Based on:


Outline

1 Introduction

2 Simple interfaces
   average magnetization, passage probability
   Interface structure; Ising & $q$-Potts

3 Double interfaces
   Tricritical $q$-Potts interfaces
   Bulk wetting transition & Ashkin-Teller

4 Interfaces at boundaries
   Wedge geometry
   Boundary wetting transition & filling transitions

5 Summary & outlook
From lattice

- Exact studies focused so far on $D = 2$ Ising, exploiting lattice solvability
**Interfaces in two dimensions**

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**From field theory**
- $T = T_c$: Interfaces are conformally invariant random curves described by SLE. Connection with CFT in $D = 2$ applied at criticality but few is known about massive deformations.

**Away from criticality?** How to avoid lattice calculations and work directly in the continuum for general models? (i.e. scaling $q$-Potts, Ashkin-Teller, . . .)
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We propose a new approach to phase separation for massive interfaces ($T < T_c$) based on local fields

Field theory yields general and exact solutions for a wider class of models with a simple language, accounting for interface structure, boundary & bulk wetting, wedge filling

Application to thermodynamic Casimir forces and its dependence on bc.s (not this talk)
Scaling limit of a system of classical statistical mechanics in $2d$ below $T_c$. $(1 + 1)$-relativistic field theory analytically continued to a 2-dim Euclidean field theory in the plane $(x, y = -it)$.

- States with minimum energy: degenerate vacua (coexisting phases)

**Ising**

- $|K_{\pm\mp}\rangle$
- $|\mp\rangle \rightarrow |+\rangle$

**3-Potts**

- $|\Omega_c\rangle$
- $|K_{a\beta}\rangle$
- $|\Omega_a\rangle \rightarrow |\Omega_b\rangle$

**Generically**

- Adjacency structure $\Omega_a |\Omega_b\rangle$: adjacent $\rightarrow$ connected by $|K_{ab}\rangle$
- $\Omega_a \rightarrow \Omega_b$: not adjacent $\rightarrow$ connected by $|K_{\bullet\bullet}\rangle$ (the lightest)
Scaling limit of a system of classical statistical mechanics in $2d$ below $T_c$. $(1 + 1)$-relativistic field theory analytically continued to a 2-dim Euclidean field theory in the plane $(x, y = -it)$.

- States with minimum energy: degenerate vacua (coexisting phases)

- Elementary excitations: kinks (domain walls or interfaces)

$$|K_{ab}(\theta)\rangle \text{ interpolates between } |\Omega_a\rangle, |\Omega_b\rangle$$ relatavistic particles with $(E, P) = (m_{ab} \cosh \theta, m_{ab} \sinh \theta)$.

- Adjacency structure

$\Omega_a |\Omega_b\rangle$: adjacent $\longrightarrow$ connected by $|K_{ab}\rangle$

$\Omega_\bullet |\Omega_\bullet\rangle$: not adjacent $\longrightarrow$ connected by $|K_{\bullet\bullet}\rangle$ (the lightest)
Phase separation for adjacent phases

Symmetry breaking boundary conditions: $a \neq b$ with $R/\xi \propto m_{ab} R \gg 1 \rightarrow \text{single interface}$

No phase separation for $a = b$

$$\therefore \langle \sigma_a \rangle = \langle \Omega_a | \sigma(x, y) | \Omega_a \rangle$$
Symmetry breaking boundary conditions: \( a \neq b \) with \( R/\xi \propto m_{ab}R \gg 1 \rightarrow \text{single interface} \)

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\[ \langle \sigma_a \rangle = \langle \Omega_a | \sigma(x, y) | \Omega_a \rangle \]

Boundary states (cf. [Ghoshal-Zamolodchokov] for the translationally invariant case)

\[
|B_{ab}(x, t)\rangle = e^{-iHt + ixP} \left[ \int_{\mathbb{R}} \frac{d\theta}{2\pi} f_{ab}(\theta) |K_{ab}(\theta)\rangle + \sum_{c \neq a, b} \int_{\mathbb{R}^2} \frac{d\theta d\theta'}{(2\pi)^2} f_{ab}^c(\theta, \theta') |K_{ac}(\theta)K_{cb}(\theta')\rangle + \ldots \right]
\]

\[
|B_a(x, t)\rangle = e^{-iHt + ixP} \left[ |\Omega_a\rangle + \sum_{c \neq a} \int_{\mathbb{R}^2} \frac{d\theta d\theta'}{(2\pi)^2} f_{aa}^c(\theta, \theta') |K_{ac}(\theta)K_{ca}(\theta')\rangle + \ldots \right]
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Simple interfaces

