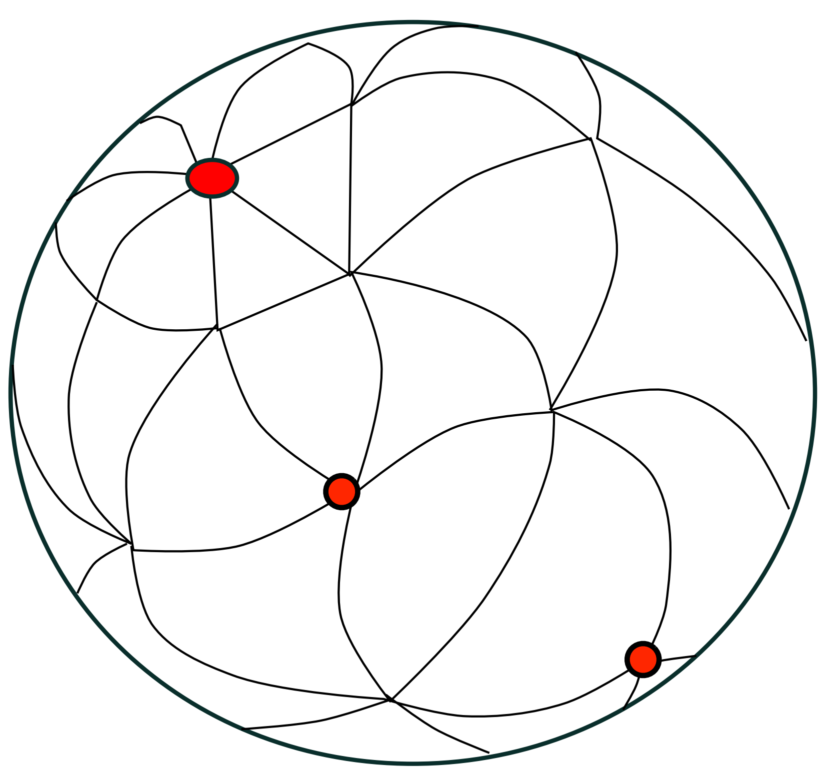
Take as initial measure on triangulations the uniform measure on triangulations and the flat measure on the angles (+ inequalities)

$$\mu(\widetilde{T}) = \mu(T, d\boldsymbol{\theta})) = \mathrm{uniform}(T) \prod_{e \in \mathcal{E}(T)} d\theta(e) \quad \prod_{v \in \mathcal{V}(T)} \delta\left(\sum_{e \mapsto v} \theta(e) - 2\pi\right)$$

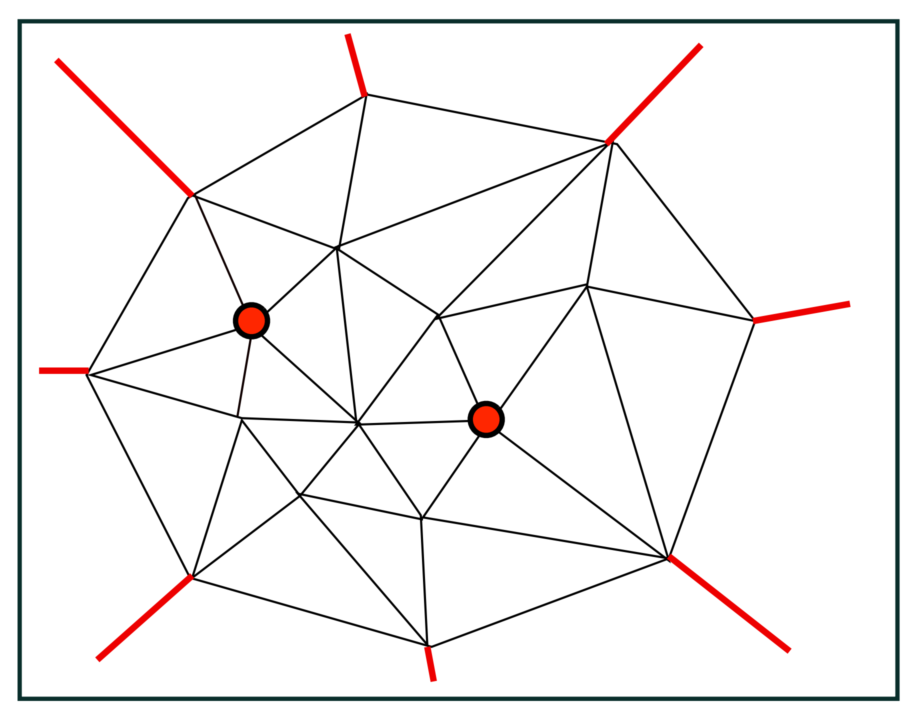
Question: which measure does this induce on Delaunay triangulations? For this consider N+3 points, 3 fixed by $\mathrm{SL}(2,\mathbb{C})$

$$\mathfrak{D}_{N+3} = \mathbb{C}^{N+3}/\mathrm{SL}(2,\mathbb{C}) \simeq \mathbb{C}^{N}$$

$$d\mu(z_4, \cdots, z_{N+3}) = ?$$



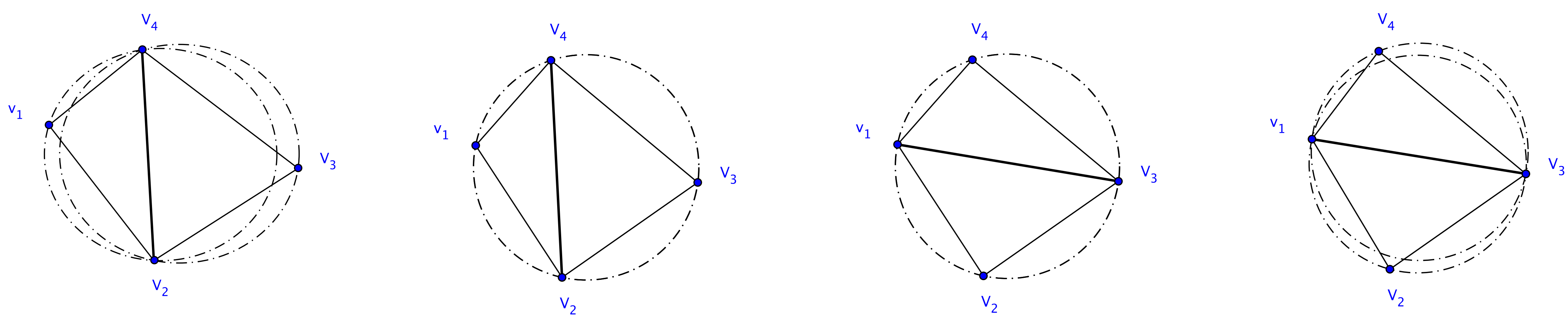
Sphere



Plane

Moving the points allows to explore the whole space of Delaunay triangulations and of dressed abstract triangulations

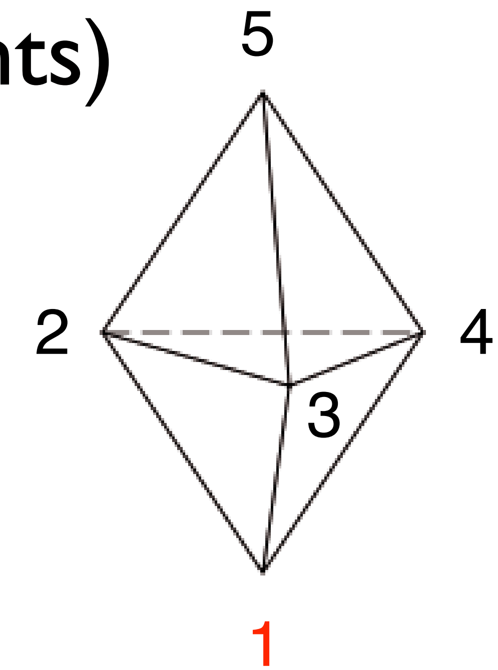
Transition between Delaunay triangulations by edge flips

