Random Matrices, Intefaces and Hydrodynamics Singularities

P. Wiegmann

review of works with friends:

Anton Zabrodin, Eldad Bettelheim, Razvan Teodorescu, Seun Yeop Lee

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List of Objects

- Random Matrix Models: Equilibrium Measure;
- Geometrical Growth Models;
- Orthogonal Polynomials: Distribution of zeros;
- Hydrodynamics Singularities;

Normal Random Matrices

Normal matrix $M \Leftrightarrow [M, M^\dagger] = 0 \Leftrightarrow$ diagonalizable by a unitary transform.

$$M = U^{-1}\mathrm{diag}(z_1, \ldots, z_N)U, \quad z_i - \text{complex}$$

The eigenvalues of $N \times N$ normal matrices with the probability distribution

$$\mathrm{Prob}(M)\mathrm{d}M = \frac{1}{\mathcal{Z}} \mathrm{e}^{-\frac{1}{\hbar}\,\mathrm{Tr}\,Q(M)} \mathrm{d}M$$

distributes by the probability density

$$P(z_1, ..., z_N) = \frac{1}{\mathcal{Z}} \left| \prod_{j<k}^{n} (z_j - z_k) \right|^2 \exp\left(-\frac{1}{\hbar} \sum_{j=1}^{N} Q(z_j) \right),$$

Q1. What is the distribution of eigenvalues for

$$\hbar \to 0, \quad N \to \infty, \quad t = \hbar N = \text{fixed?}$$

The answer depends on the potential Q.

2D Dyson's Diffusion

Brownian motion of a Normal Matrix

$$\dot{M} = M^{\dagger} + V'(M) + \text{Brownian Motion}$$

Eqenvalues (complex) perform 2D Dyson diffusion

$$\dot{z}_i = \sum_{i \neq j} \frac{\hbar}{\bar{z}_i - \bar{z}_j} + \bar{z}_i + V'(z_i) + \dot{\xi}_i, \quad \langle \xi_i(t) \bar{\xi}_j(t') \rangle = 4\delta_{ij}(t - t').$$

Probability $\frac{1}{\mathcal{Z}} e^{-\frac{1}{\hbar} \operatorname{Tr} Q(M)}$ is the Gibbs distribution of Dyson's diffusion.

Depending on V' there may or not may be Gibbs distribution.

Ginibre Ensemple and its deformations

$$P(z_1, ..., z_N) = \frac{1}{\mathcal{Z}} \left| \prod_{j<k}^{n} (z_j - z_k) \right|^2 \exp\left(-\frac{1}{\hbar} \sum_{j=1}^{N} Q(z_j) \right),$$

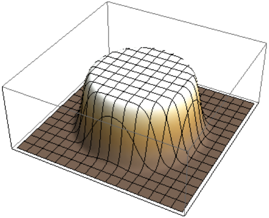
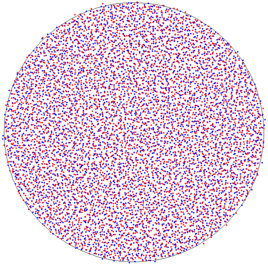
A choice of $Q(z)$ - **Gaussian** plus **harmonic** function when V is holomorphic.

Ginibre ensemble: $Q(z) = |z|^2,$

Deformed Ginibre ensemble: $Q(z) = |z|^2 + V(z) + \overline{V(z)},$
$\Delta Q = 4.$

Ginibre Ensemble

$$Q = |z|^2$$

Support is the disk of the area $\pi\hbar N$

Equilibrium measure

Continuum limit:

$$\rho(z) = \frac{1}{N}\sum_{j=1}^{N}\delta(z - z_j)$$

$$\langle\rho\rangle = \frac{\Delta Q}{4\,\text{Area}} = \frac{1}{\text{Area}} \quad \text{on the support of } \rho.$$

What is support of density?

