The Shell Model: An Unified Description of the Structure of the Nucleus (I)

ALFREDO POVES
Departamento de Física Teórica and IFT, UAM-CSIC
Universidad Autónoma de Madrid (Spain)

”Frontiers in Nuclear and Hadronic Physics”
Galileo Galilei Institute
Florence, February-Mars, 2016

Outline

- Undergraduate Nuclear Physics in a Nutshell
- The Interacting Shell Model
- Effective Interactions: Monopole, Pairing and Quadrupole
- Collectivity
  - Nuclear Phonons; Vibrational spectra
  - Superfluidity
  - Rotating Deformed Nuclei

What do the textbooks tell us about the nucleus?

- It is a system composed of Z protons and N neutrons (A=N+Z)
- Whose low energy behavior can described with non relativistic kinematics
- Bound by the strong nuclear interaction; the restriction of QCD to the space of neutrons and protons
- Which has a complicated form: Strong short range repulsion, spin-spin, spin-orbit and tensor terms, etc
- All these terms are put to good use in the description of the deuteron and of the nucleon-nucleon scattering
- However for heavier systems, typically A$>$12 the free space two body interaction is somehow forgotten and two contradictory visions emerge; the liquid drop model (LDM) and the independent particle model (IPM)

Basic experimental facts

- Which nuclei are stable?
- How much they weight? The mass of a nucleus is the sum of the masses of its constituents minus the energy due to their mutual interactions (binding energy), which is the lowest eigenvalue of its Hamiltonian
- For medium and heavy mass nuclei the binding energy per particle is roughly constant (saturation)
- What are their matter densities and radii? The nuclear radius grows as $A^{1/3}$, therefore the nuclear density is constant (saturation)

The Liquid Drop Model

- These properties resemble to those of a classical liquid drop, thus the binding energies might be reproduced by a semi empirical mass formula with its volume and surface terms: B= a_v A - a_s $A^{2/3}$
- However the drop is charged and the Coulomb repulsion a_c Z^2/ $A^{1/3}$ favors drops made only of neutrons, therefore an extra term has to be included to reproduce the experimental line of stability: the symmetry term which favors nuclei with N=Z; - a_{sym} $(N\text{-}Z)^2$/ A
- Even with this addition the LDM cannot explain the fact that there is an anomalously large fraction of even-N even-Z nuclei among the stable ones and only a few odd-odd. This requires a new ad hoc addition; the pairing term which is clearly beyond the liquid drop picture

And its limitations

- Item more, when the neutron and proton separation energies are examined, it turns out that they show peaks at very precise numbers of neutrons and protons, reminiscent of the ones found in the ionization potentials of the noble gases. This big surprise gained to these numbers the label ”magic numbers”, not a very scientific one indeed!
- In order to explain the magic numbers, the IPM (or naive shell model) of the nucleus was postulated, and the dichotomy LDM/IPM still survives in many textbooks and in common knowledge

IPM *vs* LDM

- Nuclei with proton or neutron numbers equal or very close to the magic numbers are treated by the IPM, whereas global properties and collective phenomena call for liquid drop like (quantized) excitations, or non-spherical rotating drops: All in all, the Nuclear Structure turned into Nuclear Schizophrenia
- We shall see that there is a cure; an unified view of the independent particle and the collective excitations of the nucleus based in, but going well beyond, the IPM. But this will come later, for the moment let's make an inventory of nuclear observables and recall the basic elements of the IPM

More on Experimental Data

- Nuclei are quantal objets which have discrete energy levels characterized by their total angular momentum J their parity and their isospin T. This last quantity is not an exact quantum number due to the Coulomb interaction among the protons and to the charge dependent terms of the nuclear interaction. But, only in rare cases the isospin mixing is non negligible
- Each state has a well defined excitation energy and magnetic and electric moments. It may also have a size or density distribution different from that of the ground state
- Excited states may decay by coupling to the electromagnetic field, emitting photons of different multipolarities, hence they have an associated half life and different branching ratios to different final states

More on Experimental Data

- The nuclear states couple also to the weak field and may β-decay to a more bound isobar with one more/less unit of charge. This is the most frequent decay mechanism for nuclei in their ground states, albeit they may also decay by α or proton emission. All these decays are characterized by their half-lives and branching ratios. Excited states can have even more decay modes as for instance one and two neutron emission.
- Nuclei may have resonant excitations in the continuum associated to different operators, they are dubbed "giant resonances" and are characterized by their transition strengths, their excitation energies and their widths.