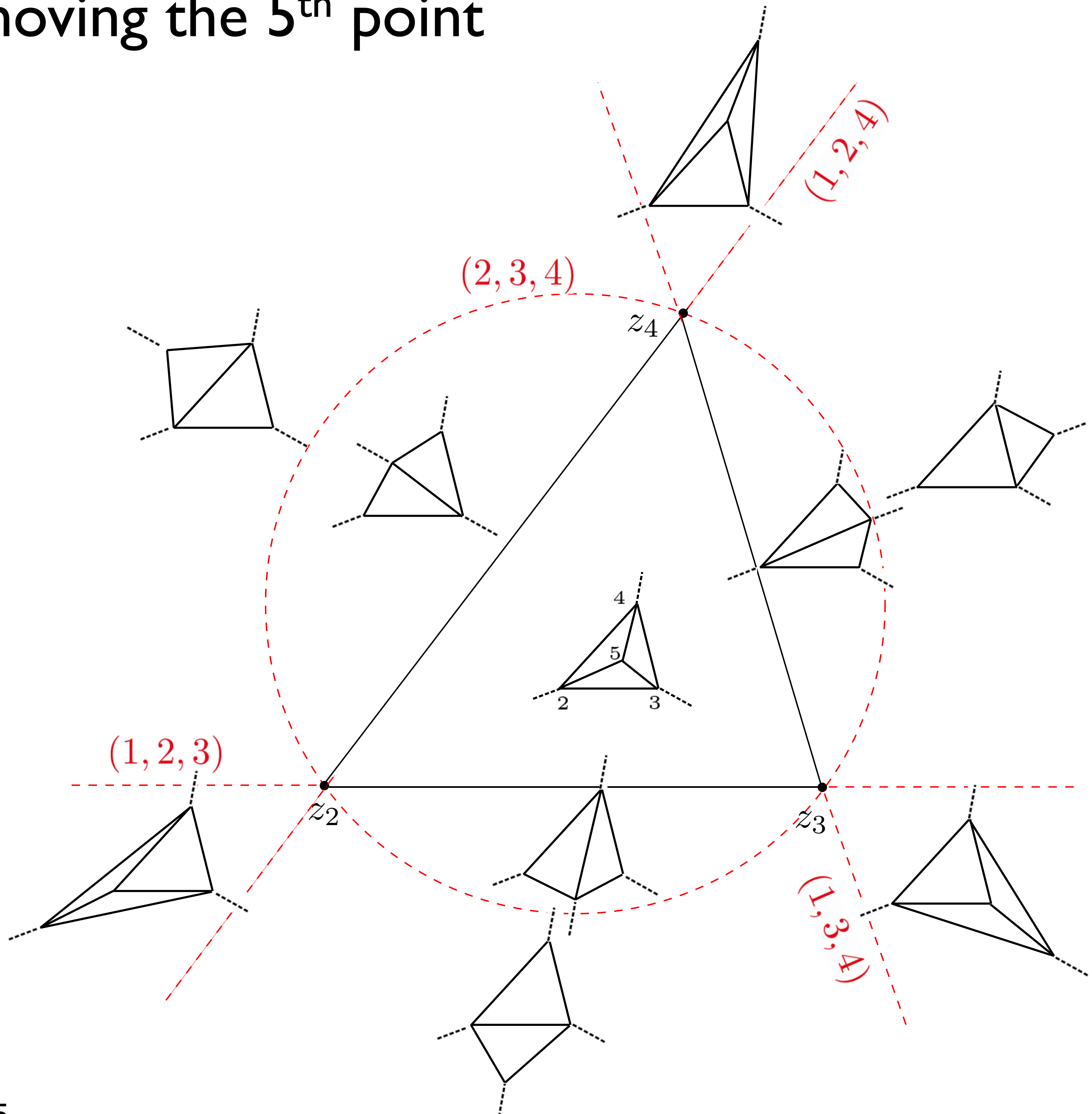


the flip of link e occurs when $\theta(e) = 0$

Elementary example: the hexahedron (5 points)

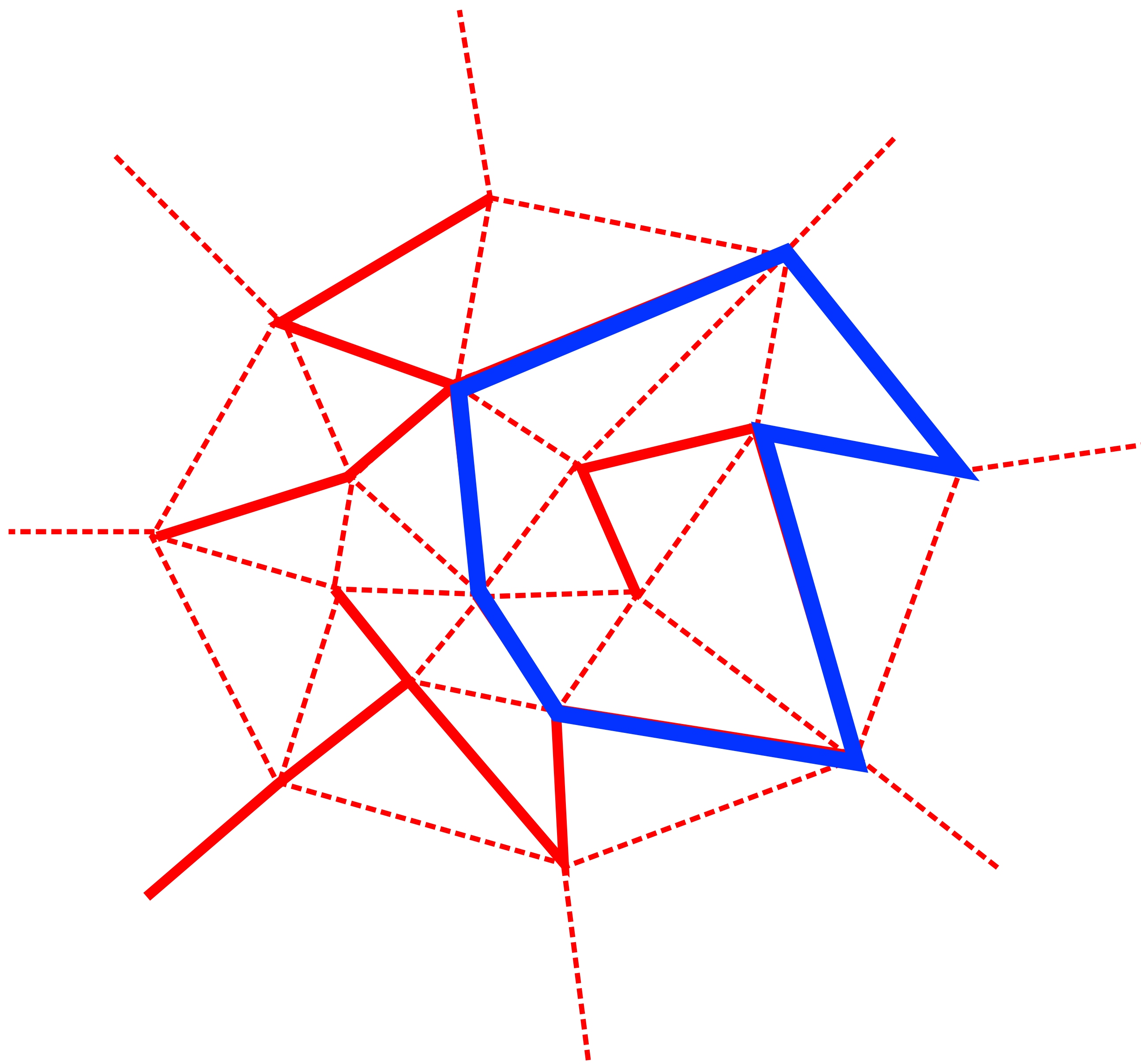
moving the 5th point





Question 1: Which sets of edges form independent basis for the angles?

Answer: The sets whose complementary form a cycle-rooted-spanning-tree of the triangulation with odd length cycle



Question 2: What is the Jacobian of the change of angle variables between two basis of edges?

Answer: Jacobian = 1 !