It depends on the deformation holomorphic function $V(z)$

The eigenvalues are **2D Coulomb interacting electrons**:

$$\frac{1}{Z_n}\mathrm{e}^{-\frac{1}{\hbar}E(z_1,\ldots,z_N)},\qquad \frac{1}{\hbar}E(z_1,\ldots z_N):=\frac{1}{\hbar}\sum_{j=1}^{N}Q(z_j)-2\sum_{j<k}\log|z_j-z_k|.$$

Continuum limit: Defining $\rho(z)=\frac{1}{N}\sum_{j=1}^{N}\delta(z-z_j)$, we have

$$E(z_1,\ldots,z_n)=\hbar N\left(\int_{\mathbb{C}}Q(z')\rho(z')\mathrm{d}^2z'-\hbar N\iint_{\mathbb{C}^2}\rho(z)\rho(z')\log|z-z'|\mathrm{d}^2z\,\mathrm{d}^2z'\right).$$

the condition for the optimal configuration is obtained when

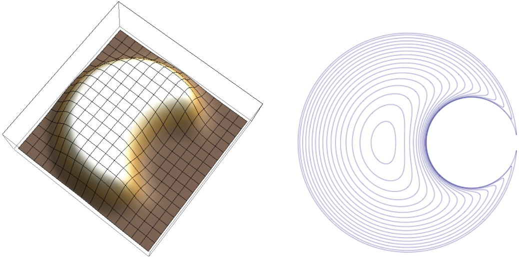
$$0=Q(z)-\hbar N\int_{\mathbf{D}}\log|z-z'|\rho(z')\mathrm{d}^2z'\quad\text{on the support of }\rho.$$

Applying Laplace operator

$$\boxed{\rho(z)=\frac{1}{\pi\hbar N}=\frac{1}{\text{Area}}\quad\text{on the support of }\rho.}$$

Bratwurst

Take $V(z) = -c \log(z-a)$ such that $Q(z) = |z|^2 - 2c \log |z-a|$ $(c > 0)$.

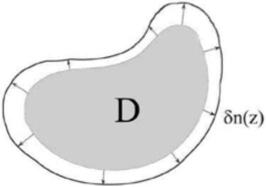


Growth

Change the size of the matrix

$$N \to N + n$$

Area of Equilibrium measure changes $t \to t + \delta t, \quad \delta t = \pi\hbar n$



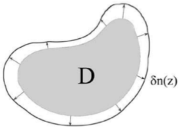
Q: What is the velocity?

Growth process

Area $t := \pi N \hbar$ is identified with time.

Define the Newtonian potential $U(z)$ by

$$U(z) = t \int_{\mathbf{D}} \log|z-w| \mathrm{d}^2 w$$



Equilibrium condition:

$$\pi Q(z) = U(z), \quad \text{inside } \mathbf{D},$$
$$\bar{z} = \partial_z U, \quad \text{inside } \mathbf{D},$$
$$\frac{d}{dt}\bar{z} = \text{velocity} = \partial_z \left[\frac{d}{dt} U(z) \right], \quad \text{on the boundary}$$

$$\frac{d}{dt}U(z) \;\; \text{is a harmonic function outside } \mathbf{D},$$
$$\frac{d}{dt}U(z) = \log|z| + \mathcal{O}(1), \quad z \to \infty,$$
$$\frac{d}{dt}U(z) = 0 \text{ on } \partial\mathbf{D},$$

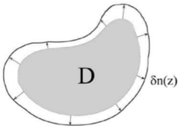
Velocity of the boundary $= \frac{d}{dt}U(z)$ is the Harmonic Measure of $\mathbf{D}$

Harmonic measure: Brownian excursion with a free boundary

A probability for BM to arrive on an element of the boundary is a harmonic measure of the boundary:

Probability to land on ds :
$$\left|\frac{\mathrm{d}f}{\mathrm{d}z}\right| = |\nabla_n G(z,\infty)| ds, \quad z \in \partial D$$
$$-\Delta G(z,z') = \delta(z-z'), \quad G|_{z\in\partial D} = 0$$

$f(z)$ is a univalent map from the exterior of the domain to the exterior of the unit circle



Hele-Shaw Problem

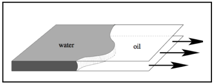
HS Hele-Shaw, inventor of the Hele-Shaw cell
(and the variable-pitch propeller)

