# Logarithmic correlations in percolation and other geometrical critical phenomena

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# Introduction

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- Scale invariance $\Rightarrow$ correlations are power-law or logarithmic
- Two possibilities for logarithms:
  1. Marginally irrelevant operator:
     Gives logs upon approach to fixed point theory.
  2. Dilatation operator not diagonalisable:
     Logs directly in the fixed point theory.

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- Happens when dimensions of two operators collide
- Resonance phenomenon produces a log from two power laws

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## Where do such logarithms appear?

- CFT with $c = 0$ [Gurarie, Gurarie-Ludwig, Cardy, ...]
  - Percolation, self-avoiding polymers ($c \to 0$ catastrophe)
  - Quenched random systems (replica limit catastrophe)
- Logarithmic minimal models [Pearce-Rasmussen-Zuber, Read-Saleur]
- For any $d \leq d_{\rm uc}$, the upper critical dimension

# Logarithms and non-unitarity [Cardy 1999]

## Standard unitary CFT

- Expand local density $\Phi(r)$ on sum of scaling operators $\varphi(r)$

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{A_{ij}}{r^{\Delta_i+\Delta_j}}$$

- $A_{ij} \propto \delta_{ij}$ by conformal symmetry [Polyakov 1970]
- $A_{ii} \geq 0$ by reflection positivity
- Hence only power laws appear

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## The non-unitary case

- Cancellations may occur
- Suppose $A_{ii} \sim -A_{jj} \to \infty$ with $A_{ii}(\Delta_i - \Delta_j)$ finite
- Then leading term is $r^{-2\Delta_i} \log r$

# Geometrical models

## $Q$-state Potts model

- Definition in terms of spins $\sigma_i = 1, 2, \ldots, Q$
$$Z = \sum_{\{\sigma\}} \prod_{(ij)\in E} \mathrm{e}^{K\delta_{\sigma_i,\sigma_j}}$$
- Reformulation in terms of Fortuin-Kasteleyn clusters
$$Z = \sum_{A\subseteq E} Q^{k(A)} (\mathrm{e}^{K} - 1)^{|A|}$$

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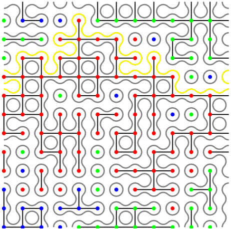

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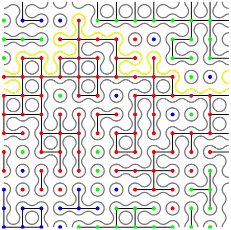
$$Z = \sum_{A\subseteq E} Q^{k(A)} (\mathrm{e}^K - 1)^{|A|}$$

- Here shown for $Q = 3$
- The limit $Q \to 1$ is percolation
- Surrounding loops (**grey**) satisfy the Temperley-Lieb algebra



# Logarithmic correlation functions for $2 \leq d \leq d_{\rm uc}$

## Reminders

- 2 and 3-point functions in any $d$ from global conformal invariance
- This is supposing only conformal invariance!
- Extra discrete symmetries must be taken into account as well
- Physical operators are irreducible under such symmetries [Cardy 1999]
  - O($n$) symmetry for polymers ($n \to 0$)
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## Correlators in bulk percolation in any dimension

- 2 and 3-point functions in bulk percolation
- Limit $Q \to 1$ of Potts model with $S_Q$ symmetry
- Structure for any $d$; but universal prefactors only for $d = 2$

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## Operators acting on one spin

- Most general one-spin operator: $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i) = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{a,\sigma_i}$

$$\underbrace{\delta_{a,\sigma_i}}_{\text{reducible}} = \underbrace{\frac{1}{Q}}_{\text{invariant}} + \underbrace{\left(\delta_{a,\sigma_i} - \frac{1}{Q}\right)}_{\varphi_a(\sigma_i)}$$

- Dimensions of representations: $(Q) = (1) \oplus (Q-1)$
  - Identity operator $1 = \sum_a \delta_{a,\sigma_i}$
  - Order parameter $\varphi_a(\sigma_i)$ satisfies the constraint $\sum_a \varphi_a(\sigma_i) = 0$