Indeed... $\mu(T, d\boldsymbol{\theta}) \ = \ \frac{1}{2}\,\mathrm{uniform}(T) \times \prod_{e \in \mathcal{E}_0(T)} d\theta(e)$

So, the measure over the points is given locally (for a given Delaunay triangulation) by a simple Jacobian

$$\mu(T, d\boldsymbol{\theta}) \ = \ d\mu(\mathbf{z}) \ = \ \prod_{v=4}^{N+3} d^2 z_v \ \left|\det\left(J_T(z)_{\smallsetminus\{1,2,3\}\times\bar{\mathcal{E}}_0}\right)\right|$$

$$J_T(z) = \left(\frac{\partial \theta_e}{\partial\left(z_v, \bar{z}_v\right)}\right)_{\substack{e \in \mathcal{E}(T) \\ v \in \mathcal{V}(T)}}$$

Question 3: What is the local form of the measure?

The matrix elements of the Jacobian are made of simple poles

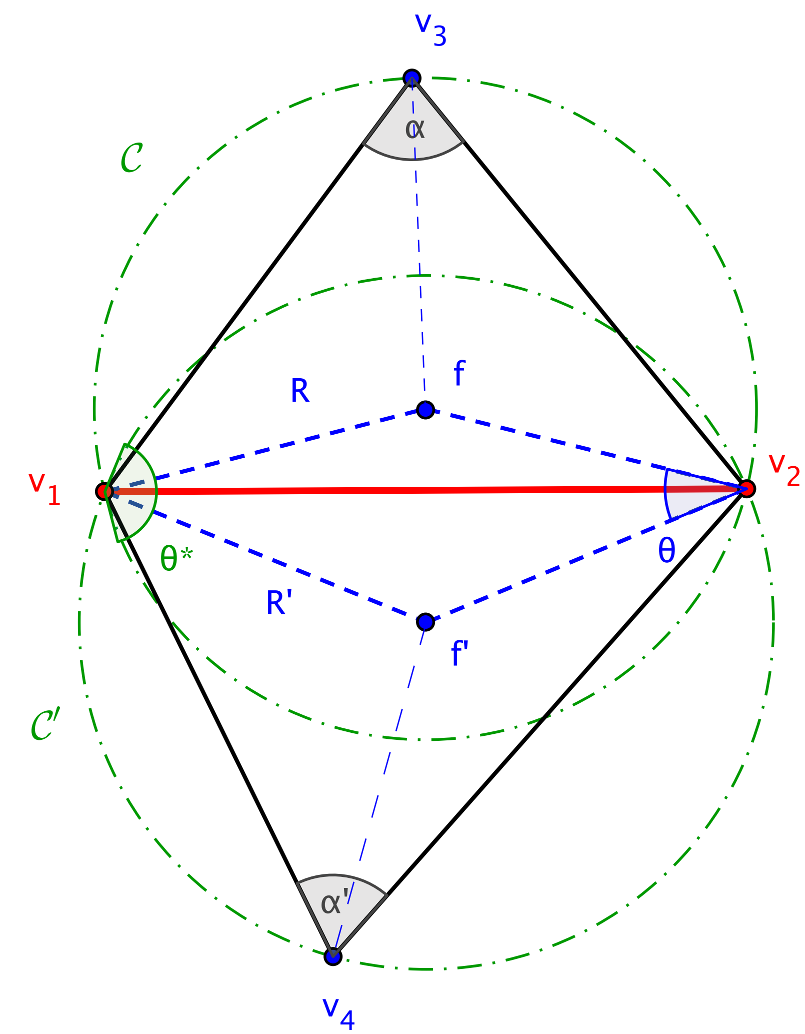
$$J_{v,e} = \frac{\partial \theta_e}{\partial z_v} \quad , \qquad J_{\bar{v},e} = \frac{\partial \theta_e}{\partial \bar{z}_v}$$

$$J_{v_1,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_4} - z_{v_1}} - \frac{1}{z_{v_3} - z_{v_1}}\right)$$

$$J_{v_3,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_3} - z_{v_1}} - \frac{1}{z_{v_3} - z_{v_2}}\right)$$

$$J_{v_2,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_3} - z_{v_2}} - \frac{1}{z_{v_4} - z_{v_2}}\right)$$

$$J_{v_4,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_4} - z_{v_2}} - \frac{1}{z_{v_4} - z_{v_1}}\right)$$



The determinant of the Jacobian matrix is locally a rational function of the z_v's and $\bar{z}_v$'s

$$\mathcal{D}_T(z)_{\setminus\{1,2,3\}} = \left|\det\left(J_T(z)_{\setminus\{1,2,3\}\times\bar{\mathcal{E}}_0}\right)\right|$$

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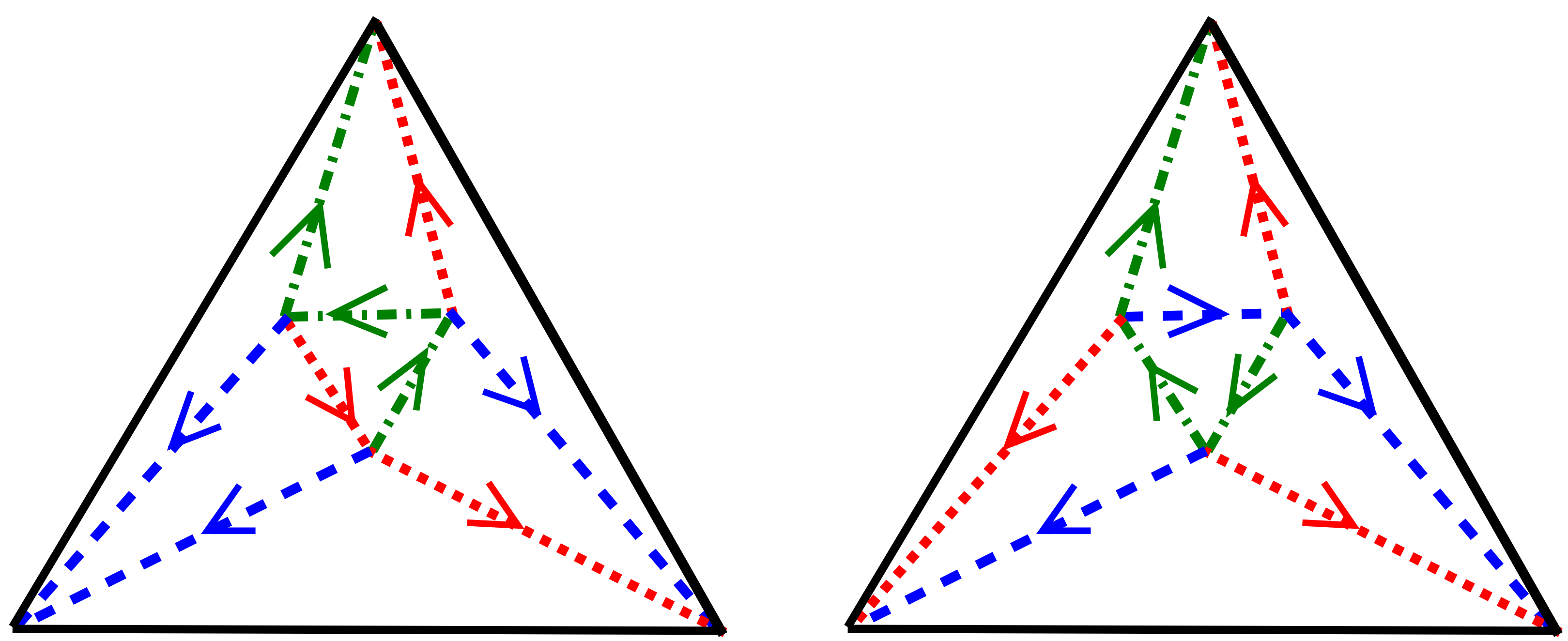
Definition 2.3 (triangle rooted spanning 3-tree)
Let T be a planar triangulation with $N+3$ vertices, and $\triangle = f_0$ be a face (triangle) of T, with 3 vertices $\mathcal{V}(\triangle) = (v_1, v_2, v_3)$ and 3 edges $\mathcal{E}(\triangle) = (e_1, e_2, e_3)$. Let $\mathcal{E}(T)_{\smallsetminus\triangle} = \mathcal{E}(T) \smallsetminus \mathcal{E}(\triangle)$ be the set of $3N$ edges of T not in $\triangle$.