Physical setup 1898

- Navier-Stokes Equation:
  $\dot{v} + (v \cdot \nabla)v = \rho^{-1}\nabla p + \mu\Delta v$

- Small Reynolds number - no inertia $0 = \rho^{-1}\nabla p + \mu\Delta v$

- incompressibility:
  $\rho = \text{const}, \quad \nabla \cdot v = 0;$



- 2D Geometry - Poiseuille's law:
  $\Delta v \approx \partial_z^2 v \approx \frac{v}{d^2} \Rightarrow v = -\frac{d^2}{12\nu} p;$

- no viscosity on the boundary:
  $\Rightarrow p = 0$ on the boundary.

Darcy Law: $v = -\nabla p, \quad \Delta p = 0; \quad p|_{\partial D} = 0; \quad p|_{\infty} = -\log|z|$

Experiment: Hele-Shaw cell, Fingering instability

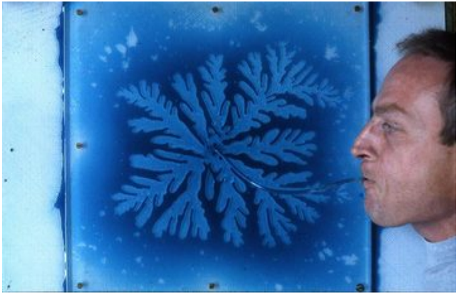
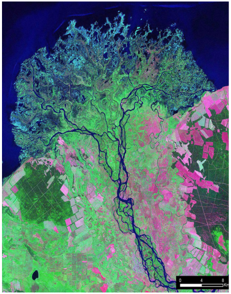


FIGURE: Viscous incompressible fluid pushed out by inviscid incompressible fluid

Blow hard, otherwise the surface tension will take over.

Fingering Instability



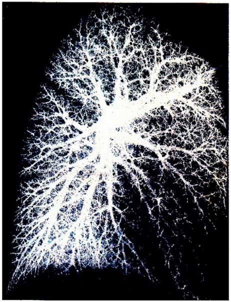


FIGURE: Flame (no convection), Serenga river (Russia), Lung vessels

Cusp-Singularities

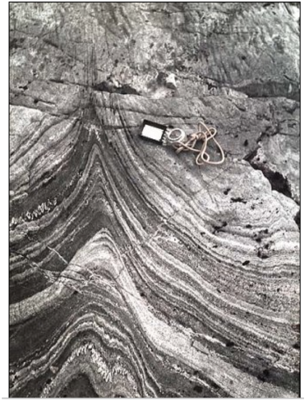
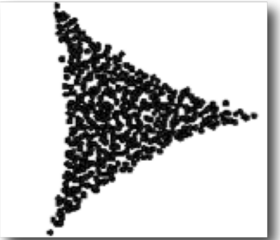


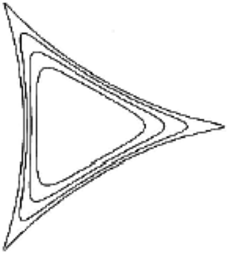
Figure: Cusp: end of a smooth growth

Cusp-Singularities: Growing Deltoid

$$P(z_1, \ldots, z_N) = \frac{1}{\mathcal{Z}} \left| \prod_{j<k}^{n} (z_j - z_k) \right|^2 \exp\left(-\frac{1}{\hbar} \sum_{j=1}^{N} Q(z_j) \right),$$

Deformed Ginibre ensemble: $Q(z) = |z|^2 + t_3 z^3 + \overline{t_3 z^3}$

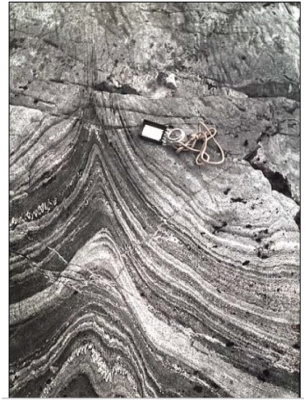




Hypotrochoid grows until it reaches a critical point.