## Operators acting symmetrically on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The $Q$ operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q-1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q-1) + \left(\frac{Q(Q-3)}{2}\right)$

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## Easy representation theory exercise

$$E = \delta_{\sigma_i \neq \sigma_j} = 1 - \delta_{\sigma_i, \sigma_j}$$

$$\phi_a = \delta_{\sigma_i \neq \sigma_j} \left( \varphi_a(\sigma_i) + \varphi_a(\sigma_j) \right)$$

$$\hat{\psi}_{ab} = \delta_{\sigma_i, a} \delta_{\sigma_j, b} + \delta_{\sigma_i, b} \delta_{\sigma_j, a} - \frac{1}{Q-2} (\phi_a + \phi_b) - \frac{2}{Q(Q-1)} E$$

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- Scalar $E$ (energy), vector $\varphi_a$ (order parameter) and tensor $\hat{\psi}_{ab}$
- Highest-rank tensor obtained from symmetrised combinations of $\delta$'s by subtracting suitable multiples of lower-rank tensors
- Constraint $\sum_{a=1}^{Q} \phi_a = 0$ and $\sum_{a(\neq b)} \hat{\psi}_{ab} = 0$

# Example for $Q = 4$

$$E = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$2\phi_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix} \quad 2\phi_2 = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

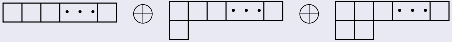
$$2\phi_3 = \begin{bmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad 2\phi_4 = \begin{bmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$6\hat{\psi}_{12} = \begin{bmatrix} 0 & 2 & -1 & -1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & 2 & 0 \end{bmatrix} \quad 6\hat{\psi}_{13} = \begin{bmatrix} 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \end{bmatrix} \quad 6\hat{\psi}_{14} = \begin{bmatrix} 0 & -1 & -1 & 2 \\ -1 & 0 & 2 & -1 \\ -1 & 2 & 0 & -1 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$6\hat{\psi}_{23} = \begin{bmatrix} 0 & -1 & -1 & 2 \\ -1 & 0 & 2 & -1 \\ -1 & 2 & 0 & -1 \\ 2 & -1 & -1 & 0 \end{bmatrix} \quad 6\hat{\psi}_{24} = \begin{bmatrix} 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \end{bmatrix}$$

$$6\hat{\psi}_{34} = \begin{bmatrix} 0 & 2 & -1 & -1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & 2 & 0 \end{bmatrix}$$

## General decomposition of symmetric $N$-spin operators

- $N = 1$ spin: 
- $N = 2$ spins: 
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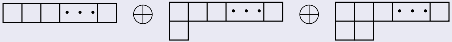
- $N = 2$ spins: 
- Rank-$k$ tensor corresponds to $k = 0, 1, \ldots, N$ boxes in second row

## General decomposition of any $N$-spin operator

- Require all spins to be different (or take $N = \#$different spins)
- Any Young diagram with $Q$ boxes, of which $Q - N$ in first row
- Boxes beyond the first row determine the $S_N$ symmetry of spins

# General setup

## Vector space

- Basis elements:
  $(E_a)_\sigma \equiv (E_{a_1,a_2,\ldots,a_N})_{\sigma_1,\sigma_2,\ldots,\sigma_N} = \delta_{a_1,\sigma_1}\delta_{a_2,\sigma_2}\cdots\delta_{a_N,\sigma_N}$
- Action of $p \in S_Q$: $pE_{a_1,a_2,\ldots,a_N} = E_{p(a_1),p(a_2),\ldots,p(a_N)}$
- Action of $\tilde{p} \in S_N$: $\tilde{p}E_{a_1,a_2,\ldots,a_N} = E_{a_{\tilde{p}(1)},a_{\tilde{p}(2)},\ldots,a_{\tilde{p}(N)}}$

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## Tensors acting on $N$ spins

- Representation of $S_Q$ corresponding to Young diagram $\lambda_Q$
- Let $n$ be number of boxes in $\lambda_Q$, *not* counting the first row
- Symmetry of $N$ spins specified by $\lambda_N \in S_N$
- Wanted tensors: $t^{\lambda_Q,\lambda_N}_{a_1,a_2,\ldots,a_n} = \frac{1}{\mathcal{N}} e^{(a)}_{\lambda_Q} \tilde{e}^{(a)}_{\lambda_N} E_{a_1,\ldots,a_n,b_1,\ldots,b_{N-n}}$
  where $e^{(a)}_{\lambda_Q}$ and $\tilde{e}^{(a)}_{\lambda_N}$ are Young symmetrisers.