We call a $\triangle$-rooted 3-tree of T ($\triangle R3T$) a family $\mathcal{F}$ of three disjoint *subsets $(\mathcal{I}, \mathcal{I}', \mathcal{I}'')$ of edges of $\mathcal{E}(T)$ such that:*

1. *$(\mathcal{I}, \mathcal{I}', \mathcal{I}'')$ are disjoint and disjoint of $\mathcal{E}(\triangle)$*
2. *Each $\mathcal{I} \cup \mathcal{E}(\triangle)$, $\mathcal{I}' \cup \mathcal{E}(\triangle)$, $\mathcal{I}'' \cup \mathcal{E}(\triangle)$ is a cycle rooted spanning tree of T with cycle $\triangle$.*



nb: Δ-rooted spanning 3-trees are NOT Schnyder woods !

Theorem 2.3 *Let T be a planar triangulation of the plane with $N+3$ vertices. If the 3 fixed points (v_1, v_2, v_3) belong to a triangle (face) of T, the measure determinant takes the form*

$$\mathcal{D}_T(z)_{\smallsetminus\{1,2,3\})} = 4^{-N} \sum_{\substack{\mathcal{F}=(\mathcal{I},\mathcal{I}',\mathcal{I}'') \\ \triangle R3T\ of\ T}} \epsilon(\mathcal{F}) \times \prod_{\vec{e}=(v\to v')\in\mathcal{I}} \frac{1}{z_v - z_{v'}} \times \prod_{\vec{e}=(v\to v')\in\mathcal{I}'} \frac{1}{\bar{z}_v - \bar{z}_{v'}} \qquad (2.10)$$

where $\epsilon(\mathcal{F}) = \pm 1$ is a sign factor, coming from the topology of T and of $\mathcal{F}$, that will be defined later.

- The measure can be written as a sum over these spanning 3-trees
- Tree 1: analytic part - Tree 2: anti-analytic part
- This is a non trivial extension of the spanning tree representation of the determinant of scalar Laplacians.
- This representation is specific to this Jacobian matrix *D*.
- It is useful for proof of convergence, factorization properties, etc.
- Can we use it for more?

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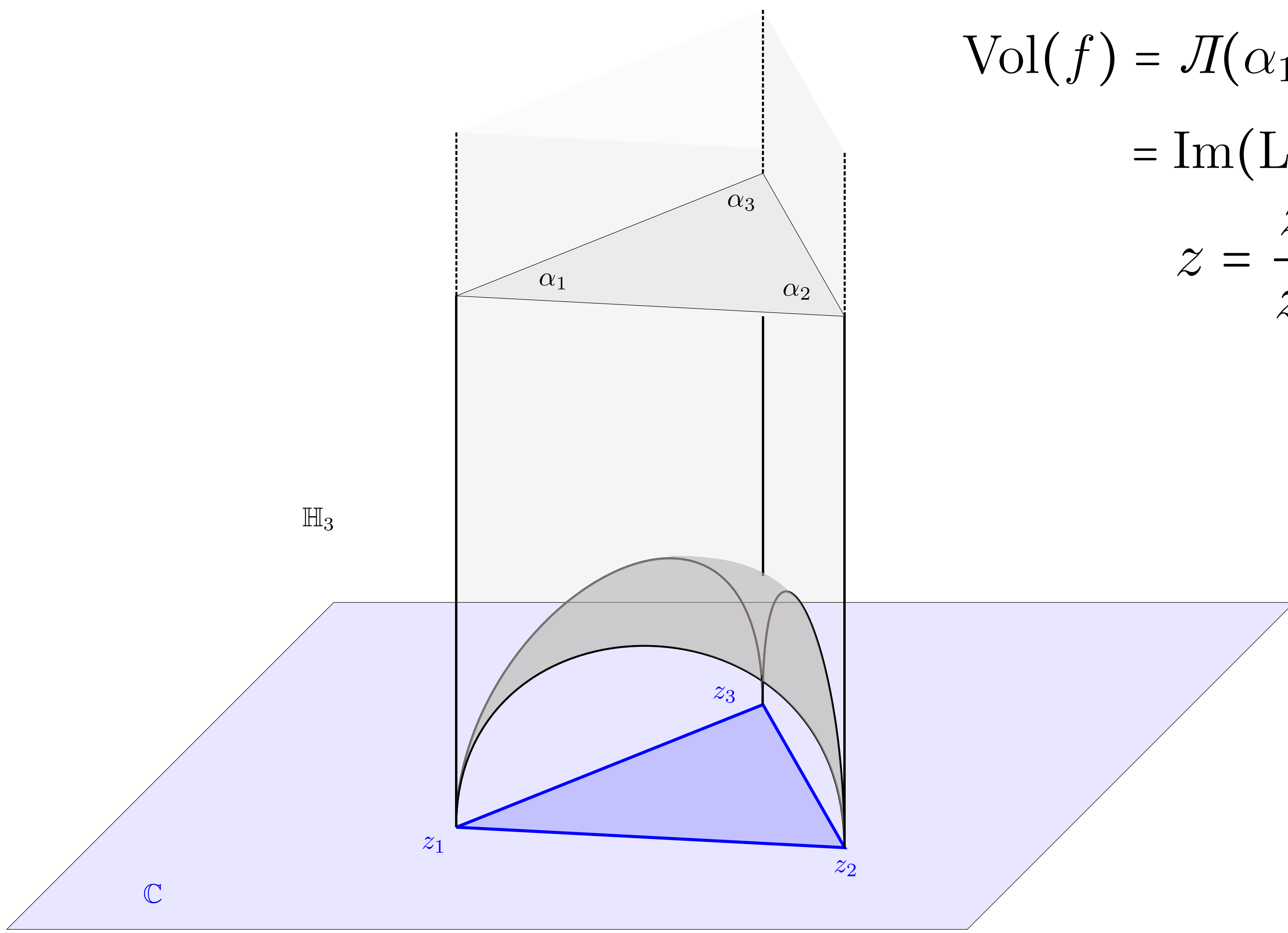
Hyperbolic volume of triangle = volume of ideal tetrahedron above the triangle in hyperbolic Poincaré half-space



$$\mathrm{Vol}(f) = Л(\alpha_1) + Л(\alpha_2) + Л(\alpha_3)$$
$$= \mathrm{Im}(\mathrm{Li}_2(z)) + \ln(|z|)\mathrm{Arg}\,(1-z)$$
$$z = \frac{z_3 - z_1}{z_2 - z1}$$

"action" of a triangulation = (minus) sum over h-volumes

$$\mathcal{A}_T \;=\; - \sum_{triangles\;\; f \in \mathcal{F}(T)} \mathrm{Vol}(f)$$

The mesure as a Kähler form!

Define the *N x N* matrix $D_{u\bar{v}}(z) = \dfrac{\partial}{\partial z_u}\dfrac{\partial}{\partial \bar{z}_v}\mathcal{A}_T(z)$

1. $D_{u\bar{v}}$ is a Kähler form on $\mathfrak{D}_{N+3}$ i.e. $D > 0$

2. $D_{u\bar{v}}$ is countinuous (no discontinuity when a flip occurs)

