THE EVALUATION OF STRING AMPLITUDES

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Abstract

The geometrical approach to the evaluation of amplitudes for the ground state of strings is reviewed, and related developments are considered A case is made that most of the ideas in string theory were already discovered in embryo in the fruitful years 1969-1974.

1 The Beginnings

In the middle of year 1968 I was feeling very pessimistic about the possibility of theorists ever being able to say anything about scattering amplitudes for hadrons, beyond the simple tree and Regge pole approximations and was contemplating changing fields. However, to everyone's complete surprise Gabriele Veneziano came up with his famous compact form for a dual amplitude; The amplitude which describes the scattering of four identical scalar particles, $A(s,\ t,\ u)$ where is given by the sum of three terms

$$\boxed{A(s,t) + A(s,u) + A(t,u)}$$

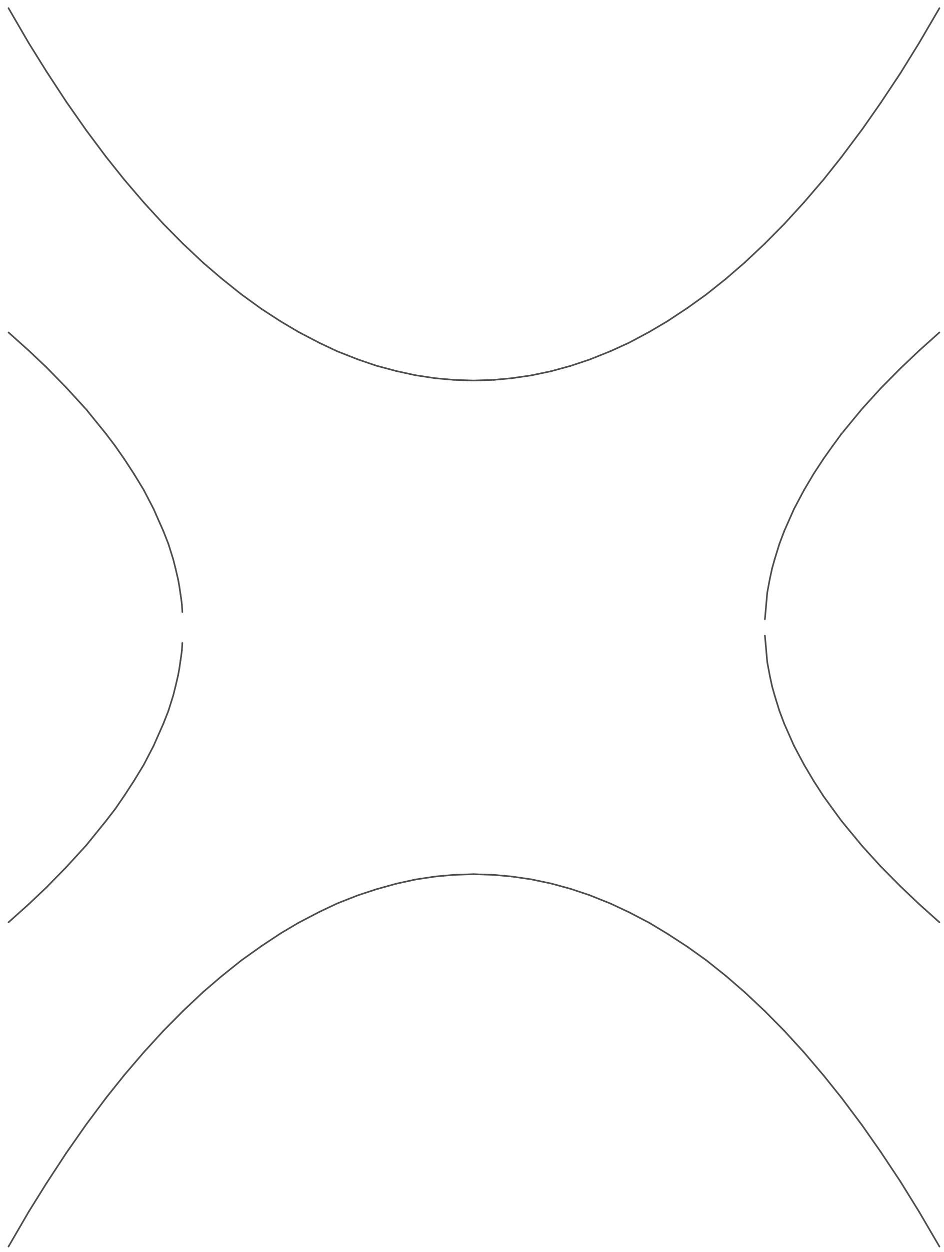
and $s\,t,\ u$ are the energies in the three possible ways of looking at the scattening process; If the process is considered as one where the initial momenta are $p_1^\mu \text{and} p_2^\mu$ and the final as $-p_3^\mu \text{and} - p_3^\mu$ then $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, etc. See Figure. Each contribution can be expressed as an integral representation