Cusp-Singularities

Deformed Ginibre ensemble: $Q(z) = |z|^2 + V(z) + \overline{V(z)}$



Almost any deformation leads to a cusp singularity: $y^p \sim x^q$

The most generic is (2,3)- singularity

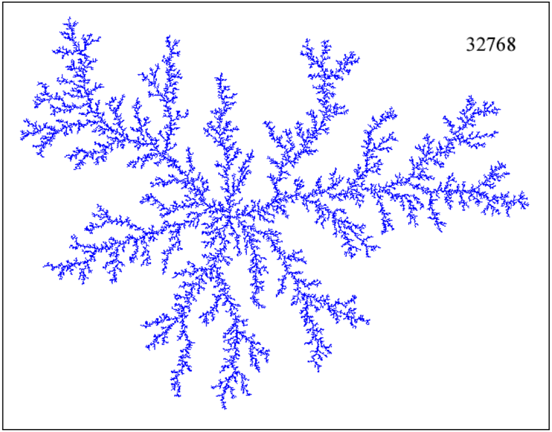
$$\boxed{y^2 \sim x^3}$$

Diffusion limited aggregation (DLA)

Fractal pattern with (numerically computed) dimension

$$D_H = 1.71004...$$

Structure of this pattern is the main problem one the subject



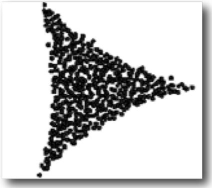
Zeros of Complexified Orthogonal Polynomials

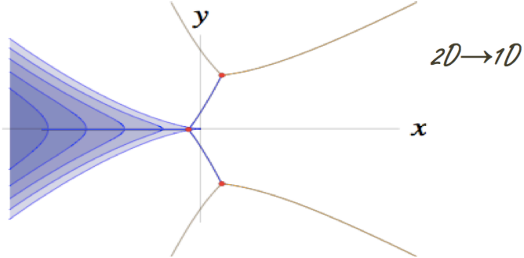
Unstable Diffusion

$V = t_3 z^3$ - an example when the integral $\int e^{-\frac{1}{\hbar}\mathrm{tr}Q} dM$ diverges, there is no Gibbs distribution:

$$\dot{z}_i = \sum_{i \neq j} \frac{\hbar}{\bar{z}_i - \bar{z}_j} + \bar{z}_i + V'(z_i) + \dot{\xi}_i, \quad \langle \xi_i(t) \bar{\xi}_j(t') \rangle = 4\delta_{ij}(t - t').$$

Particle escape. One keeps to pump particles to compensate escaping particles.





Bi-orthogonal polynomials and growth process

The measure for the subset of the eigenvalues, $z_1, ..., z_k$, $(k \leqslant n)$, is given by

$$P(z_1, ..., z_N) = \frac{1}{\mathcal{Z}} \left| \prod_{j<k}^{n} (z_j - z_k) \right|^2 \exp\left(-\frac{1}{\hbar} \sum_{j=1}^{N} Q(z_j) \right),$$

Bi-orthogonal polynomials $p_j = z^j + ...$

$$h_j \delta_{ij} = \int_{\mathbb{C}} p_i(z) \overline{p_j(z)} e^{-\frac{1}{\hbar} Q(z)} d^2 z.$$

Polynomial

$$p_n(z) = \langle \prod_j (z - z_j) \rangle = \int \prod_j (z - z_j) P(z_1, \ldots, z_N) d^2 z_1 \ldots d^2 z_N$$

Q: What is the asymptotic distribution of the roots of $p_n(z)$ for $n \to \infty, \quad \hbar \to 0$?

Christoffel - Darboux formula

Density

$$\rho_N(z) = \frac{1}{N}\langle\sum_j \delta(z - z_j)\rangle = \int P(z; z_2, \dots, z_N)\mathrm{d}^2 z_2 \dots \mathrm{d}^2 z_N$$

Christoffel - Darboux formula

$$\boxed{\rho_{N+1} - \rho_N(z) = |\Psi_N(z)|^2}$$

where

$$\Psi_n(z) = h_n^{-1/2} e^{\frac{1}{\hbar}\left(-\frac{1}{2}|z|^2 + V(z)\right)} p_n(z)$$

are weighted orthogonal polynomials

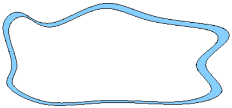
$$\delta_{nm} = \int \Psi_n(z)\overline{\Psi_m(z)}\mathrm{d}^2 z$$

$|\Psi_n|^2$ can be seen as a velocity of growth.