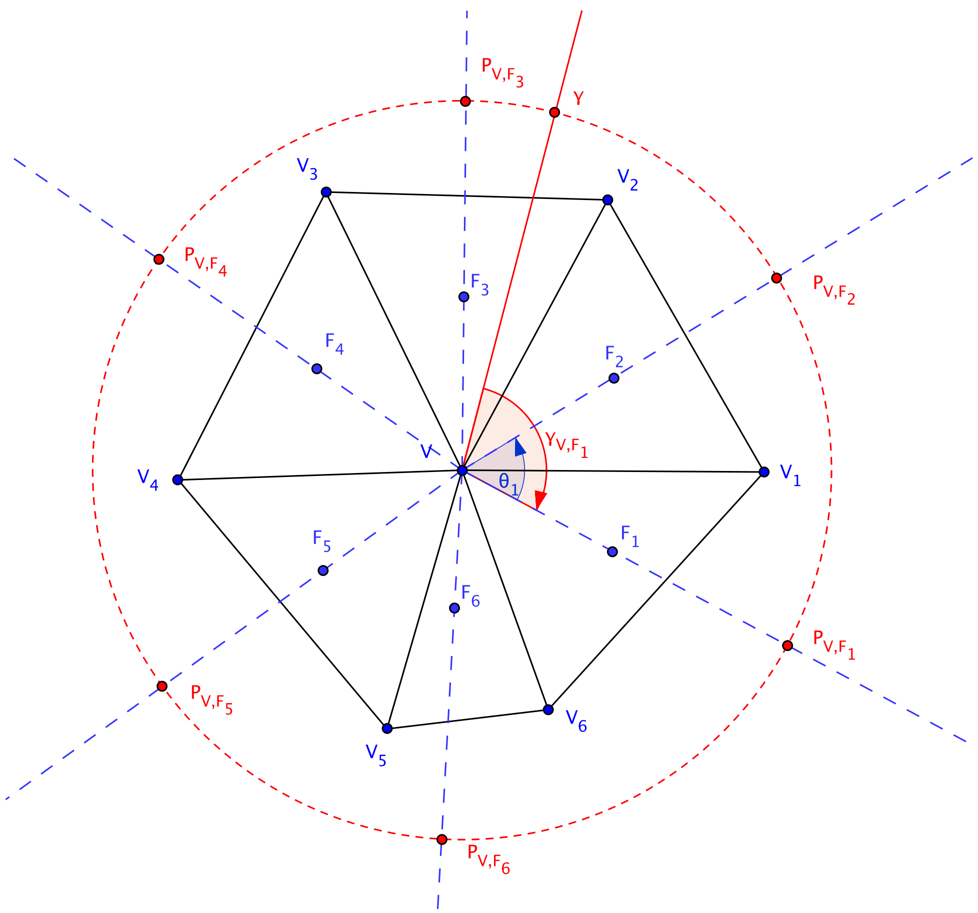
3. The measure determinant is the Kähler volume form

$$\mathcal{D}_T(z)_{\setminus\{1,2,3\})} = \det\left[\left(D_{u,\bar{v}}\right)_{\substack{u,v\neq\\ \{1,2,3\}}}\right]$$

The *(2N x 2N)* Jacobian has been reduced to a *N x N* Kähler determinant!

But it is not a determinantal process!

This is not too surprising, the initial measure over independent angles can be written as a combination of Chern classes

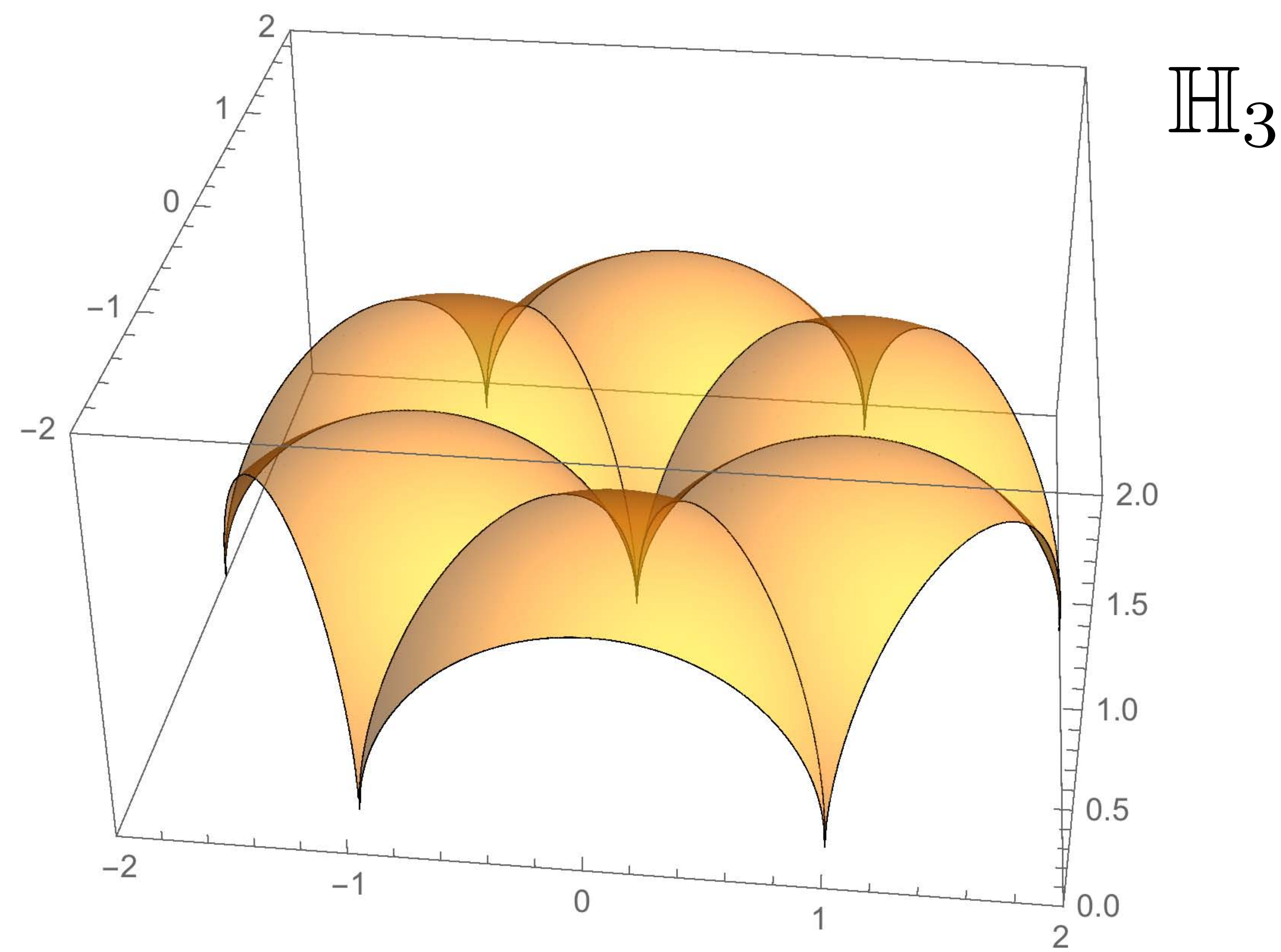


$$\psi_v = c_1(\mathcal{L}_v) = du_v = -\frac{1}{4\pi^2}\sum_{f=2}^{n}\sum_{f'=1}^{f-1} d\theta_{v,f_+}\wedge d\theta_{v,f'_+}$$

$$\left(\sum_v 4\pi^2\,\psi_v\right)^N = \pm\, N!\,2^{2N+1}\prod_e d\theta_e$$

and the angle measure is a measure on the moduli space of the punctured Riemann sphere $\mathcal{M}_{0,N+3}$
So this measure is related (and quite possibly equivalent) to the Weil-Peterson measure over $\mathcal{M}_{0,N+3}$ and our model is related to topological 2D gravity

The 2-d metric on the boundary of the hyperbolic tetrahedra induced by the 2d-metric in H_3 makes it a constant negative curvature ball with cusps singularities (punctured sphere)



If our measure is the W-P measure, this implies that our model is in the same universality class (pure gravity) than planar maps

$$\gamma_{\mathrm{string}} = -1/2$$

Conformal invariance is exact and explicit

$d\mu(z) = d^2 z \ \det(D)$ is a conformal point process

Independence of the 3 fixed points and $\mathrm{SL}(2,\mathbb{C})$ invariance

$$H = \frac{\det\left(D_{\backslash a,b,c}(z)\right)}{\left|\Delta_3(z_a, z_b, z_c)\right|^2}$$