# Some examples

## $N = 2$ spins in representation $\lambda_Q = [Q-2, 2]$

- Recall: $\hat{\psi}_{ab} = \delta_{\sigma_i,a}\delta_{\sigma_j,b} + \delta_{\sigma_i,b}\delta_{\sigma_j,a} - \frac{1}{Q-2}(\phi_a + \phi_b) - \frac{2}{Q(Q-1)} E$
- Obtained by imposing $\sum_{a \neq b} \hat{\psi}_{ab} = 0$. Correct, but a bit *ad hoc*.
- In the general setup we find (with present notation):

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\begin{aligned}
t_{ab}^{[Q-2,2],[2]} &= E_{ab} + E_{ba} - \frac{1}{Q-2}\left(t_a^{[Q-1,1],[2]} + t_b^{[Q-1,1],[2]}\right) \\
&+ \frac{2}{(Q-1)(Q-2)} t^{[Q],[2]}
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## Conclusions this far

- Subtracted tensors have same $\lambda_N$ representation
- But $\lambda_Q$ representations, stripped of the first row, are *smaller*

## $N = 3$ spins in representation $\lambda_Q = [Q-3,2,1]$

$$
\begin{aligned}
t_{abc}^{[Q-3,2,1],[2,1]} &= E_{abc} + E_{bac} - E_{cba} - E_{cab} \\
&- \frac{1}{2(Q-1)} (2t_{ab} - t_{ca} - t_{cb})^{[Q-2,2],[2,1]} \\
&- \frac{1}{4(Q-3)} (2t_{ac} + 2t_{bc})^{[Q-2,1,1],[2,1]} \\
&- \frac{1}{Q(Q-2)} (2t_c - t_a - t_b)^{[Q-1,1],[2,1]}
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&- \frac{1}{Q(Q-2)} (2t_c - t_a - t_b)^{[Q-1,1],[2,1]}
\end{aligned}
$$

## Confirms the general picture

- Note that we cannot eliminate $> 1$ box in any given column.
- This can be understood from the antisymmetrisation.

# General result

$$
\begin{aligned}
t^{\lambda_Q,\lambda_N}_{a_1,\ldots,a_n} &= e^{(a)}_{\lambda_Q} \tilde{e}^{(a)}_{\lambda_N} \sum_{i_k \neq a_m} E_{a_1,\ldots,a_n,i_1,\ldots,i_{N-n}} - \sum_{\lambda'_Q \subset \lambda_Q} \frac{1}{A_{\lambda'_Q}(Q)} e^{(a)}_{\lambda_Q} t^{\lambda'_Q,\lambda_N}_{a(\lambda'_Q)} \\
\lambda_Q &= (\lambda_0, \lambda_1, \ldots, \lambda_p) \\
\lambda'_Q &= (\lambda'_0, \lambda'_1, \ldots, \lambda'_p) \\
A_{\lambda'_Q}(Q) &\propto \prod_{i=1}^{p} \frac{(Q-n+i-1-\lambda'_i)!}{(Q-n+i-1-\lambda_i)!}
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**Poles for $Q = 0, 1, 2, \ldots$**

- What does this mean, and how do we cure these divergences?

# Geometrical interpretation in terms of FK clusters

## One-spin results

$$\langle I(r) I(0) \rangle = 1 \,,$$

$$\langle \varphi_a(r) \varphi_b(0) \rangle = \frac{1}{Q} \left( \delta_{a,b} - \frac{1}{Q} \right) \mathbb{P} \left( \quad \right) \,.$$

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- In general we do not know exactly (even in $d = 2$) the probability $\mathbb{P}\Bigg($  $\Bigg)$ that the two spins belong to the same FK cluster.
- But its large-distance asymptotics is predicted from CFT.