Average magnetization, passage probability

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\]

Partition functions (leading order)

\[
Z_a(R) = \langle \mathcal{B}_a(0, iR/2) | \mathcal{B}_a(0, -iR/2) \rangle \sim \langle \Omega_a | \Omega_a \rangle = 1
\]

\[
Z_{ab}(R) = \langle \mathcal{B}_{ab}(0, iR/2) | \mathcal{B}_{ab}(0, -iR/2) \rangle \sim \frac{|f_{ab}(0)|^2}{\sqrt{2\pi m R}} e^{-mR}
\]

\[
\Sigma_{ab} = - \lim_{R \to \infty} \frac{Z_{ab}(R)}{Z_a(R)} = m
\]

\( \leftrightarrow \) \( mR \to \infty \) \( \implies \) projection to low-energy physics: \( \theta \ll 1 \)
Simple interfaces: order parameter profile

One-point function of the spin operator along the horizontal axis \((x, y = 0)\)

\[
\langle \sigma(x, 0) \rangle_{a b} = \frac{1}{Z_{a b}} \langle \mathcal{B}_{a b}(0, iR/2) | \sigma(x, 0) | \mathcal{B}_{a b}(0, -iR/2) \rangle
\]

\[
\approx \frac{|f_{a b}(0)|^2}{Z_{a b}} \int_{\mathbb{R}^2} \frac{d\theta_1 d\theta_2}{(2\pi)^2} \mathcal{M}_{a b}^{\sigma}(\theta_1 | \theta_2) e^{-mR \left(1 + \frac{\theta_1^2 + \theta_2^2}{4}\right)} - imx\theta_{12}
\]
Single interfaces: order parameter profile

One-point function of the spin operator along the horizontal axis \((x, y = 0)\)

\[
\langle \sigma(x, 0) \rangle_{ab} = \frac{1}{\mathcal{Z}_{ab}} \langle \mathcal{B}_{ab}(0, iR/2)|\sigma(x, 0)|\mathcal{B}_{ab}(0, -iR/2) \rangle
\]

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\]

Matrix element: 2-kink Form Factor + disconnected

\[
\mathcal{M}_{ab}^\sigma(\theta_1 | \theta_2) = \langle K_{ab}(\theta_1) | \sigma(0, 0) | K_{ba}(\theta_2) \rangle = F_{aba}(\theta_{12} + i\pi) + 2\pi\delta(\theta_{12}) \langle \sigma \rangle_a
\]
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\]

\[
\simeq \frac{|f_{ab}(0)|^2}{Z_{ab}} \int_{\mathbb{R}^2} \frac{d\theta_1 d\theta_2}{(2\pi)^2} \mathcal{M}_{ab}^\sigma(\theta_1 | \theta_2) e^{-mR(1 + \frac{\theta_1^2 + \theta_2^2}{4}) - imx\theta_{12}}
\]

- **Matrix element:** 2-kink Form Factor + disconnected

- **Crossing symmetry**

  Two kinks can annihilate \(\rightarrow\) kinematic pole of the FF: *does not require integrability*

  [Berg-Karowski-Weisz '78; Smirnov 80's; Delfino-Cardy '98]

  \[
  K_{ab}(\theta_1) + K_{ba}(\theta_2) \to \emptyset \quad \text{as} \quad \theta_1 - \theta_2 \to i\pi
  \]

  \[
  \sim -i \text{Res}_{\theta = i\pi} F_{\sigma}^\theta = \langle \sigma \rangle_a - \langle \sigma \rangle_b
  \]
low-energy expansion

\[
F_{\sigma}^{\sigma}(\theta + i\pi) = \frac{i\Delta \langle \sigma \rangle}{\theta} + \sum_{k=0}^{\infty} c^{(k)}_{\theta} \theta^k
\]
Single interfaces (cont’d)