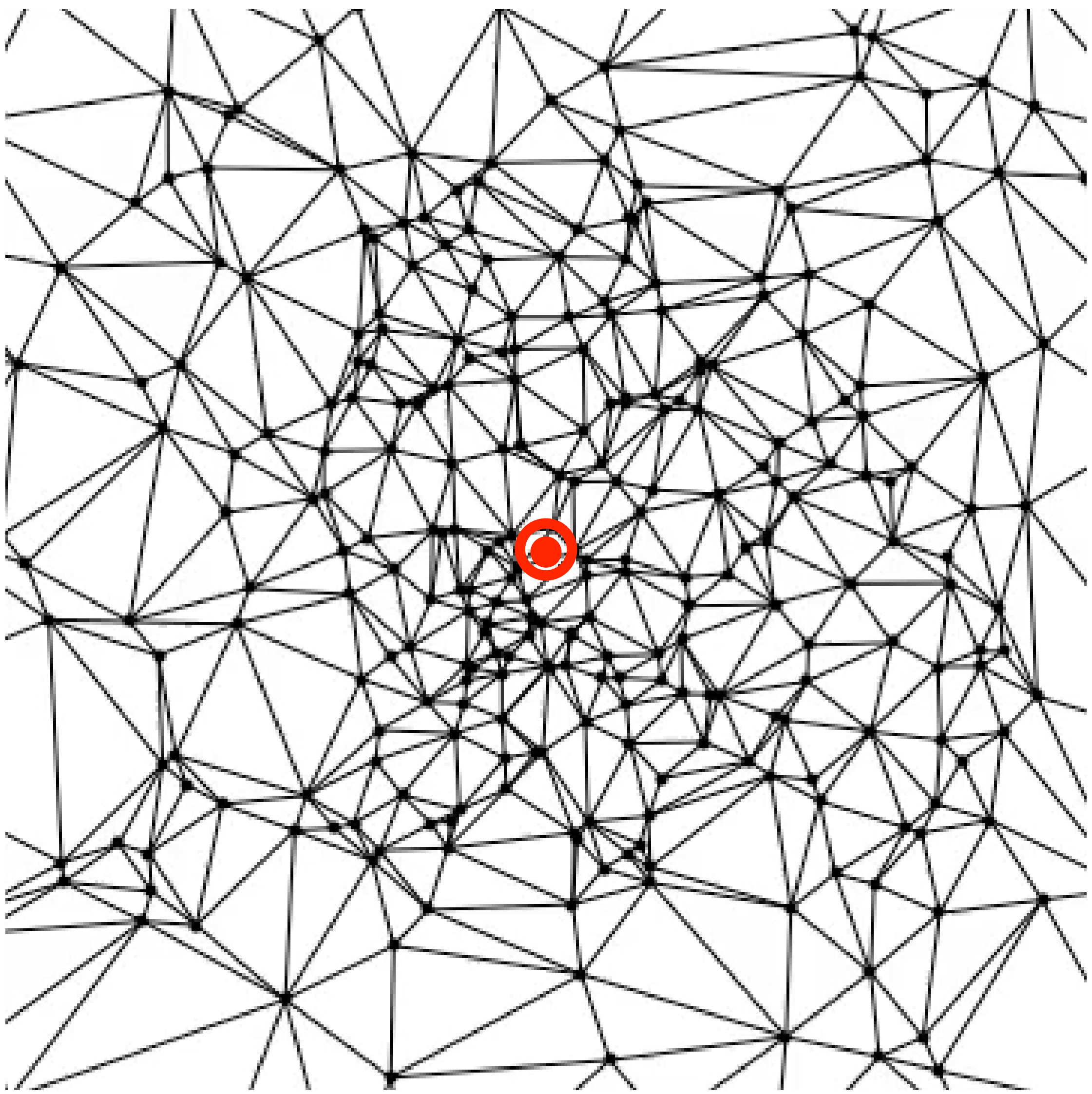
$$\Delta_3(z_a, z_b, z_c) = (z_a - z_b)(z_a - z_c)(z_b - z_c)$$

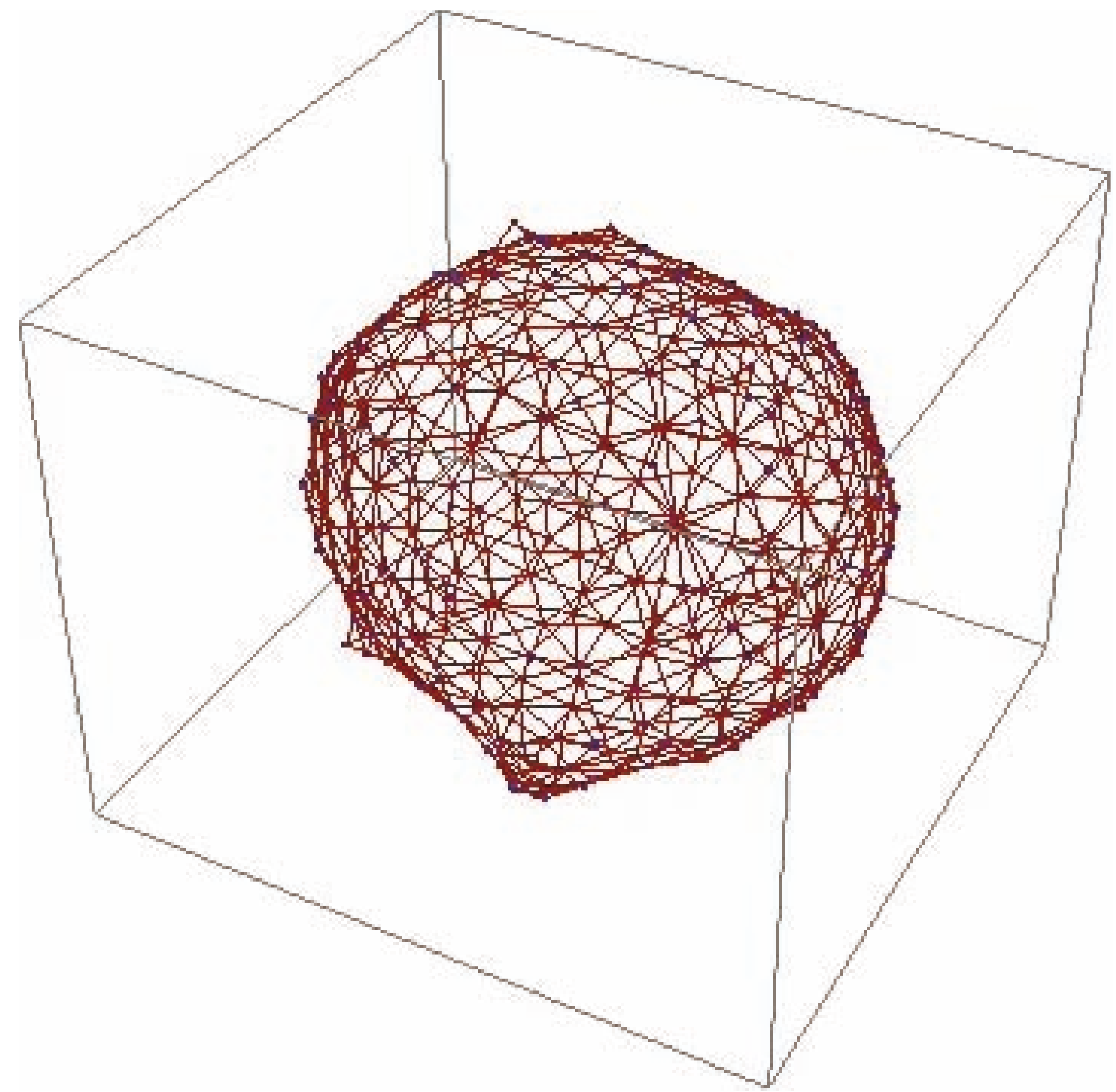
is independent of the choice of points

$$z \to w = \frac{az+b}{cz+d} \quad \text{with} \quad ad - bc = 1$$

$$H(z) = \left|\prod_{i=1}^{N+3} w'(z_i)\right|^2 H(w) = \prod_{i=1}^{N+3} \frac{1}{|cz_i + d|^2} H(w)$$

Consequence: D is a very singular but integrable measure
One expects large fluctuations of the density of points at all scales, consequence of conformal invariance





Poisson process on the sphere $\longrightarrow$ collapse of half of the points

The sum of angles around the collapsed points $\displaystyle\sum_{e\to V}\theta_e \to 2\pi_+$