$$\boxed{A(s,t) = \int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}dx = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}}$$

with $\alpha(s) = \alpha_0 + \alpha' s$ The result is that the Veneziano amplitude [1] with these linear trajectories implies the existence of an infinite set of poles with multiple degeneracies. The Veneziano amplitude displays the property of duality; the same amplitude may be expressed as a sum of s channel poles with residues decomposable into positive angular functions of the scattering angle given by

$$\boxed{t = -2p^2(1 - \cos(\theta)}$$

where p is the centre of mass momentum. Two avenues of research developed out of this; one was the operator approach to the subject, the foundations to which were laid by Veneziano himself, together with Fubini and Gordon [2],[3]. I shall not say very much myself about this as I expect it will be comprehensively covered by other speakers. The other line of inquiry lay in the direction of generalising the integral representation of the



amplitude: Bardakci and Ruegg [4] gave an integral representation for the five-point function, which they, and Chan Hong-Mo and his wife [5] generalised further to the N point amplitude. I myself was occupied in these activities and believe that Keith Jones and I were the first to notice the tachyon condition; that if one imposes the (unphysical) requirement that the ground state is a tachyon, then the four and five point amplitudes can be expressed as integrals of a single integrand over the whole of $\mathbb{R}$ and $\mathbb{R}^2$ respectively [6]. The big advance came with the marvellous formula of Koba and Nielsen [7] giving an elegant formula for the N point tree amplitude;

$$A(s,t)t = \int_{-\infty}^{\infty} \frac{1}{dV_{abc}} \prod_{1}^{N} \theta(z_i - z_{i+1})(z_i - z_{i+1})^{\alpha_0 - 1} \prod_{j>i} (z_i - z_j)^{-2\alpha' p_i . p_j}$$

with

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_b - z_a)(z_c - z_a)(z_a - z_c)}$$

This integration measure is introduced as a consequence of conformal invariance; to account for the property that the real axis along which the integrationis performed is invariant under transformations of the M‘obius group,

$$\boxed{z' \mapsto \frac{az+b}{cz+d}, \quad ad - bc = 1}$$

Note that the tachyon condition $\alpha_0 = 1$ makes the amplitude completely permutation invariant in all subenergies. As an aside, while I was thumbing through 'Whittaker and Watson' looking for inspiration, I came across an exercise in the chapter on hypergeometric functions, which might be paraphrased as; 'Prove that the 5 point function may be expressed in terms of the Hypergeometric Function $F_{3:2}$', referring to a paper by A.C. Dixon [8] in 1905! A parallel development was the study by Nambu [9] and independently Goto [10] of the propagation of an object with a one dimensional extension i.e a string instead of a particle. As a particle moves in space time it traces out a curve and the action may be described by the reparametrisation invariant expression

$$\boxed{S = \int \sqrt{\frac{dx^{\mu}}{d\tau}\frac{dx_{\mu}}{d\tau}}\, d\tau}$$

A string sweeps out a worldsheet in space-time, and the action is proportional to the area swept out by the sheet.