low-energy expansion

\[ F_{aba}^{\sigma}(\theta + i\pi) = \frac{i\Delta \langle \sigma \rangle}{\theta} + \sum_{k=0}^{\infty} c_{ab}^{(k)} \theta^k \]

after some manipulations

\[ \sigma(x, 0)_{ab} = \langle \sigma \rangle_a + \frac{i \Delta \langle \sigma \rangle}{2} \int_{\mathbb{R}} \frac{d\theta}{\theta} e^{-\frac{\theta^2}{2} + i\eta\theta} + \ldots \quad \left( \eta \equiv \frac{x}{\lambda}, \quad \lambda \equiv \sqrt{\frac{R}{2m}} \right) \]
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The simple pole is essential but it needs to be regularized (\( \lim_{\epsilon \to 0} \frac{1}{\theta \pm i\epsilon} = \mp \pi i\delta(\theta) + \mathcal{P} \frac{1}{\theta} \)).
Single interfaces (cont’d)

Final result:

\[
\langle \sigma(x, 0) \rangle = \frac{\langle \sigma \rangle_a + \langle \sigma \rangle_b}{2} - \frac{\langle \sigma \rangle_a - \langle \sigma \rangle_b}{2} \text{erf}(\eta) + c^{(0)}_{ab} \sqrt{\frac{2}{\pi m R}} e^{-\eta^2} + \ldots
\]

[Delfino-Viti 12]
Final result:

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\]

- the non-local term is generated by the pole. It reflects non-locality of kinks w.r.t. spin field
- subleading local corrections \( \propto c^{(k)}_{ab} \): interface structure
- extend the derivation to \( y \neq 0 \): replacement \( \eta \rightarrow \chi \equiv \eta / \kappa \), \( \kappa \equiv \sqrt{1 - 4y^2/R^2} \).
  
The profile depends only on \( \chi \equiv \eta / \kappa \), Contour lines are arcs of ellipses.  
  \[
  \frac{x^2}{R^2 m (\text{const.})} + \frac{y^2}{(R/2)^2} = 1
  \]
- Midpoint fluctuation \( \sim \sqrt{R} \)
Examples: broken $\mathbb{Z}_2$ & broken $S_q$

- **Ising model:**
  \[ \langle \sigma \rangle_+ = -\langle \sigma \rangle_- \]
  \[ \langle \sigma(x, y) \rangle_\mp = \langle \sigma \rangle \pm \text{erf}(\chi) \]

  Perfect match with scaling of lattice solution, cf [Abraham, 81].
  Next correction is $\propto c_{\pm}^{(1)} \neq 0$ (3-furcation, by parity)
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- **$q$-state Potts model:** The scattering theory is integrable [Chim-Zamolodchikov] and Form Factors are known [Delfino-Cardy]

\[
\sigma_c(x) = \delta_s(x), c - \frac{1}{q} \\
\langle \sigma_c \rangle_a = \frac{q\delta_{ac} - 1}{q - 1} M \\
c_{ab,c}^{(0)} = [2 - q(\delta_{ac} + \delta_{bc})] MB(q)
\]

with $B(3) = \frac{1}{4\sqrt{3}}, B(4) = \frac{1}{3\sqrt{3}}$.

For $q = 3$: $\langle \sigma_3(0, 0) \rangle_{12} \propto \frac{1}{\sqrt{mR}} \rightarrow \text{“island”: branching & recombination of the interface}$
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For $q = 3$: $\langle \sigma_3(0, 0) \rangle_{12} \propto \frac{1}{\sqrt{mR}} \rightarrow$ "island": branching & recombination of the interface

Branching is a general phenomenon not due to integrability
For integrable theories we can compute the amplitude of the island (i.e. $B(q)$)
The interface will cross the horizontal axis ($y = 0$) in $x \in (u, u + du)$, with passage probability $p(u; 0)du$, how is the magnetization affected in $x$?

$$\langle \sigma(x, 0) \rangle_{ab} = \int_{\mathbb{R}} du \sigma_{ab}(x|u)p(u; 0)$$

$$\sigma_{ab}(x|u) = \theta(u - x)\langle \sigma \rangle_a + \theta(x - u)\langle \sigma \rangle_b + A_{ab}^{(0)} \delta(x - u) + A_{ab}^{(1)} \delta'(x - u) + \ldots$$
Passage probability and interface structure

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\]

