Towards a Non-equilibrium Bethe Ansatz for the Kondo Model

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  - ✘ Fermi golden rule (kinetic equation): $W = \delta(E_{\text{in}} - E_{\text{out}} - \hbar\omega)|\langle\text{in}|\mathcal{O}|\text{out}\rangle|^2$
  - ✘ Coherent evolution: $\langle\psi|e^{\frac{\imath}{\hbar}Ht}Je^{-\frac{\imath}{\hbar}Ht}|\psi\rangle = \sum_{i,j}\langle\psi|i\rangle\langle j|\psi\rangle\langle i|J|j\rangle e^{\frac{\imath}{\hbar}(E_i-E_j)t}$

Models

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- The Bethe ansatz solution involves finding a set of N complex rapidities λ_i to the Bethe ansatz equations, given a set of inhomogeneities, z_i, the twist, κ, and the shift η.

Slavnov Overlaps

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- $|\boldsymbol{a}\rangle$ satisfies the Bethe equations with inhomogeneities, $\boldsymbol{w}$.
- $|\boldsymbol{b}\rangle$ is generic.
- Slavnov (extending Korepin, Gaudin) $(Q_{\boldsymbol{\alpha}}(x) = \prod_i (x - \alpha_i))$:

$$\langle \boldsymbol{a}|\boldsymbol{b}\rangle = \det_{i,j} \frac{1}{a_i - b_j} - \frac{1}{a_i - b_j + \eta} + \\ + \frac{Q_{\boldsymbol{w}}(b_j - \eta)Q_{\boldsymbol{a}}(b_j + \eta)}{Q_{\boldsymbol{w}}(b_j)Q_{\boldsymbol{a}}(b_j - \eta)}\left[\frac{1}{a_i - b_j} - \frac{1}{a_i - b_j - \eta}\right]$$

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$\Rightarrow$ Slavnov det may serve as a starting point, on which additional formalism must be built.

Alternative Approaches

- An axiomatic approach may be taken instead of a direct approach F. A. Smirnov *Form Factors Completely Integrable Models QFT*.

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- In certain cases overlaps satisfy enough conditions to fix them entirely. Worked through in the case of Sine-Gordon, qKdV and reductions.
- This approach may be combined with semiclassics (classical KdV) to obtain results.

Babelon,Bernard, Smirnov, Comm. Math. Phys. 182,186 (1996). Smirnov hep-th/9802132

Functional-analytic Approach to Slavnov Determinants

Sutherland Limit

- A symmetric I. Kostov, Y. Matsuo, JHEP 1210 (2012) 168 representation can be given EB, I. Kostov JPhysA 47(2014) 25401. Take $\boldsymbol{u} = \boldsymbol{a} \cup \boldsymbol{b}$, $Q_{\boldsymbol{a}}(x) = \prod_i (x - a_i)$:

$$\langle \boldsymbol{a} | \boldsymbol{b} \rangle = \det_{i,j} \left(\delta_{i,j} + \frac{Q_{\boldsymbol{z}}(u_i) Q_{\boldsymbol{u}}(u_i + \eta)}{Q_{\boldsymbol{z}}(u_i + \eta) Q'_{\boldsymbol{u}}(u_i)} \frac{1}{u_i - u_j + \eta} \right)$$

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- We must take det of $\mathbb{1} + K$, where $K = \frac{Q_{\boldsymbol{z}}(u_i) Q_{\boldsymbol{u}}(u_i + \eta)}{Q'_{\boldsymbol{u}}(u_i) Q_{\boldsymbol{z}}(u_i + \eta)} \frac{1}{u_i - u_j + \eta}$

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- $\frac{Q_{\boldsymbol{z}}(x)Q_{\boldsymbol{u}}(x+\eta)}{Q'_{\boldsymbol{u}}(x)Q_{\boldsymbol{z}}(x+\eta)}\psi(x+\eta)$ is a candidate for $\mathcal{K}\psi$.
- More precisely:

$$(\mathcal{K}\psi)(y) = \oint_{\boldsymbol{u}} \frac{1}{y-x}\frac{Q_{\boldsymbol{z}}(x)Q_{\boldsymbol{u}}(x+\eta)}{Q_{\boldsymbol{u}}(x)Q_{\boldsymbol{z}}(x+\eta)}\psi(x+\eta)$$

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The Slavnov matrix is just $1+\mathcal{K}$ with:

$$\mathcal{K} = \mathcal{P} e^{-\Phi} e^{\eta\partial} e^{\Phi}$$

with

$$e^{\Phi} = \frac{Q_{\boldsymbol{u}}}{Q_{\boldsymbol{z}}}, \quad (\mathcal{P}f)(x) = \oint \frac{f(y)}{x-y}$$

The Kondo Model

The Kondo Problem

- We consider the Kondo model with the Hamiltonian:

$$\int \psi^\dagger_\sigma(x)\left(-\imath\hbar\partial_x\right)\psi_\sigma(x) + g\psi^\dagger_\sigma(0)\vec{\sigma}_{\sigma\sigma'}\psi_{\sigma'}(0)\cdot\vec{S}.$$