More on Experimental Data

- Different nuclear reactions provide access to these resonances and to a lot of complementary information, like the spectroscopic factors
- Nuclear effective theories and/or models should be able to explain quantitatively this large body of experimental data and to predict the nuclear behavior in regions unexplored experimentally yet

The Independent Particle Model

The basic idea of the IPM is to assume that, at zeroth order, the result of the complicated two body interactions among the nucleons is to produce an average self-binding potential. Mayer and Jensen (1949) proposed an spherical mean field consisting in an isotropic harmonic oscillator plus a strongly attractive spin-orbit potential and an orbit-orbit term.

$$H = \sum_i h(\vec{r}_i)$$

$$h(r) = -V_0 + t + \frac{1}{2}m\omega^2 r^2 - V_{so}\vec{l} \cdot \vec{s} - V_B l^2$$

The Independent Particle Model

Later, other functional forms , which follow better the form of the nuclear density and have a more realistic asymptotic behavior, *e.g.* the Woods-Saxon well, were adopted

$$V(r) = V_0 \left(1 + e^{\frac{r-R}{a}}\right)^{-1}$$

with

$$V_0 = \left(-51 + 33\frac{N-Z}{A}\right) MeV$$

and

$$V_{ls}(r) = \frac{V_0^{ls}}{V_0}(\vec{l}\cdot\vec{s})\frac{r_0^2}{r}\frac{dV(r)}{dr} \; ; \; V_0^{ls} = -0.44V_0$$

The Independent Particle Model

The eigenvectors of the IPM ($h\phi_{nljm} = \epsilon_{nlj}\phi_{nljm}$) are characterized by the radial quantum number n, the orbital angular momentum l, the total angular momentum j and its Z projection m. With the choice of the harmonic oscillator, the eigenvalues are:

$$\epsilon_{nlj} = -V_0 + \hbar\omega(2n + l + 3/2)$$

$$-V_{so}\frac{\hbar^2}{2}(j(j+1) - l(l+1) - 3/4) - V_B\hbar^2 l(l+1)$$

In order to reproduce the nuclear radii,

$$\hbar\omega = 45A^{-1/3} - 25A^{-2/3}$$

we shall denote (2n+l) by p, the principal quantum number of the oscillator.

Vocabulary

- STATE: a solution of the Schrödinger equation with a one body potential; e.g. the H.O. or the W.S. It is characterized by the quantum numbers *nljm* and the projection of the isospin t_z
- ORBIT: the ensemble of states with the same *nlj*, e.g. the 0d5/2 orbit. Its degeneracy is (2j+1)
- SHELL: an ensemble of orbits quasi-degenerated in energy, e.g. the *pf* shell
- MAGIC NUMBERS: the numbers of protons or neutrons that fill orderly a certain number of shells
- GAP: the energy difference between two shells
- SPE, single particle energies, the eigenvalues of the IPM hamiltonian

The wave function of the nucleus in the IPM

- The WF of the ground state of a nucleus (N, Z) is the product of one Slater determinant for the protons and another for the neutrons, built with the N/Z states ϕ_{nljm} of lower energy
- Except if N and Z are such that they correspond to the complete filling of a set of orbits, the solution is not unique. If we have one particle in excess or in defect, this is not a problem because of the magnetic degeneracy. In all the remaining cases the many body solutions of the IPM do not have a well defined total angular momentum J, as they should due to the rotation invariance of the Hamiltonian.

The wave function of the nucleus in the IPM plus schematic pairing

- Thus, already at this stage, it is necessary to incorporate dynamical effects that go beyond the spherical mean field obtain physically sound solutions. The minimal choice is to assume that pairs of identical particles on top of a filled orbit are always coupled to total angular momentum zero, due to the strong residual two body pairing interaction
- Lets work out the case of the Calcium isotopes as a textbook example

The IPM description of the Calcium isotopes

- ^{40}Ca is doubly magic. All the orbits of the p=1, 2, and 3 HO shells are filled for neutrons and protons. Therefore the WF of its ground state is a single Slater determinant and ”a fortiori” has J^{π}=0^{+} a fact borne out by experiment. A nice, if trivial, triumph of the IPM.
- The next IPM orbit is the $0f_{7/2}$ followed by $1p_{3/2}$: if we add a neutron, we have several candidates for the GS, (j=7/2, m), but all of them are degenerate in energy, what makes the choice of m irrelevant. Definitely the IPM prediction for the GS of ^{41}Ca is J^{π}=$7/2^{-}$, and, trivially its first excited state has J^{π}=$3/2^{-}$. A new success of the IPM.

The IPM description of the Calcium isotopes

- Let's move to ^{42}Ca. Now we have more choices; (j=7/2, m), (j=7/2, m'). This gives 28 combinations which correspond to the values of J allowed by the Pauli principle $J^{\pi}=0^{+}$, 2^{+}, 4^{+} and 6^{+} with M degeneracies 2J+1, 1+5+9+13=28. At the spherical mean field level all have the same energy.
- What one should do now is to compute the expectation value of the residual interaction in these states, to break the degeneracy. And indeed, the effective residual neutron neutron interaction privileges the 0^{+} over the other couplings. Again this is what the experiments tell us.
- If we disregard the other possible couplings, the GS of ^{43}Ca would be $J^{\pi}=7/2^{-}$, as it is. We can continue applying the same recipe as far as we want in neutron number. What will be your the prediction for ^{57}Ca?