Matching with field theory yields

\[
p(x; y) = \frac{1}{\sqrt{\pi \kappa \lambda}} e^{-x^2} \implies \text{Gaussian Bridge (\textasteriskcentered)}
\]

\[
A_{ab}^{(0)} = \frac{c_{ab}^{(0)}}{m} \implies \text{Bifurcation amplitude}
\]

\((\text{\textasteriskcentered})\) rigorously known for Ising and Potts \[\text{Greenberg, Joffe, '05; Campanino, Joffe, Velenik, '08}\]
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“RG” perspective: large \(R/\xi\) expansion

- \(R/\xi = \infty\): sharp interface picture
- \(R/\xi \gg 1\): proliferation of inclusions: bubbles of different phases

\[
\propto A_{ab}^{(0)} + A_{ab}^{(1)} + \ldots
\]
If the vacua $|\Omega_a\rangle$ and $|\Omega_b\rangle$ cannot be connected by a single kink

$$|B_{ab}\rangle = \sum_{c \neq a, b} \left[ \begin{array}{c} a \ \ c \ \ b \\ c \ \ a \ \ b \end{array} \right] + \sum_{d \neq c, b} \left[ \begin{array}{c} a \ \ c \ \ d \ \ b \\ d \ \ a \ \ b \end{array} \right] + \ldots$$
Double interfaces (I)

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4-kink matrix element

$$\langle K_{bd}(\theta_3)K_{da}(\theta_4) | \sigma | K_{ac}(\theta_1)K_{cb}(\theta_2) \rangle = \sigma = \sigma + \ldots$$
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4-kink matrix element

$$\langle K_{bd}(\theta_3)K_{da}(\theta_4)|\sigma|K_{ac}(\theta_1)K_{cb}(\theta_2)\rangle = \begin{array}{c}
\sigma \\
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4
\end{array} + \begin{array}{c}
\sigma \\
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4
\end{array} + \ldots$$

Connected part: low-energy limit

$$\mathcal{M}_{ab,cd}^{\sigma,\text{conn}}(\theta_1, \theta_2|\theta_3, \theta_4) = \left[ 2\langle \sigma \rangle_c - \langle \sigma \rangle_a - \langle \sigma \rangle_b \right] \frac{\theta_{12}\theta_{34}}{\theta_{13}\theta_{14}\theta_{23}\theta_{24}}$$
Double interfaces (I)

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this structure is inherited from the kinematic poles

- Average spin field

$$\langle \sigma(x, y) \rangle^{\text{conn}} \sim \int_{\mathbb{R}^4} d\theta_1 \ldots d\theta_4 \mathcal{M}_{ab,cd}(\theta_1, \theta_2|\theta_3, \theta_4) Y^{-}(\theta_1) Y^{-}(\theta_2) Y^{+}(\theta_3) Y^{+}(\theta_4)$$

$$Y^{\pm}(\theta) = \exp\left[ -\frac{1 \pm \epsilon}{2} \theta^2 \pm i\eta\theta \right]$$

then: regularization and integration over all the rapidities
Disconnected parts: each annihilation (leg contraction) produces a Dirac delta

\[
\begin{align*}
\theta_1 & \quad \theta_2 \\
\sigma & \quad \theta_3 \\
\theta_4 & \\
\sigma & \\
\theta_1 & \quad \theta_2 \\
\theta_4 & \quad \theta_3 \\
a & \quad \theta_3 \\
b & \\
c & \\
d & \\
= & \quad 2\pi \delta(\theta_{13}) \frac{i(\langle \sigma_c \rangle - \langle \sigma_b \rangle)}{\theta_{24}} \\
\theta_1 & \quad \theta_2 \\
\sigma & \quad \theta_3 \\
\theta_4 & \\
\sigma & \\
\theta_1 & \quad \theta_2 \\
\theta_4 & \quad \theta_3 \\
a & \quad \theta_3 \\
b & \\
c & \\
d & \\
= & \quad 2\pi \delta(\theta_{14}) \sum_{e \neq a, b} S_{ab}^{ce}(0) S_{ab}^{ed}(0) \frac{i(\langle \sigma_a \rangle - \langle \sigma_e \rangle)}{\theta_{23}} \\
\theta_1 & \quad \theta_2 \\
\theta_4 & \quad \theta_3 \\
a & \quad \theta_3 \\
b & \\
c & \\
d & \\
\text{then: sum up all the contributions}
For arbitrary models