Asymptotes of Orthogonal Polynomials solve the growth problem solve

Important result: At a properly defined $n \to \infty$

$|\Psi_n(z)|^2$ is localized on $\partial\mathbf{D}$ and proportional to the width of the infinitesimal strip:

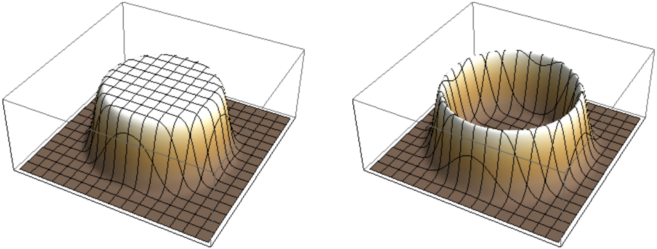


$$z \in \partial D : \quad |\Psi_n(z)|^2 |dz| \sim |f'(z)dz| \approx \text{Harmonic measure}$$

The simplest example: Circle

When $V(z) = 0$ the orthogonal polynomials are simply

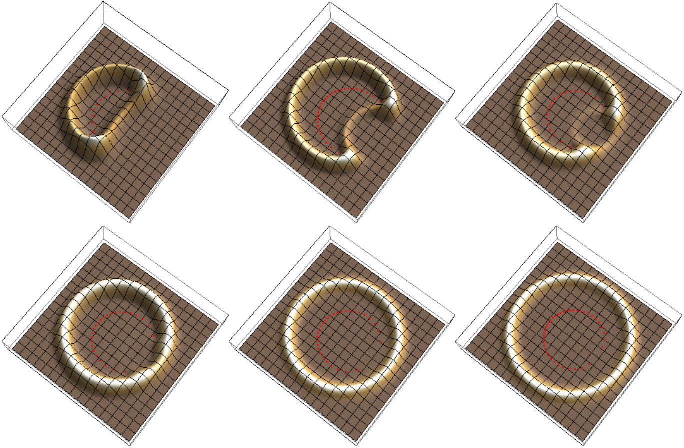
$$\Psi_n(z) \propto z^n e^{-\frac{1}{\hbar}|z|^2}$$



The difference between the consecutive kernels $|\Psi_n(z)|^2$ is localized on $\partial\mathbf{D}$ and proportional to the width of the infinitesimal strip.

Another example: Bratwurst

Take $V(z) = -c \log(z-a)$ such that $Q(z) = |z|^2 - 2c \log|z-a|$ ($c > 0$).



The plots of $p_n(z)\overline{p_n(z)}e^{-NQ(z)}$ for various times.

Zeros of Orthogonal Polynomials

- Szego theorem:

  Zeros of Orthogonal Polynomials with real coefficients defined on $\mathbb{R}$ are distributed on $\mathbb{R}$.
- Zeros of Orthogonal Polynomials with real coefficients defined on $\mathbb{C}$ are distributed on $\mathbb{C}$.

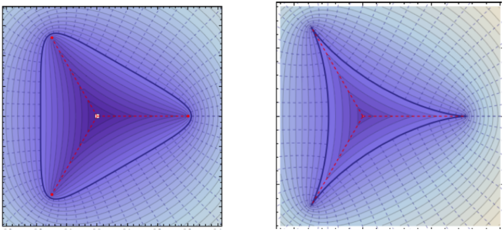


FIGURE: Deltoid: $Q(z) = |z|^2 + t_3 z^3 + \overline{t_3 z^3}$

Balayage

A minimal body (an open curve) which produces the same Newton potential as a domain D - *mother body* - Γ

$$\int\int_D \log|z-w|\mathrm{d}^2 w = \oint_\Gamma \log|z-w|\sigma(w)|\mathrm{d}w|$$

$$z \in \Gamma : \quad S(z)dz = \sigma(z)|dz|$$

A graph Γ:

$$\Omega = \int^z S(z')\mathrm{d}z'$$

Level lines of Ω:

$$\mathrm{Re}\Omega(z)|_\Gamma = 0, \quad \mathrm{Re}\Omega(z)|_{z\to\Gamma} > 0;$$

are branch cuts drawn such that jump of $S(z)$ is imaginary.