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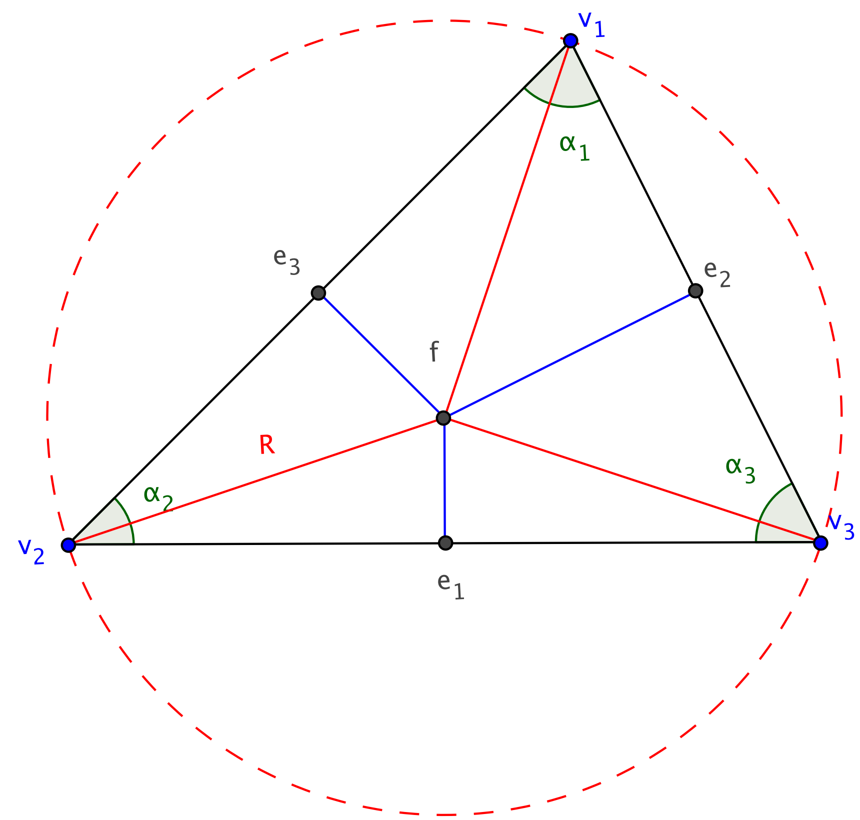
Local geometrical representation of D as a sum over the triangles of local operators

$$D_{u,\bar v} = \sum_f D_{u,\bar v}(f) \quad , \qquad D_{u,\bar v}(f) = -\frac{\partial^2}{\partial z_u \partial \bar z_v}\mathrm{Vol}(f)$$



$$D(f) = \frac{1}{8\,R(f)^2}\begin{pmatrix} \cot(\alpha_2)+\cot(\alpha_3) & -\cot(\alpha_3)-\mathrm{i} & -\cot(\alpha_2)+\mathrm{i} \\ -\cot(\alpha_3)+\mathrm{i} & \cot(\alpha_3)+\cot(\alpha_1) & -\cot(\alpha_1)-\mathrm{i} \\ -\cot(\alpha_2)-\mathrm{i} & -\cot(\alpha_1)+\mathrm{i} & \cot(\alpha_1)+\cot(\alpha_2) \end{pmatrix}$$

This suggest D is a "discretized" Fadeev-Popov determinant!

The Hermitean form D can be written as

$$\Phi\cdot D(f)\cdot\overline{\Psi} = \sum_{i,j\ \text{vertices of}\ f} \Phi(v_i) D_{i\bar{j}}(f)\overline{\Psi}(v_j) = \frac{\mathrm{Area}(f)}{R(f)^2}\ \overline{\nabla}\Phi(f)\ \nabla\overline{\Psi}(f)$$

with local derivative operator from $\Phi(\text{vertices}) \to \Pi(\text{faces})$

$$\nabla\Phi(f) = \frac{1}{2\mathrm{i}}\frac{\Phi(v_1)(\bar{z}_3-\bar{z}_2)+\Phi(v_2)(\bar{z}_1-\bar{z}_3)+\Phi(v_3)(\bar{z}_2-\bar{z}_1)}{\mathrm{Area}(f)}$$

$$\overline{\nabla}\Phi(f) = -\frac{1}{2\mathrm{i}}\frac{\Phi(v_1)(z_3-z_2)+\Phi(v_2)(z_1-z_3)+\Phi(v_3)(z_2-z_1)}{\mathrm{Area}(f)}$$

simple geometric characterization of ∇

$$\nabla_{f,v} = \frac{\partial\log(\mathrm{Area}(f))}{\partial z_v} = \frac{1}{z_{v''}-z_v}+\frac{1}{z_{v'}-z_v}-\frac{\partial\log(\mathrm{Radius}(f))}{\partial z_v}$$

If complex functions are identified with real vector fields

$$\Phi = \Phi^z \ , \ \bar{\Phi} = \Phi^{\bar{z}} \qquad \mathrm{Area}(f) = d^2 w_f \qquad \frac{1}{R(f)^2} = \mathrm{e}^{\,\phi(w_f)}$$

One gets $$\Phi \cdot D(f) \cdot \bar{\Psi} = \sum_f \frac{\mathrm{Area}(f)}{R(f)^2} \bar{\nabla}\Phi(f) \ \nabla\bar{\Psi}(f)$$

$$\Phi \cdot D \cdot \overline{\Psi} = \int d^2 w \ \mathrm{e}^{\,\phi(w)} \ \partial_{\bar{z}}\Phi^z(w) \ \partial_z \Psi^{\bar{z}}(w)$$

with $$\phi(f) = -2 \log(R(f))$$

One can see the Kähler form as a discretized version of the Faddeev-Popov determinant in Polyakov’s formulation of two dimensional gravity and of non-crititical string theory !

Indeed...

Back to the functional integral over 2d Riemannian metrics, in the conformal gauge

$$g_{ab}(z) = \delta_{ab}\ \mathrm{e}^{\phi(z)} \qquad \int \mathcal{D}[g_{ab}] \ = \ \int \mathcal{D}[\phi]\ \det(\nabla_{\mathrm{FP}})$$

with Faddeev-Popov ghost systems $(\mathbf{b},\mathbf{c})$

$$\det(\nabla_{\mathrm{FP}}) \ = \ \int \mathcal{D}[\mathbf{c},\mathbf{b}]\ \exp\left(\int d^2z\ \mathrm{e}^{\phi}\ (b_{zz}(\nabla c)^{zz} + b_{\bar{z}\bar{z}}(\nabla c)^{\bar{z}\bar{z}})\right)$$

Integrating over the b's (the ghosts) only gives

$$\det(\nabla_{\mathrm{FP}}) \ = \ \int \mathcal{D}[\mathbf{c}]\ \exp\left(\int d^2z\ \mathrm{e}^{\phi}\,\partial_z c^{\bar{z}}\ \partial_{\bar{z}} c^{z}\right)$$

The D operator is nothing but a discretised FP determinant

$$D \ = \ \nabla_{\mathrm{FP}}$$

This suggest to identify the scalar function on faces

$$\phi(f) = -2\,\log(R(f))$$

with a "discretized" Liouville field on the Voronoï lattice ...

1. Continuum and discrete 2D gravity: what remains to be understood?
2. Circle packings and circle patterns
3. Delaunay circle patterns and planar maps
4. A measure over planar triangulations
5. Spanning 3-trees representation
6. Kähler geometry over triangulation space and 3D hyperbolic geometry
7. Discretized Faddev-Popov operator and Polyakov's 2D gravity
8. **Local uniform bounds and the continuum limit**

Up to now, existence results at finite N (number of points)

We are interested in the continuum, large N limit.

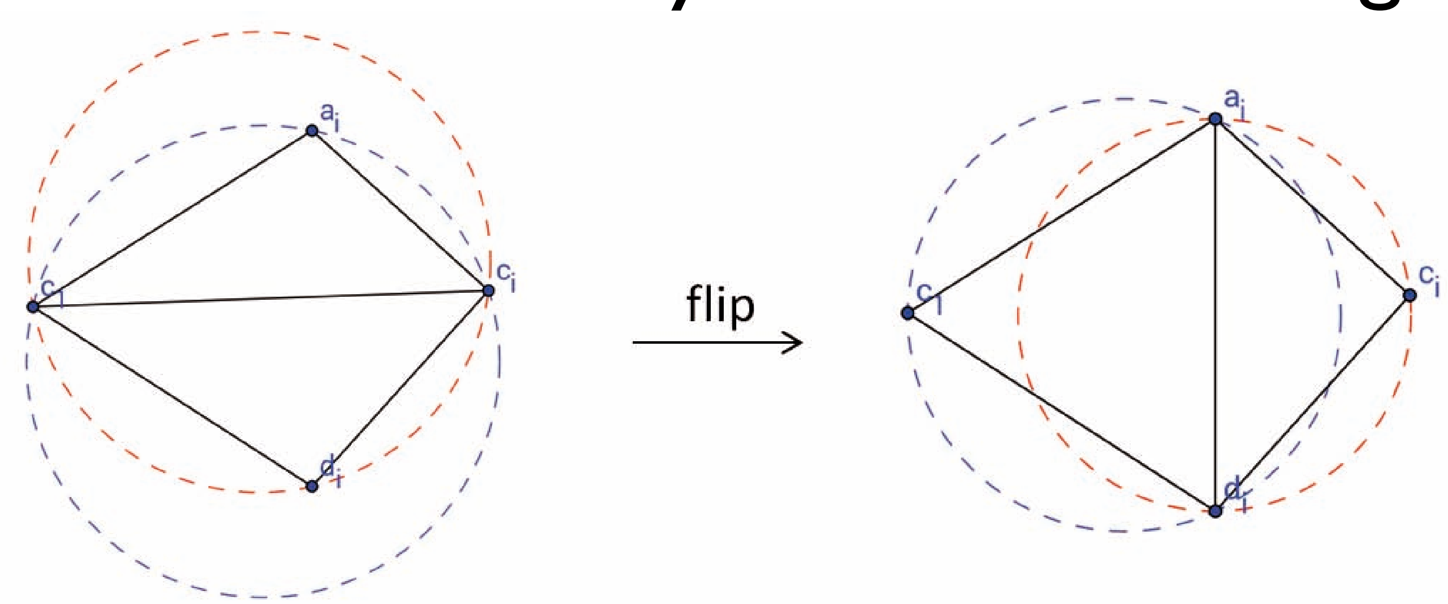
First step, get bounds as a function of N.