\[
\langle \sigma(x, y) \rangle_{ab} = \frac{\langle \sigma \rangle_a + \langle \sigma \rangle_b - 2\langle \sigma \rangle_c}{4} G(\chi) - \frac{\langle \sigma \rangle_a - \langle \sigma \rangle_b}{2} L(\chi) + \frac{\langle \sigma \rangle_a + \langle \sigma \rangle_b + 2\langle \sigma \rangle_c}{4}
\]

\[
G(\chi) = -\frac{2}{\pi} e^{-2\chi^2} - \frac{2\chi}{\sqrt{\pi}} e^{-\chi^2} + \text{erf}^2(\chi)
\]

\[
L(\chi) = -\frac{\chi}{\sqrt{\pi}} e^{-\chi^2} + \text{erf}(\chi)
\]
For arbitrary models

\[\langle \sigma(x, y) \rangle_{ab} = \frac{\langle \sigma \rangle_a + \langle \sigma \rangle_b - 2\langle \sigma \rangle_c}{4} G(\chi) - \frac{\langle \sigma \rangle_a - \langle \sigma \rangle_b}{2} L(\chi) + \frac{\langle \sigma \rangle_a + \langle \sigma \rangle_b + 2\langle \sigma \rangle_c}{4}\]

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\[L(\chi) = -\frac{\chi}{\sqrt{\pi}} e^{-\chi^2} + \text{erf}(\chi)\]

Universal scaling form. Specific features of the models enters through the vev.s \(\langle \sigma_\alpha \rangle_\beta\)

A “forced” example: Ising bubble (we have only two vacua!)

\[\langle \sigma(x, y) \rangle_{\pm\pm} = \langle \sigma \rangle_{\pm \pm} G(\chi)\]

perfect match with lattice Ising [Abraham-Upton, 93]
Tricritical $q$-state Potts

Annealed vacancies are allowed (if no vacancies: pure $q$-state Potts).

- vacua connectivity

**Continuous transitions**

$T < T_c$, $\rho = 0$, $q \leq 4$

**First-order transitions**

$T = T_c$, $\rho > \rho_c$, $q \leq 4$

$T = T_c$, $q > 4$

$(q$ not too large $\xi(q = 10) \approx 10)$
**Tricritical $q$-state Potts**

Annealed vacancies are allowed (if no vacancies: pure $q$-state Potts).

- vacua connectivity

![diagram](image)

- continuous transitions
  $T < T_c$, $\rho = 0$, $q \leq 4$

- first-order transitions
  $T = T_c$, $\rho > \rho_c$, $q \leq 4$

Dilute regime: Star-graph-like vacua structures. The continuum limit is described by an integrable scattering theory whose spectrum is known. Elementary excitations: $K_{i0}$, $K_{0j}$. The process

$$|K_{i0}\rangle + |K_{0j}\rangle \rightarrow |\tilde{K}_{ij}\rangle$$

cannot take place (absence of a pole of $S_{ij}^{00}$ in the physical strip [Delfino, '99]) $\rightarrow$ the vacua connectivity for the dilute case is a star graph.
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![Diagram of continuous and first-order transitions]

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- Order parameter profiles

$$\langle \sigma_1(x,y) \rangle_{12} = \frac{\langle \sigma_1 \rangle_1}{2} \left[ \frac{q-2}{2(q-1)} (1 + G(\chi)) + \frac{q}{q-1} L(\chi) \right] \quad \text{(smooth-step-like)}$$

$$\langle \sigma_3(x,y) \rangle_{12} = -\frac{\langle \sigma_1 \rangle_1}{2(q-1)} \left[ 1 + G(\chi) \right] \quad \text{(bubble-like)}$$
Tricritical $q$-state Potts