# Two-spin results

$$\langle E(r)E(0)\rangle = \left(\frac{Q-1}{Q}\right)^2 \left( \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ & \\ \bullet & \bullet\end{smallmatrix}\right) + \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ | & \\ \bullet & \bullet\end{smallmatrix}\right) \right) + \frac{Q-1}{Q}\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ | & | \\ \bullet & \bullet\end{smallmatrix}\right),$$

$$\langle \phi_a(r)\phi_b(0)\rangle = \frac{Q-2}{Q^2}\left(\delta_{a,b} - \frac{1}{Q}\right)\left( \frac{Q-2}{Q}\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ | & \\ \bullet & \bullet\end{smallmatrix}\right) + 2\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ | & | \\ \bullet & \bullet\end{smallmatrix}\right) \right),$$

$$\left\langle \hat{\psi}_{ab}(r)\hat{\psi}_{cd}(0) \right\rangle = \frac{2}{Q^2}\left( \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2}(\delta_{ac}+\delta_{bd}+\delta_{ad}+\delta_{bc}) + \frac{2}{(Q-2)(Q-1)} \right)\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ | & | \\ \bullet & \bullet\end{smallmatrix}\right).$$

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## Remark on notation

Operators are symmetric, so $\mathbb{P}(\vert\,\vert)$ is short-hand for $\mathbb{P}(\vert\,\vert) + \mathbb{P}(\times)$, etc.

E.g. $\left\langle t_{ab}^{[Q-2,1,1],[1,1]} t_{cd}^{[Q-2,1,1],[1,1]}\right\rangle$ would be proportional to $\mathbb{P}(\vert\,\vert) - \mathbb{P}(\times)$.

# Physical interpretation

## Classification of $d > 2$ Potts operators in by $S_Q$ and $S_N$

- $\left\langle t_a^{\lambda_Q^1,\lambda_N^1} t_b^{\lambda_Q^2,\lambda_N^2} \right\rangle = 0$ if $\lambda_Q^1 \neq \lambda_Q^2$.
- Akin to symmetry classification of quasi-primaries in $d > 2$ CFT.
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- $t_{a_1,\ldots,a_N}^{[Q-N,N],[N]} = \varphi_{0,N} \otimes \varphi_{0,N}$ for $N \geq 2$ symmetric clusters
  - Also known as 2$N$-leg watermelon operator (cf. Coulomb gas)
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- $t_{a_1,a_2,a_3}^{[Q-3,2,1],[2,1]} = \varphi_{1/3,3} \otimes \varphi_{-1/3,3}$ for three $[2,1]$ clusters

# Physical interpretation

## Classification of $d > 2$ Potts operators in by $S_Q$ and $S_N$

- $\left\langle t_a^{\lambda_Q^1,\lambda_N^1} t_b^{\lambda_Q^2,\lambda_N^2} \right\rangle = 0$ if $\lambda_Q^1 \neq \lambda_Q^2$.
- Akin to symmetry classification of quasi-primaries in $d > 2$ CFT.
- Highest-rank ($k = N$) tensor makes $N$ clusters propagate.

## Interpretation as Kac operators $\varphi_{r,s}$ in $d = 2$ bulk CFT

- $t_{a_1,\ldots,a_N}^{[Q-N,N],[N]} = \varphi_{0,N} \otimes \varphi_{0,N}$ for $N \geq 2$ symmetric clusters
  - Also known as $2N$-leg watermelon operator (cf. Coulomb gas)
- $t_a^{[Q-1,1],[1]} = \varphi_{1/2,0} \otimes \varphi_{-1/2,0}$ for one cluster (which can wrap)
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- $t_{a_1,a_2,a_3}^{[Q-3,2,1],[2,1]} = \varphi_{1/3,3} \otimes \varphi_{-1/3,3}$ for three $[2,1]$ clusters
- Makes sense within Jones-Temperley-Lieb representation theory.