Theorem 1: Delaunay triangulations extremize the Kähler volume measure.

If T Delaunay and T' not Delaunay with same vertices {z}

$$\mathcal{D}_T(z)_{\setminus\{1,2,3\}} \;>\; \mathcal{D}_{T'}(z)_{\setminus\{1,2,3\}}$$

Proof: Use the Lawson Flip Algorithm to go from T' to T and show that the measure always increases during a flip.



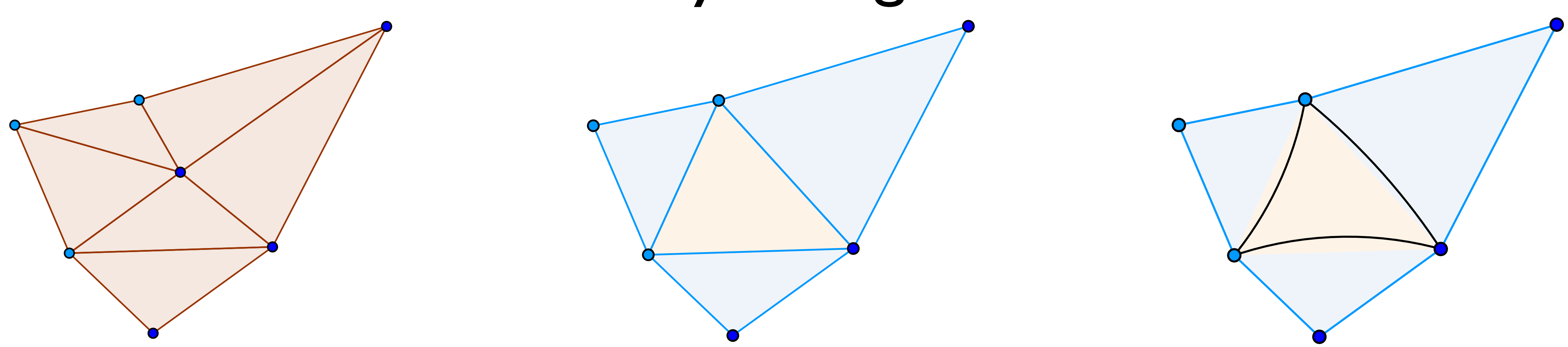
Consider the total integral over N(+3) points

$$A_N = \int_{\mathbb{C}^N} \prod_{v=4}^{N+3} d^2 z_v \mathcal{D}_{T_N^D}(z)_{\setminus\{1,2,3\}}$$

Theorem 2: The sequence is growing as

$$A_{N+1} \geq (N+1)\frac{\pi^2}{8} A_N$$

Proof: Compare integrands when adding/removing a point, using theorem 1 and proper decomposition of the plane into some kind of "conformal Delaunay triangulation"



Local inequalities potentially interesting to prove convergence for probability measures, not only integrals

Corollary: The partition function defined by the series

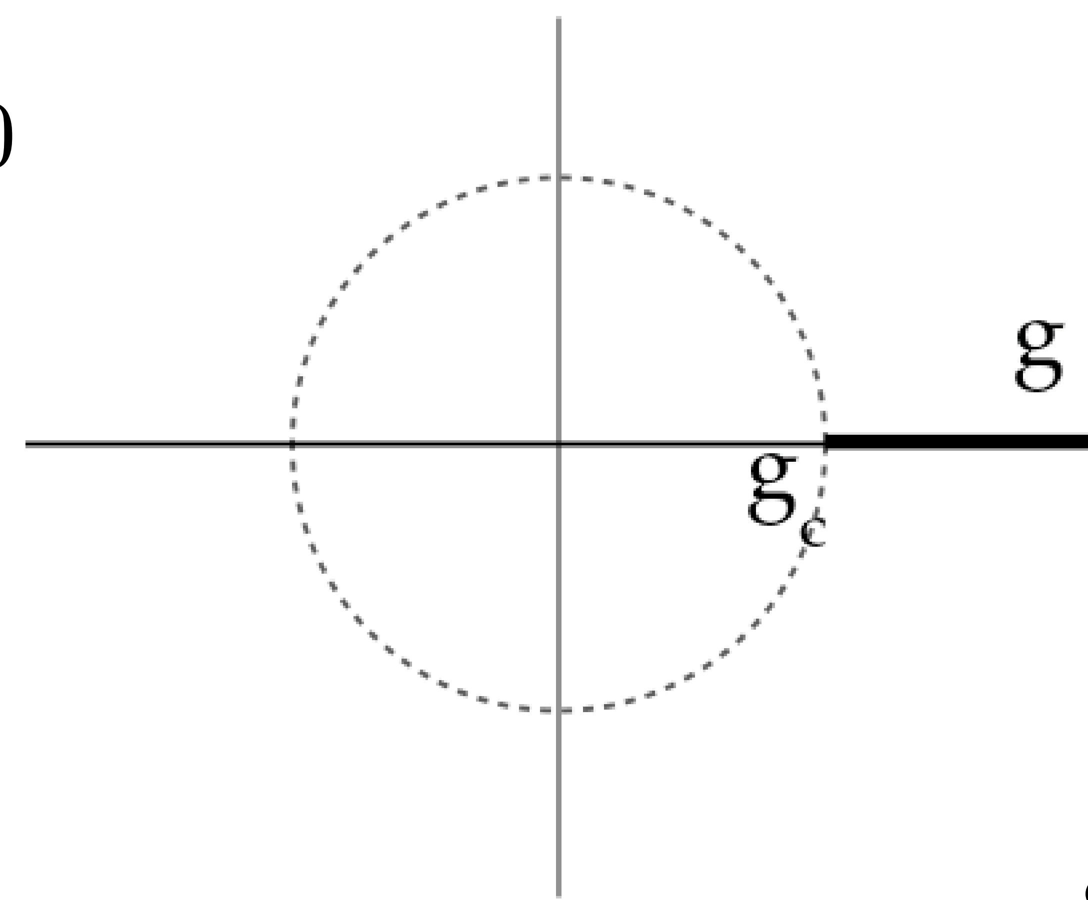
$$Z(g) = \sum_N A_N/N!\, g^N$$

has a finite non-zero radius of convergence

$$|g| < g_c$$

and a critical point on the real axis, which is expected to correspond to continuum 2D gravity in the scaling limit

$$g/g_c = \mathrm{e}^{-\mu_R a^2} \qquad a \to 0$$



Conclusion

- We have an explicit quasi-conformal embedding of planar «dressed admissible» 2-dimensional maps onto the complex plane
- This point process is well defined for finite number of points
- It has many interesting properties (conformal invariance)
- It is what is expected from continuum 2d quantum gravity
- But we would like to be able to characterise its "continuum limit", namely the limit when the density of points become infinite, and the corresponding statistical system
- This is more difficult... renormalization group methods needed (work in progress)
- Any help and ideas from mathematicians is welcome
- Thank you for your interest!