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$\boldsymbol{\sigma}\otimes s = (\sigma_1, \sigma_2, \ldots, \sigma_N, s)$.
- Let Q order x: $x_{Q(1)} < x_{Q(2)} < \cdots < x_{Q(N)}$.
Define Non-standard representation: $\psi_{\boldsymbol{\sigma}\otimes s}(\boldsymbol{x})$

$$\psi_{Q\boldsymbol{\sigma}\otimes s}(\boldsymbol{x}) = \tilde{\psi}_{\boldsymbol{\sigma}\otimes s}(\boldsymbol{x}),$$

The Kondo Model

- The Bethe Ansatz solution:

$$\psi_{\boldsymbol{\sigma}\otimes s}(\boldsymbol{x}) = \sum_{P\in S_N} \mathrm{sign}(P)\Psi_{P\circ Q}(\boldsymbol{\sigma}\otimes s)e^{\imath \boldsymbol{x}\cdot P\boldsymbol{k}}.$$

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- Ψ is an *inhomogeneous* Heisenberg Bethe ansatz wavefunction.

Relation Between the Kondo and Heisenberg Problems

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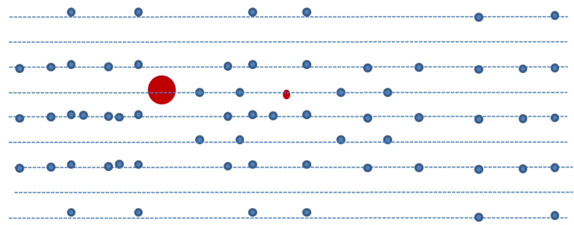
- In the thermodynamic limit strings form:



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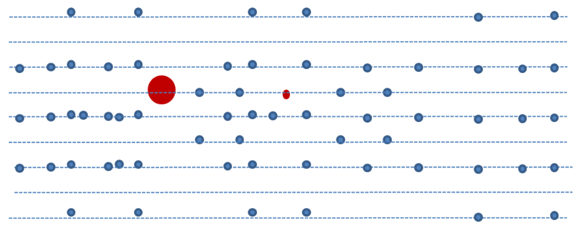

- Independent density $\boldsymbol{\sigma}(\lambda)$ and dependent density $\boldsymbol{\sigma}_h(\lambda)$.

Thermodynamic Bethe Ansatz

- The partition function

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- $E = \int \varepsilon(\lambda)\boldsymbol{\sigma}(\lambda)$

$S = \int (\boldsymbol{\sigma} + \boldsymbol{\sigma}_h) \log(\boldsymbol{\sigma} + \boldsymbol{\sigma}_h) - \boldsymbol{\sigma} \log(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h \log(\boldsymbol{\sigma}_h)\,.$

$\frac{\delta \boldsymbol{\sigma}_h(\lambda)}{\delta \boldsymbol{\sigma}(\lambda')} = \boldsymbol{K}(\lambda - \lambda')$

Quench Action Approach

- We wish to compute a non-equilibrium version, e.g., the amount of energy absorbed:

$$P(T) = \sum_f |\langle i|f\rangle|^2 e^{-\frac{E_f - E_i}{T}}.$$

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- We write integral equations for $\frac{\delta\log|\langle i|\boldsymbol{\sigma}\rangle|^2}{\delta\boldsymbol{\sigma}(\lambda)}$, the *'Non-equilibrium source'*.

Integral Eqs. for Non-equilibrium Source

- $\frac{\delta \log \det(1+\mathcal{K})}{\delta \boldsymbol{\sigma}} = \mathrm{tr}(1+\mathcal{K})^{-1}\frac{\delta \mathcal{K}}{\delta \boldsymbol{\sigma}}$, where $\mathcal{K} = \mathcal{P} e^{-\Phi} e^{\eta \partial} e^{\Phi}, \quad \mathcal{P} = \oint \frac{1}{x-y}$

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explicitly:

$$\mathcal{R}(x,y) - \oint_{\boldsymbol{u}} \frac{Q_{\boldsymbol{u}}(x'+\imath)Q_{\boldsymbol{z}}(x')}{Q_{\boldsymbol{u}}(x')Q_{\boldsymbol{z}}(x'+\imath)}\frac{\mathcal{R}(x'+\imath,y)}{(x'-x)} = \frac{1}{x-y}$$

Summary

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- Compute source as $\mathrm{tr}\mathcal{R}\frac{\delta \mathcal{K}}{\delta \boldsymbol{\sigma}}$, with $\mathcal{K} = \mathcal{P}e^{-\Phi}e^{\eta\partial}e^{\Phi}$, $\mathcal{R} = (1+\mathcal{K})^{-1}$ by solving a linear integral equation depending on $\boldsymbol{\sigma}$.

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- Fast convergence? Validity? Numerics?

The End