# Continuum limit: Making sense of $\hat{\psi}_{ab} = t_{ab}^{[Q-2,2],[2]}$

## Energy operator $\varepsilon_i = E - \langle E \rangle$, with $E = \delta_{\sigma_i \neq \sigma_{i+1}}$ invariant

$$\langle \varepsilon(r) \varepsilon(0) \rangle = (Q-1) \tilde{A}(Q) r^{-2\Delta_\varepsilon(Q)},$$

- All correlators of $\varepsilon_i$ vanish at $Q = 1$ (true already on the lattice)
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## Two-cluster operator $\hat{\psi}_{ab}(\sigma_i, \sigma_{i+1})$

$$\langle \hat{\psi}_{ab}(r)\hat{\psi}_{cd}(0) \rangle = \frac{2A(Q)}{Q^2} \left( \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2}(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) + \frac{2}{(Q-1)(Q-2)} \right) \times \underbrace{r^{-2\Delta_2(Q)}}_{\text{CFT part}},$$

- In 2D: exponent $\Delta_2 = \frac{(4+g)(3g-4)}{8g}$ known from Coulomb gas

# Percolation limit $Q \to 1$

## Avoiding the $Q \to 1$ catastrophe

- The "scalar" part of $\langle \hat{\psi}_{ab}(r)\hat{\psi}_{cd}(0)\rangle$ diverges
- But $\Delta_2 = \Delta_\varepsilon = \frac{5}{4}$ at $Q = 1$ in 2D
  - And actually $\Leftrightarrow$ $d^F_{\rm red\ bonds} = \nu^{-1}$ for all $2 \leq d \leq d_{\rm u.c.}$ [Coniglio 1982]
- So we can cure the divergence by mixing the two operators:

$$\tilde{\psi}_{ab}(r) = \hat{\psi}_{ab}(r) + \frac{2}{Q(Q-1)}\varepsilon(r).$$

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## Using $\langle \hat{\psi}_{ab}\varepsilon\rangle = 0$, we find a finite limit at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r)\tilde{\psi}_{cd}(0)\rangle = 2A(1)r^{-5/2}\left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) + 4A(1)\frac{2\sqrt{3}}{\pi}r^{-5/2} \times \log r,$$

where we assumed that $A(1) = \tilde{A}(1)$.

## Where does the log come from?

$$\frac{1}{Q-1}\left(r^{-2\Delta_\varepsilon(Q)} - r^{-2\Delta_2(Q)}\right) \sim 2\,\frac{\mathrm{d}(\Delta_2 - \Delta_\varepsilon)}{\mathrm{d}Q}\bigg|_{Q=1} r^{-5/2}\log r$$

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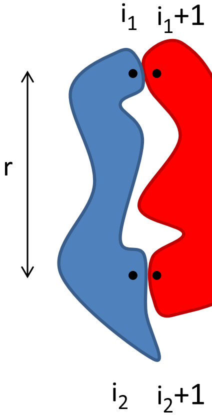
## Geometrical interpretation of this logarithmic correlator?

- Idea: Translate the spin expressions into FK cluster formulation
- In addition to the above results, it follows from the representation theory that $\langle \varepsilon \hat{\psi}_{ab} \rangle = \langle \varepsilon \phi_a \rangle = \langle \hat{\psi}_{ab} \phi_c \rangle = 0$, and also $\langle \hat{\psi}_{ab} \rangle = \langle \phi_a \rangle = \langle \varepsilon \rangle = 0$.
- All correlators take a simple form in terms of FK clusters

Recall that:

$$\langle \hat{\psi}_{ab}(\sigma_{i_1}, \sigma_{i_1+1}) \hat{\psi}_{cd}(\sigma_{i_2}, \sigma_{i_2+1}) \rangle \propto \mathbb{P}_2(r = r_1 - r_2).$$




$$\mathbb{P}_2(r_1 - r_2) = \mathbb{P}\begin{bmatrix} (i_1, i_1+1) \notin \text{ same cluster} \\ (i_2, i_2+1) \notin \text{ same cluster} \\ \text{two clusters } 1 \to 2 \end{bmatrix}.$$