Dilute 3-Potts: plot of $\langle \sigma_0(x, y) \rangle_{12}$

Dilute case: the bubble is not suppressed for $mR \gg 1$ (cf. pure 3-Potts)
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- Passage probability matches field theory with

$$P(x_1, x_2; y = 0) = \frac{2m}{\pi R} (\eta_1 - \eta_2)^2 e^{- (\eta_1^2 + \eta_2^2)}$$

the interfaces $\Omega_1|\Omega_0$, $\Omega_0|\Omega_2$ are mutually avoiding curves anchored in $(0, \pm R/2)$. 
Ising spins $\sigma, \tau$ on a lattice

$$\mathcal{H}_{AT} = - \sum_{\langle x_1, x_2 \rangle} \left[ J\sigma(x_1)\sigma(x_2) + J\tau(x_1)\tau(x_2) + J_4\sigma(x_1)\sigma(x_2)\tau(x_1)\tau(x_2) \right]$$
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scaling AT($J_4$) renormalizes into Sine-Gordon($\beta$) $\Rightarrow$ $J_4 \leftrightarrow \beta$ & kinks $\leftrightarrow$ solitons

$$\frac{4\pi}{\beta^2} = 1 - \frac{2}{\pi} \sin^{-1} \left( \frac{\tanh 2J_4}{\tanh 2J_4 - 1} \right) \text{ on square lattice} \quad \text{[Kadanoff]}$$
Ising spins $\sigma, \tau$ on a lattice

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on square lattice [Kadanoff]

- Vacua connectivity

We can tune $J_4$ to change the vacua connectivity and the phase separation pattern $\rightarrow$ Transition!
Bulk wetting transition

- $J_4 > 0$: drops of $\pm \mp$ phase are adsorbed along $(++)|(- -)$ with contact angle $\gamma$
- $J_4 \to 0^+$, $\gamma \to 0^+$: wetting
- $J_4 \leq 0$: drops spreading, $(++)|(- -)$ is wetted by $\pm \mp (\gamma = 0)$
Bulk wetting transition: Ashkin-Teller (II)

- **Bulk wetting transition**

  \[ J_4 > 0: \text{drops of } \pm \mp \text{ phase are adsorbed along } (++)|(−−) \text{ with contact angle } \gamma \]

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- Decoupling point \( J_4 = 0 \)
  Ising results are recovered

- Equilibrium condition for the triple line \( \implies \) contact angle

\[ \gamma = 2\pi \frac{4\pi - \beta^2}{8\pi - \beta^2} \]
Bulk wetting transition: Ashkin-Teller (II)

- **Bulk wetting transition**

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  \[ J_4 \leq 0: \text{drops spreading, } ++|-- \text{ is wetted by } \pm \pm \text{ (} \gamma = 0 \text{)} \]

  \[ \gamma(J_4) = \pi + \frac{\pi^2}{\pi - 4 \sec^{-1}(1 - \coth(2J_4))} \]

  **here AT=4-Potts**

  \[ \alpha = \beta = \gamma \]

- **Decoupling point** \( J_4 = 0 \)

  Ising results are recovered

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  \[ \gamma = 2\pi \frac{4\pi - \beta^2}{8\pi - \beta^2} \]

- **Observables are sensitive only of the interaction sign**:

  from \( J_4 < 0 \) to \( J_4 > 0 \)

  \[ \langle \sigma_i(x, y) \rangle_{(++,-)} \propto \mathcal{L}(\chi) \quad \rightarrow \quad \propto \text{erf}(\chi) \]

  \[ \langle \sigma \tau(x, y) \rangle_{(++,-)} \propto \mathcal{G}(\chi) \quad \rightarrow \quad \propto \text{erf}^2(\chi) \]

  \[ P(x; y) = (\chi_1 - \chi_2)^2 p(\chi_1)p(\chi_2) \quad \rightarrow \quad = p(\chi_1)p(\chi_2) \]
Phenomenological description in terms of contact angle and surface tensions

Equilibrium condition for the contact line $C$:

$$\Sigma_{Ba} = \Sigma_{Bb} + \Sigma_{ab} \cos \theta_0$$

(Young’s law, 1802)

$\theta_0 \to 0$: wetting transition (spreading of the drop)
Boundary field theory

- Vertical b.dry. Pinned interface selected with a b.dry changing field $\mu_{ab}(y)$: switches from $B_a$ to $B_b$

$$0 \langle \Omega_a | \mu_{ab}(y) | K_{ba}(\theta) \rangle_0 = e^{-my \cosh \theta} F^{\mu}_0(\theta)$$

linear behavior for small rapidities:

$$F^{\mu}_0(\theta) = c\theta + o(\theta)$$
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\]

