NUCLEAR REACTIONS
Instructor: A. Volya, e-mail: avolya@fsu.edu
Homeworks
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1 Effective Radius in Square Well Potential

(a) Calculate the scattering length a and effective range r_0 in the effective range expansion at low energies

$$k \cot \delta_0(k) \approx -\frac{1}{a} + \frac{1}{2} r_0 k^2,$$

for a square attractive potential (depth U_0, radius R).

(b) When the potential depth U_0 is very close to critical $U_{\rm cr}$ (at which a new bound state is formed and a becomes infinite) find the dependence of the binding energy E_0 as a function of $\delta U = U_0 - U_{\rm cr}$.

2 Wigner inequality

Consider two regular solutions $u(k_1, r)$ and $u(k_2, r)$ of the radial Schrödinger equation at slightly different energies, the corresponding magnitudes of wave vectors are k_1 and k_2.

(a) Show that the following equation is satisfied for an arbitrary location R.

$$\left. u(k_1, r)\frac{du(k_2, r)}{dr} - u(k_2, r)\frac{du(k_1, r)}{dr}\right|_{r=R} = (k_1^2 - k_2^2)\int_0^R u(k_1, r)u(k_2, r)dr \tag{1}$$

(b) For s-wave states in a potential of a finite range R the radial function, normalized by delta function in k, is $u(k, r) = \sqrt{2/\pi}\sin(kr + \delta(k))$ at $r \geq R$. Examine an infinitesimal change $k_2 = k$ and $k_1 = k + dk$ and show that

$$\int_0^R u^2(k, r)dr = \frac{1}{\pi}\left[R + \frac{d\delta}{dk} - \frac{1}{2k}\sin\left(2kR + 2\delta\right)\right]. \tag{2}$$

(c) Using the effective range expansion

$$k \cot(\delta) \approx -\frac{1}{a} + \frac{1}{2} r_0 k^2 \tag{3}$$

in the limit $k \to 0$ show that

$$2R\left[1 - \frac{R}{a} - \frac{1}{3}\left(\frac{R}{a}\right)^2\right] > r_0 \tag{4}$$

(c) In the vicinity of a resonance at energy E_0 the phase shift is rapidly changing as a function of energy

$$\delta(E) = \delta_0 - \arctan\left(\frac{\Gamma/2}{E - E_0}\right),$$

where δ_0 is a constant and Γ is the width of the resonance. Demonstrate that the fast change in the phase shift shows that near the resonance energy the continuum states have increasingly large amplitude in the interior region $r < R$, namely

$$\int_0^R u^2(k,r)dr \approx \frac{\hbar}{\pi}\sqrt{\frac{2E_0}{m}}\frac{\Gamma/2}{(E-E_0)^2 + \Gamma^2/4} \tag{5}$$

3 Spherical shell potential

Assume a potential of a spherical shell,

$$U(r) = g\delta(r - R). \tag{6}$$

(a) Calculate the cross section for low energy particles.

(b) For large g find resonances and their lifetimes.

4 Decay rate at short times

Consider time evolution of a decaying state $\psi(t)$ at very short times. The state $\psi(0)$ is prepared and evolves with time under Hamiltonian H.

(a) Let us define the survival probability as $S(t) = |\langle\psi(t)|\psi(0)\rangle|^2$. Show that for short times $S(t)$ is non-exponential, find the characteristic time scale of non-exponentiallity.

(b) Consider a more general definition of the survival probability that measures the probability of the system to remain in some internal subspace $\mathcal{P}$, namely $S(t) = \langle\mathcal{P}\psi(t)|\mathcal{P}\psi(0)\rangle$. Here $\mathcal{P}$ is a projection operator, for the initial state we assume $\mathcal{P}\psi(0) = \psi(0)$. Show that at the initial moment the rate of the decay $R(t) = -dS(t)/dt$ is zero.

5 Decay rate at long times

Consider s-waves states in a spherically symmetric potential. The potential has no bound states and the continuum eigenstates states can be labeled by the asymptotic momentum $|k\rangle$, here $k > 0$. Suppose that in an interior region there is an initial resonant state $|\alpha\rangle$, normalized as $\langle\alpha|\alpha\rangle = 1$. This state is embedded in the continuum and decays. Using the formalism of the standard scattering theory find the remote time behavior of the survival probability.

6 Porter-Tomas distribution

For narrow resonances the reduced width of a state is given by the square of the overlap between the wave function $|\psi\rangle$ of this state and decay channel wave function $|c\rangle$; $\gamma = |\langle c|\psi\rangle|^2$.

(a) If Hilbert space of the model is two dimensional and eigenstate $|\psi\rangle$ can be seen as randomly oriented vector with real components (we assume here time reversal invariance). What is the distribution of reduced widths γ.

(b) Generalize part (a) assuming Ω dimensional space.

(c) Assuming a large Ω limit express the limiting distribution in terms of average reduced width $\overline{\gamma}$.

7 Unstable spin-system in magnetic field

Consider two interacting distinguishable spin-1/2 molecules, $s_1 = s_2 = s = 1/2$, with the spin-spin interaction

$$H^\circ = \alpha\, \vec{s}_1 \cdot \vec{s}_2. \tag{7}$$

The system is placed in the magnetic field that produces an additional term in the Hamiltonian

$$H^{\mathrm{B}} = \epsilon s_1^z + \epsilon s_2^z = \epsilon S^z \tag{8}$$

which leads to Zeeman splitting. In addition to that this two-spin system in the magnetic field becomes open; in the presence of the field the first molecule in its excited polarized state can dissociate. This means that the molecule when in the state with $s_1^z = 1/2$ state decays exponentially. The decay is modeled by an additional non-Hermitian term in the Hamiltonian

$$W = -i\frac{\gamma}{4}(s_1^z + s), \tag{9}$$

so that the first molecule in the state with $s_1^z = 1/2$ would have decay width γ and this width would be zero in the state $s_1^z = -1/2$. As a result, in the magnetic field the effective Hamiltonian for the two-spin system becomes

$$\mathcal{H} = H^\circ + H^{\mathrm{B}} + W = \alpha\, \vec{s}_1 \cdot \vec{s}_2 + \epsilon s_1^z + \epsilon s_2^z - i\frac{\gamma}{4}(s_1^z + s). \tag{10}$$

Find the non-stationary eigenstates of the system; determine their resonance energies and widths as a function of decay strength γ. Discuss limits where γ is very small and very large. Is it possible for the levels to cross in the complex plane?