Balayage reduces the domain to a curve Γ

Zeros of Orthogonal Polynomials

Important result:

A locus of zeros of Orthogonal Polynomials is identical to balayage

$$\Psi \sim f'(z) \sum_{\text{all branches of } \Omega} (\text{Stokes coefficients})_k \, e^{-\frac{1}{\hbar}\Omega_k(z)}$$

A graph of zeros is identical to level lines of Ω

$$\mathrm{Re}\Omega(z)|_{\Gamma} = 0, \quad \mathrm{Re}\Omega(z)|_{z\to\Gamma} > 0;$$

Boutroux Curves

Definition:

$$(\bar{z}, S(z)) : \quad \text{Real Riemann surface}$$

$$d\Omega = S(z)dz$$

$$\mathrm{Re} \oint_{B-cycles} \mathrm{d}\Omega = 0 - \text{all periods are imaginary}$$

number of conditions - number of parameters $= g$ - there is no general proof that these curves exist.

Important result:

Zeros of Orthogonal Polynomials are distributed along levels of Boutroux curves

A graph Γ : $\quad \mathrm{Re}\Omega(z)|_{\Gamma} = 0, \quad \mathrm{Re}\Omega(z)|_{z \to \Gamma} > 0;$

Summary: Geometrical aspects of Random Matrix ensemble

- Given a holomorphic function $V(z)$ construct a domain D whose exterior Cauchy transform $\frac{1}{\pi}\int\frac{\mathrm{d}^2w}{z-w} = V'$. Domain D is the support of the equilibrium measure;

- Weighted polynomial $|\Psi_N| = e^{-\frac{1}{2\hbar}Q}p_N$ achieves the maximum on the boundary of the domain.

  Its height is a harmonic measure of the domain.
- Harmonic measure $|f'|$ gives the evolution of the domain with increasing $t = \pi\hbar N$;

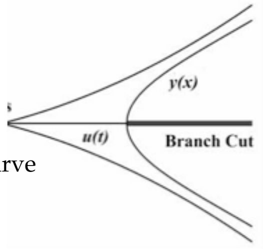
- Balayage of the domain is the support of zeros of orthogonal polynomials

- Balayage is a Boutroux curve

Evolution of the cusp

$$y(x,t) = -4\,(x - u(t))\left(x + \frac{1}{2}u(x)\right)^2,$$

$$u(t) = --2(t - t_c)^{1/2}$$



$y(x)$ - is a degenerate elliptic Boutroux curve
- a pinched torus.

After the singularity -
the curve becomes non-degenerate!

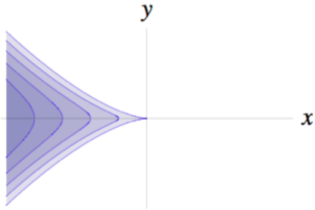
$$y^2 = (x - e_1(t))\,(x - e_2(t))\,(x - e_3(t))$$

Unique Elliptic Boutroux Curve

$$y^2 = (x - e_1(t))\,(x - e_2(t))\,(x - e_3(t))$$

found by Krichever, Gamsa, Rodnisco, David (early 90s).