This probability should thus behave as $r^{-2\Delta_2}$

- Recall also the divergence-curing combination

$$\tilde{\psi}_{ab}(r_i) \equiv \tilde{\psi}_{ab}(\sigma_i, \sigma_{i+1}) = \hat{\psi}_{ab}(\sigma_i, \sigma_{i+1}) + \frac{2}{Q(Q-1)} \varepsilon(\sigma_i, \sigma_{i+1})$$

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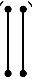
- Expression in terms of simple percolation probabilities

$\mathbb{P}_2 = \mathbb{P}\left(\right.$  $\left.\right)$, $\mathbb{P}_1 = \mathbb{P}\left(\right.$  $\left.\right)$, $\mathbb{P}_0 = \mathbb{P}\left(\right.$  $\left.\right)$, and $\mathbb{P}_{\neq} \equiv \mathbb{P}(\sigma_i \neq \sigma_{i+1})$

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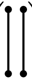
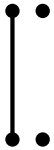
$\mathbb{P}_2 = \mathbb{P}\Big($  $\Big)$, $\mathbb{P}_1 = \mathbb{P}\Big($  $\Big)$, $\mathbb{P}_0 = \mathbb{P}\Big($  $\Big)$, and $\mathbb{P}_{\neq} \equiv \mathbb{P}(\sigma_i \neq \sigma_{i+1})$

## Exact two-point function of $\tilde{\psi}_{ab}$ at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r_1) \tilde{\psi}_{cd}(r_2) \rangle = 2 \left( \delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) \times \mathbb{P}_2(r) + 4 \left[ \mathbb{P}_0(r) + \mathbb{P}_1(r) - 2\mathbb{P}_2(r) - \mathbb{P}_{\neq}^2 \right].$$

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## Reminder: CFT expression

$$\langle \tilde{\psi}_{ab}(r)\tilde{\psi}_{cd}(0)\rangle = 2A(1)r^{-5/2}\left(\delta_{ac}+\delta_{ad}+\delta_{bc}+\delta_{bd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}\right) + 4A(1)\frac{2\sqrt{3}}{\pi}r^{-5/2}\times\log r,$$

Comparison with the CFT expression yields geometrical interpretation

$$F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}^2_{\neq}}{\mathbb{P}_2(r)} \sim \underbrace{\frac{2\sqrt{3}}{\pi}}_{\text{universal}} \log r,$$

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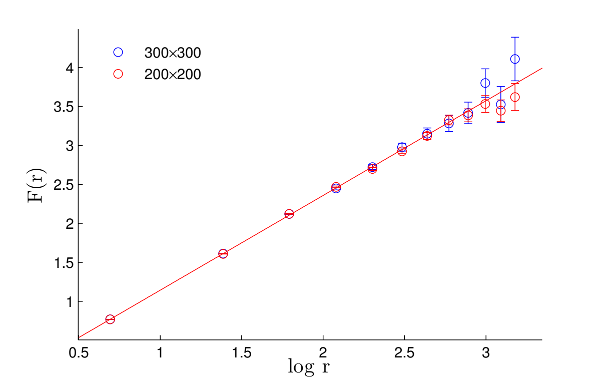
## Numerical check



# Conclusion

- Logarithmic observables in Potts model for $2 \leq d \leq d_{\rm uc}$
  - Occurs for all $Q = 0, 1, 2, \ldots$
- Prediction of logarithmic structure for any $d$
- Universal prefactor given by derivative of critical exponents
  - Hence only explicit values in $d = 2$
- Logarithmic dependence can be checked numerically
- Classification of all $(S_Q, S_N)$ operators (cf. Young diagrams)
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# Thank you!

Firenze è come un albero fiorito
che in piazza dei Signori ha tronco e fronde,
ma le radici forze nuove apportano
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E Firenze germoglia ed alle stelle
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canta baciando piazza Santa Croce,
e il suo canto è sì dolce e sì sonoro
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Così scendanvi dotti in arti e scienze
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So may experts in arts and sciences descend here
to make Florence richer and more splendid!