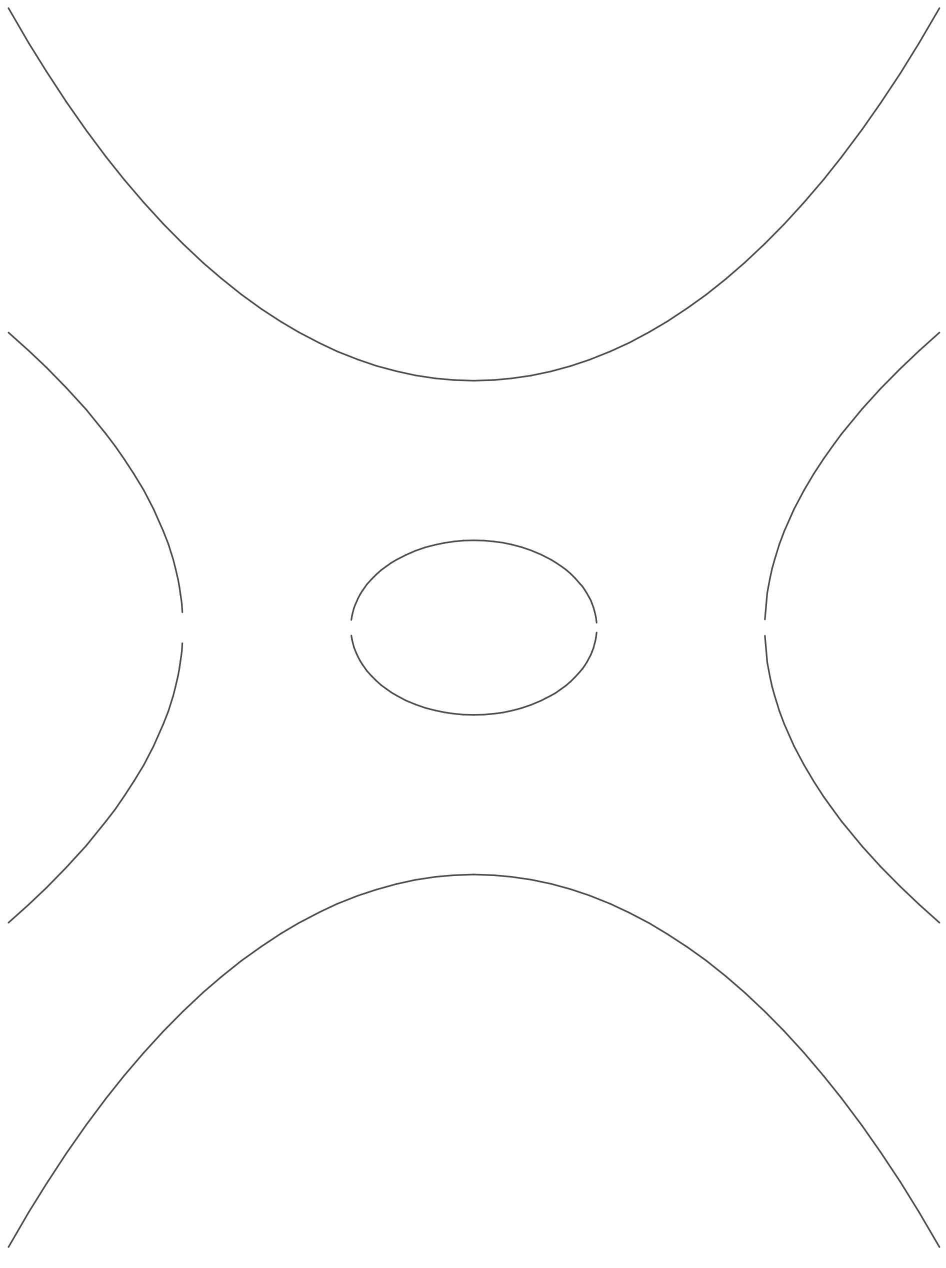
$$\boxed{S = \int \sqrt{\left(\frac{\partial x^{\mu}}{\partial \tau}\frac{\partial x_{\mu}}{\partial \sigma}\right)^2 - \left(\frac{\partial x_{\mu}}{\partial \sigma}\right)^2 \left(\frac{\partial x^{\nu}}{\partial \tau}\right)^2}\, d\sigma d\tau}$$