The IPM description of other observables

- Within the IPM some properties of the nucleus stem just from those of the odd nucleon alone, for instance their ground state magnetic moments
- It is also useful to define the single particle limit of the γ and β decay transition probabilities. In the former case these are called Weisskopft units. Transitions which carry many WU's indicate the onset of collectivity.

	λ=1	λ=2	λ=3	λ=4
E	$1. \times 10^{14} A^{2/3} E^3$	$7.3 \times 10^7 A^{4/3} E^5$	$34. \times A^2 E^7$	$1.1 \times 10^{-5} A^{8/3} E^9$
M	$5.6 \times 10^{13} E^3$	$3.5 \times 10^7 A^{2/3} E^5$	$16 \times A^{4/3} E^7$	$4.5 \times 10^{-6} A^2 E^9$

(energies in MeV)

- Allowed and super allowed β decays have reduced transition probabilities O(1) corresponding to log ft values 3-5

The IPM supersedes the LDM

- The IPM explains the magic numbers, the spins and parities of the ground states and some excited states of doubly magic nuclei plus or minus one nucleon, their magnetic moments, etc. As we have just seen, with the addition of an schematic pairing tern it can go a bit further in semi-magic nuclei (Schmidt lines).
- What is less well known is that in the large A limit, the IPM can reproduce the volume, the surface and the symmetry terms of the semi-empirical mass formula as well.

The IPM and the semi-empirical mass formula

- Let's take the IPM with an HO potential and neglect the spin orbit term. Then:

$$H = \sum_i t_i - V_0 + \frac{1}{2} m\omega^2 r_i^2$$

- the single particle energies are: $\epsilon_i = -V_0 + \hbar\omega(p_i + 3/2)$
- and $< r_i^2 >= b^2(p_i + 3/2)$ with $b^2 = \dfrac{\hbar}{m\omega}$
- The degeneracy of each shell is d=(p+1)(p+2) for protons and for neutrons

The IPM and the semi-empirical mass formula

- Assume N=Z. To accommodate A/2 identical particles we need to fill the shells up to p=p_F
- Experimentally, the radius of the nucleus is given by $< r^2 >= \frac{3}{5}R^2 = \frac{3}{5}(1.2A^{1/3})^2$
- And in the IPM by:

$$< r^2 >= \sum_i^{A/2} < r_i^2 > \frac{2}{A} = \sum_{p=0}^{p_F} b^2(p+3/2)(p+1)(p+2)$$

From

$$\frac{A}{2} = \sum^{p_F}(p+1)(p+2)$$

it obtains at leading order, $p_F = (\frac{3}{2}A)^{3/2}$

The IPM and the semi-empirical mass formula

- Putting everything together we find, at leading order in p_F, $b^2 = A^{1/3}$ and $\hbar\omega = 41 \cdot A^{-1/3}$
- We can now compute the total binding energy as:

$$B = \sum_{i=1}^{A} (-V_0 + \hbar\omega(p_i + 3/2))$$

- that gives at leading order

$$\frac{B}{A} + V_0 = \hbar\omega \cdot \frac{p_F^4}{4} \cdot \frac{2}{A} = \hbar\omega(3/2A)^{4/3} \frac{1}{2A} = \hbar\omega A^{1/3}$$

Finally we have $\frac{B}{A} = -V_0 + 41$ and we recover the volume term of the semi empirical mass formula for $V_0 \sim 60$ MeV

The IPM and the semi-empirical mass formula

- If we go to next to leading order, keeping the terms in p_F^3, we recover the surface term with the correct coefficient
- We can repeat the calculation at leading order but with N$\neq$Z, and obtain

$$B = -AV_0 + \frac{\hbar\omega}{4}((p_F^\nu)^4 + (p_F^\pi)^4) = -AV_0 + \frac{\hbar\omega}{4}((3N)^{4/3} + (3Z)^{4/3})$$

- Making a Taylor expansion around the minimum at N=Z and using the previously determined values we find an extra term of the form $(N\text{-}Z)^2/A$ with a coefficient a_{sym}=16 MeV.

- This coefficient is roughly one half of the one resulting from the fit of the semi empirical mass formula to the experimental binding energies. The reduced amount of symmetry energy which we get reflects the fact that the nuclear two body neutron-proton interaction is in average more attractive than the neutron-neutron and the proton proton ones.
- To account for that we have an experimental anchor: the evidence that for N$\neq$Z the neutron and proton radii are roughly equal. Therefore we should use different values of $\hbar\omega$ for protons and neutrons in this derivation. This complicates a bit the calculation but I invite you to verify that it solves the problem

The limits of the IPM

- When a nucleus is such that it has both neutrons and protons outside closed shells, the IPM fails completely
- This is mainly due to the very strong residual interaction between neutrons and protons
- Dominated by its quadrupole quadrupole components
- Which may favor energetically that the nucleus acquire a permanent deformation and exhibit rotational spectra. This is a case of spontaneous symmetry breaking.
- In other cases collective states of vibrational type may also develop