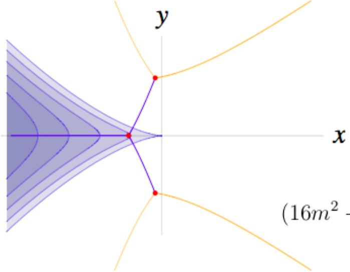
Branch points are transcedental obtained through solution of algebraic equation involving elliptic integrals.



$$y^2 = -(x - e(t))\left(x + \frac{e(t)}{2}\right)^2$$

$$e(t) = -\sqrt{t_c - t}$$

$$y^2 = (x - e_1)(x - e_2)(x - e_3)$$



$$(e_1, e_2, e_3) = \sqrt{\frac{3}{h^2 - \frac{3}{4}}}(-1, \tfrac{1}{2} + \mathrm{i}h, \tfrac{1}{2} - \mathrm{i}h)\sqrt{t}$$

$$h \approx 3.2463822537442788756 76.$$

$$m = \tfrac{1}{2} + \tfrac{3}{4}\frac{1}{\sqrt{\frac{9}{4}+h^2}}$$

$$(16m^2 - 16m + 1)E(m) = (8m^2 - 9m + 1)K(m).$$

More about Boutroux curves: How to plant and grow trees

- Start with a polynomial $V'(x) = t_g x^g + \dots$ of a degree g
- Determine a degenerate hyper elliptic Boutroux curve

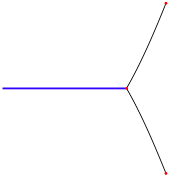
$$y = \sqrt{x - e(t)} \prod_{k=1}^{g} (x - d_k(t))$$

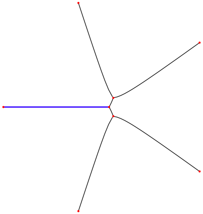
such that a positive part of Laurent expansion is $\sqrt{x}V'(\sqrt{x})$

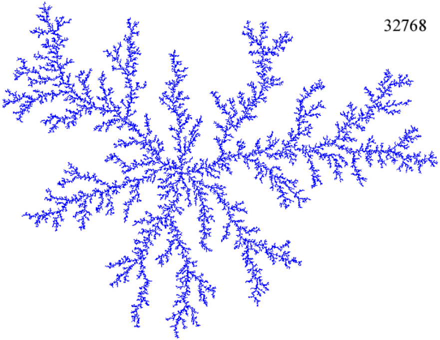
$$y = \sqrt{x} \left(\overbrace{x^g + t_{g-1}x^{g-1} + \dots}^{\text{fixed}=V'} + \overbrace{\frac{t}{x}}^{\text{time}} + \overbrace{\frac{C(t)}{x^2}}^{\text{capacity}} + \text{negative powers} \right)$$

- Run t keeping positive part fixed. Negative powers follow. Pinched cycles begin to open. Level graph branches. When all double points open the process stabilizes;

Numerical plot of first two generations







Capacity $C(t)$ is the measure of the size of the pattern, t is its mass

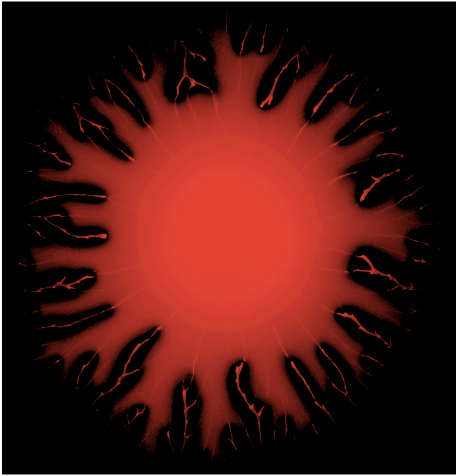
$$y = \sqrt{x}V' + \overbrace{\frac{t}{\sqrt{x}}}^{\text{time}} + \overbrace{\frac{C(t)}{\sqrt{x^3}}}^{\text{capacity}} + \text{negative powers}$$

At every genus transition - branch of the tree capacity jumps by universal (transcendental) value

$$\eta = \frac{\dot{C}_{\text{after branching}}}{\dot{C}_{\text{before branching}}} = 0.91522030388$$

- Conjecture: Capacity grows with the mass as $C \sim t^{1/D_H}$, where D_H is the fractal dimension of the pattern
- Conjecture: D_H is a simple function of η;
- Conjecture: $\frac{1}{D_H} - \frac{1}{2} = 1 - \eta \Rightarrow D_H = \underbrace{1.71004}_{\text{numerical digits in DLA}} 56918$

c



Mahech Bandi (OIST) observed suggestive structures in miscible fluids where 2D pattern evolves into 1D patterns