The striking resemblance of this process to the duality diagrams of the dual resonance model led to the quest for a closer connection, and an interpretation of the Koba Nielsen amplitude Nielsen and I developed an approach relying upon an electrostatics. I hit upon the idea as a result of an advertisement for Philips in *Scientific American* in the form of a research report on conformal methods in potential theory.I was very familiar with the use of complex analysis in solving 2-dimensional electrostatic problems as an undergraduate from Jean's book on Electromagnetism. Our idea was a method for computing the structure of the amplitudes corresponding to the Feynman diagrams for the ground state scattering in String Theory by means of an electrical analogue in which the amplitude is related to the heat generated in a plate of uniform resistivity corresponding to the world sheet associated with each diagram, with currents related to the components of particle four momenta.

Holger Nielsen had a rather more physical approach to the same idea, more closely related to the path integral formalism and talked about it at the Kiev Conference in 1970. We got together and wrote a paper describing this idea to which I gave the unfortunately recondite title, ‘ An Analogue Model for K.S.V. Theory’[11]! This paper demonstrated that the Veneziano amplitude describes the elementary process of string scattering. and reproduced the Koba Nielsen multiparticle amplitude for many particles. We also computed the one loop contribution up to a measure and thus opened up the possibility of calculating a String perturbation theory This was then followed by the work of Amati, Alessandrini and Lovelace.

Susskind also claimed to identify the amplitudes in the dual resonance model with string scattering [12] at about the same time. Holger visited Susskind at Yeshiva for one month in that year. The last few pages of this paper acknowledge a Nordita preprint of Holger Nielsen (the text of his Kiev talk?) in which he describes the fishnet approach to dual resonance theory, and takes the continuum limit. In earlier papers Susskind [13] employs more of an operator approach to calculating amplitudes.

The replacement of the distance function $|z_i - z_j|$ by $|z_i - z_j + \theta_i\theta_j|$, where the θ_i are Grassmannian, with subsequent integration over these additional variables was shown in [20],[21] to give rise to the Neveu Schwarz amplitudes. The alternative operator method gained prominence however, although Mandel-



stam used our approach to give the first calculation of the four Fermion dual amplitude [15] and later in the mid 80's used it again in his proof of finiteness of String perturbation theory. At about the same time Corrigan, Olive, Goddard and Smith calculated the same process using the operator method. [16] The excellent review of Paolo di Vecchia [17] contains a detailed discussion of the operator approach to the calculation of amplitudes, starting with the papers of Fubini, Gordon and Veneziano [3] and Susskind [13]. The idea is to introduce an operator

$$\boxed{Q_\mu \;=\; Q_\mu^{(+)}(z) + Q_\mu^{(0)}(z) + Q_\mu^{(-)}(z)}$$

with

$$\boxed{Q_\mu^{(+)} \;=\; i\sqrt{2\alpha'}\,\Sigma_{n=1}\frac{a_n}{\sqrt{n z^{-n}}};\quad Q_\mu^{(0)} = \hat{q} + 2\alpha'\hat{p}\log(z)}$$

and $Q_\mu^{(-)}(z)$ the complex conjugate of $Q_\mu^{(+)}(z)$ Corresponding to the external leg with momentum p a vertex operator

$$\boxed{V(z:\;p)\;\;: \exp(ip.Q)}$$

is introduced which serves to create a string in terms of creation operators $a_n^\dagger$ Manipulation of the formalism shows that the vacuum expectation value of a product of such vertex operators gives essentailly the Koba Nielsen integrand;

$$\boxed{\left\langle 0,0\right|\Pi_{i=1}^{N} V(z_i;p_i)\left|\right\rangle \;=\; \Pi_{i>j}(z_i - z_j)^{-2\alpha' p_i.p_j} 2\pi^d\,\Sigma_{i=1}^{N} p_i}$$

(After 1070)