- Tilted b.dry: take an imaginary Lorentz boost ($B_\Lambda : \theta \to \theta + \Lambda$)

\[
\mathcal{B}_{-i\alpha} : F^\mu_0(\theta) \longrightarrow F^\mu_{\alpha}(\theta) = F^\mu_0(\theta + i\alpha)
\]

at small rapidities: \( F^\mu_{\alpha}(\theta) \simeq c(\theta + i\alpha) \)
Interfaces at boundaries

Interfaces in a shallow wedge

Order parameter in the wedge

\[ \langle \sigma(x,y) \rangle \]

\[ = \alpha \langle \Omega_a | \mu_{ab}(0,R/2) \sigma(x,y) \mu_{ba}(0,-R/2) | \Omega_a \rangle \]

\( \alpha \ll 1 \)

\[ \rightarrow \]

Passage probability density

\[ P(x;y) = 8 \frac{\sqrt{2}}{\sqrt{\pi}} \frac{mR^3}{3} \left( x + \alpha R/2 \right)^2 - \left( \alpha y \right)^2 
+ \frac{mR^2}{2} e^{-\chi^2} \]

Vanishes along the boundary.

Midpoint fluctuations \( \sim \sqrt{R} \).

[Delfino-AS, PRL '13]
Interfaces in a shallow wedge

Order parameter in the wedge

\[ \langle \sigma(x, y) \rangle_{W_{aba}} = \frac{\alpha \langle \Omega_a | \mu_{ab}(0, R/2) \sigma(x, y) | \mu_{ba}(0, -R/2) | \Omega_a \rangle_\alpha}{\alpha \langle \Omega_a | \mu_{ab}(0, R/2) \mu_{ba}(0, -R/2) | \Omega_a \rangle_\alpha} \]

\[ (\alpha \ll 1) = \langle \sigma \rangle_b + (\langle \sigma \rangle_a - \langle \sigma \rangle_b) \left[ \text{erf}(\chi) - \frac{2}{\sqrt{\pi}} \frac{\chi + \sqrt{2mR\alpha}}{1 + mR\alpha^2} e^{-\chi^2} \right] \]

\[ \rightarrow \text{recover results for lattice Ising with } \alpha = 0 \] [Abraham, ’80]

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Passage probability density

\[ P(x; y) = \frac{8\sqrt{2}}{\sqrt{\pi}\kappa^3} \left( \frac{m}{R} \right)^{3/2} \left( x + \alpha R/2 \right)^2 \left( \alpha y \right)^2 e^{-\chi^2} \]

- Vanishes along the boundary.
- Midpoint fluctuations \( \sim \sqrt{R} \).
Half plane

The boundary amplitude may exhibit a simple pole at $\theta = i\theta_0$

$$kink + boundary \rightarrow bound \ state \ |\Omega_a\rangle'$$

with binding energy: $E'_0 - E_0 = m\cos\theta_0$

kink unbinding $\rightarrow$ wetting transition

$$\theta_0(T_0) = 0 \quad , \quad T_0 < T_c$$

resonant angle $\leftrightarrow$ contact angle
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**Wedge**

Lorentz invariance

$$\theta_0 \rightarrow \theta_0 - \alpha \quad \text{(wedge covariance)}$$

condition encountered in effective hamiltonian theories

Kink unbinding $\rightarrow$ filling condition

$$E'_\alpha - E_\alpha = m \cos(\theta_0 - \alpha) \rightarrow \theta_0(T_\alpha) = \alpha$$

condition known from macroscopic thermodynamic arguments [Hauge '92]
A new method: exact and general field-theoretic formulation of phase separation and related issues (passage probabilities, interface structure (branching), interfaces at boundaries, wetting & filling)

Phase separation is investigated for general models for the first time directly in the continuum, the known solutions from lattice for Ising are recovered as a particular case.

Extended observables (interfaces) captured by local fields

The validity of the technique does not rely on integrability but rather on the fact that domain walls are particle trajectories

Although $mR \gg 1$ projects to low energies, relativistic particles are essential for kinematical poles and contact angles
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Perspectives

- Extensions to higher dimensions are possible (e.g. 3D XY vortex profile [Delfino, 14]); what about more vortices?
- Connection with critical point &SLE?
- Different geometries
- ...
Thank you for your attention!