In 1970 Daniele Amati assembled most of the active European string theory workers at Cern. were present at various times, Peter Goddard, Gabriele Veneziano, Ed Corrigan, David Olive, Claude Lovelace,Eugene Cremmer, Joel Scherk, Bernard Julia, Werner Nahm, Jean Loup Gervais, Jeffrey Goldstone Paul Frampton, Lars Brink, Claudio Rebbi,Richard Brower, Charles Thorn, R. Walz, Victor Alessandrini, Paulo di Vecchia, Stefano Sciuto F. Gliozzi, E. Del Guidice and myself. (Apologies to those I have forgotten; there were more Italians than I have listed.) The high points of the next three years at CERN were the papers of Goddard and Thorn, and independently, Richard Brower, proving the No Ghost Theorem, the papers of Alessandrini, Amati and Lovelace extending the analogue calculation to world sheets of arbitrary genus and the famous paper of Goddard, Goldstone Rebbi and Thorn quantising the bosonic string in 26 dimensions [18]. (Lovelace had already shown that 26 was a significant dimension, as that in which the pomeron singularity becomes a pole, rather than a cut, ensuring unitarity) [19].

One of the goals of the earlier period was to construct an off shell theory, so that the string states could couple to currents. Ed Corrigan and I had a solution to this problem, by the introduction of Dirichlet boundary conditions; in our paper [22] we exploited both the analogue and the operator methods to construct amplitudes with the correct properties. A general bosonic

boson	fermion
a_n^μ	$b_{n+1/2}^\mu$
$c_{n+1/2}^\mu$	d_n^μ

Table 1: caption
arXiv:0704.0101

state can be expressed as

$$\boxed{x^\mu(\sigma,\tau) \;=\; q + 2ip\tau + \sum a_n^\mu \exp(in\tau)\cos(n\sigma)}$$

with integrally moded a_n^μ bosonic oscillator creation operators; Neveu and Schwarz employed half integrally moded $b_{n+1/2}^\mu$ fermionic oscillators and Ramond introduced the integrally moded d_n^μ fermionic operators. It is obvious that what is missing is a set of half integrally moded bosonic $c_{n+1/2}^\mu$'s! These are just what is required to construct the off-shell states. 1

Actually such oscillators had been introduced earlier by John Schwarz, but without the application we had in mind. I display a page from our paper showing a scattering of 3 on-shell and one off-shell string states into 2 on and 2 off-shell states. The important point is that the off shell string states stop at a finite time, on what would now be called a $\mathrm{D-instanton}$. This paper was written at the time of the first demise of string theory and nobody took any notice of it except for Mike Green, who extended our theory to closed strings and Joe Polchinski. Almost 30 years later, Polchinski also came up with an idea which we should kick ourselves for not having thought of; namely to to apply Dirichlet boundary conditions in only a subset d of the dimensions and thus invent $\mathrm{D-branes}$. In our earlier paper, we encountered a paradox; it seemed that in the bosonic case the critical dimension ought to be 16 instead of 26, till Ed noticed that if we included the fermionic sector then the dimension was the expected 10. In this context it is worth noting that Ed Witten showed that in the case of arbitrary d, the inclusion of fermions restored the critical dimension to 10.

Conclusion

As I see it there are several interesting features about the 5 years from 1969-1974. As I hope I have shown, many of the ideas which have come to be recognised in the canon of string theory were already discovered in this period; the concept of duality, its implementation in the geometric and operator factorisation of amplitudes, the critical dimension, even the genesis of the d-brane and most importantly the idea that string amplitudes for ground state closed strings can be identified with graviton amplitudes. With every resurrection of string theory, the possibility of confrontation with experimental evidence has unfortunately retreated. In the dual resonance period, when the theory was envisaged as atheory of hadrons, theory and experiment were very close, and people like Chan Hong Mo and Claude Lovelace were developing phenomenological as well as theoretical aspects of the subject. Also, at the time I saw it as a last hope for a tractable physics As an undergraduate, I was intrigued by the tractability of many areas of physics, thanks to the linearity of Electromagnetic Theory and Quantum Mechanics. I had great hopes for String Theory, but these hopes have not been realised.

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