The Shell Model: An Unified Description of the Structure of the Nucleus (IV)

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Outline

- The Nilsson model
- Quadrupole Collectivity; SU3 and its variants
- Applications: ^{40}Ca case
- Other examples of LSSM calculations: The N=20, 28, 40 islands of inversion

Deformed nuclei; The Nilsson model

The Nilsson model is an approximation to the solution of the IPM plus a quadrupole-quadrupole interaction.

$$H = \sum_i h(\vec{r}_i) + \hbar\omega\kappa \sum_{i<j} Q_i \cdot Q_j$$

$$h(r) = -V_0 + t + \frac{1}{2} m\omega^2 r^2 - V_{so} \vec{l} \cdot \vec{s} - V_B l^2$$

Which amounts to linearizing the quadrupole quadrupole interaction, replacing one of the operators by the expectation value of the quadrupole moment (or by the deformation parameter).

Deformed nuclei; The Nilsson model

Thus, the resulting physical problem is that of the IPM subject to a quadrupole field, which, obviously breaks rotational symmetry.

$$H_{Nilsson} = \sum_i h(\vec{r}_i) - \frac{1}{3}\hbar\omega\delta Q_0(i)$$

Which is just the diagonalization of the quadrupole operator in the basis of the IPM eigenstates. The resulting (Nilsson) levels are characterized by their magnetic projection on the symmetry axis m, also denoted K and the parity.

Deformed nuclei; The Nilsson model

The formulae below make it possible to build the relevant matrices.

$$\langle pl|r^2|pl\rangle = p + 3/2 \quad : \quad \langle pl|r^2|pl+2\rangle = -[(p-l)(p+l+3)]^{1/2}$$

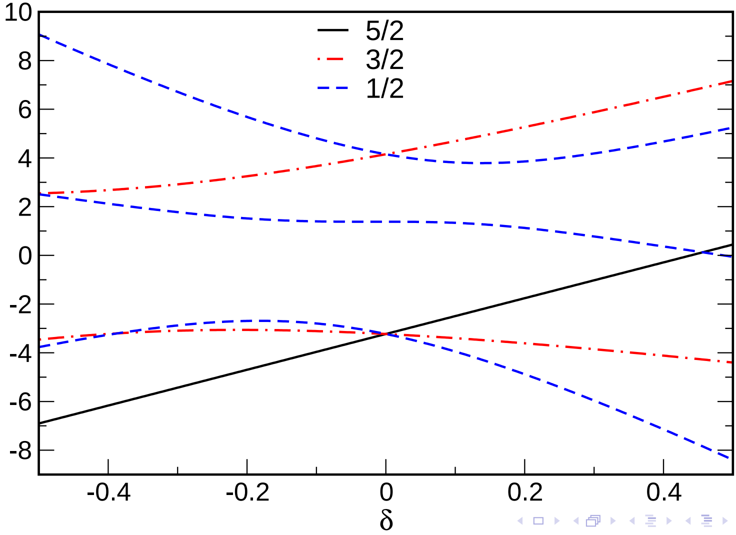
$$Q_0 = 2r^2C_2 = 2r^2\sqrt{4\pi/(2l+1)}\,Y^{20} \quad : \quad \langle jm|C_2|jm\rangle = \frac{j(j+1)-3m^2}{2j(2j+2)}$$

$$\langle jm|C_2|j+2m\rangle = \frac{3}{2}\left\{\frac{[(j+2)^2-m^2][(j+1)^2-m^2]}{(2j+2)^2(2j+4)^2}\right\}^{1/2}$$

$$\langle jm|C_2|j+1m\rangle = -\frac{3m[(j+1)^2-m^2]^{1/2}}{j(2j+4)(2j+2)}$$

Deformed nuclei; Nilsson Diagrams

Diagramas de Nilsson para la capa p=2



Intrinsic and Laboratory frame wave functions

The intrinsic wave functions provided by the Nilsson model correspond to the Slater determinants built putting the neutrons and the protons in the lowest Nilsson levels (each one has degeneracy two, $\pm$m). Therefore, for even even nuclei K=0, for odd nuclei K=m of the last half occupied orbit, and for odd-odd, there are different empirical rules, not always very reliable.
The laboratory frame wave functions are obtained rotating the intrinsic frame with the Wigner matrices, i.e. correspond to the solutions of the rigid rotor problem. In the even-even case this leads trivially to the energy formula for a rotor:

$$E(J) = \sum_i \epsilon_i(Nilsson) + \frac{\hbar^2}{2\mathcal{I}} J(J+1)$$

The SU3 symmetry of the harmonic oscillator

The mechanism that produces permanent deformation and rotational spectra in nuclei is much better understood in the framework of the SU(3) symmetry of the isotropic harmonic oscillator spherical and its implementation in Elliott's model. The basic simplification of the model is threefold; i) the valence space is limited to one major HO shell; ii) the monopole hamiltonian makes the orbits of this shell degenerate and iii) the multipole hamiltonian only contains the quadrupole-quadrupole interaction. This implies (mainly) that the spin orbit splitting and the pairing interaction are put to zero. Let's then start with the spherical HO which in units m=1 ω=1 can be written as:

$$H_0 = \frac{1}{2}(p^2 + r^2) = \frac{1}{2}(\vec{p} + i\vec{r})(\vec{p} - i\vec{r}) + \frac{3}{2}\hbar = \hbar(\vec{A}^{\dagger}\vec{A} + \frac{3}{2})$$

The SU3 symmetry of the harmonic oscillator

$$\vec{A}^{\dagger} = \frac{1}{\sqrt{2\hbar}}(\vec{p} + i\vec{r}) \;\; \vec{A} = \frac{1}{\sqrt{2\hbar}}(\vec{p} - i\vec{r})$$

which have bosonic commutation relations. H_0 is invariant under all the transformations which leave invariant the scalar product $\vec{A}^{\dagger}\vec{A}$. As the vectors are three dimensional and complex, the symmetry group is U(3). We can built the generators of U(3) as bi-linear operators in the A's. The anti-symmetric combinations produce the three components of the orbital angular momentum L_x, L_y and L_z, which are on turn the generators of the rotation group O(3). From the six symmetric bi-linears we can retire the trace that is a constant; the mean field energy. Taking it out we move into the group SU(3). The five remaining generators are the five components of the quadrupole operator:

The SU3 symmetry of the harmonic oscillator

$$q^{(2)}_{\mu} = \frac{\sqrt{6}}{2\hbar}(\vec{r} \wedge \vec{r})^{(2)}_{\mu} + \frac{\sqrt{6}}{2\hbar}(\vec{p} \wedge \vec{p})^{(2)}_{\mu}$$

The generators of SU(3) transform single nucleon wavefunctions of a given **p** (principal quantum number) into themselves. In a single nucleon state there are **p** oscillator quanta which behave as l=1 bosons. When we have several particles we need to construct the *irreps* of SU(3) which are characterized by the Young's tableaux (n_1, n_2, n_3) with $n_1 \geq n_2 \geq n_3$ and $n_1+n_2+n_3=N$**p** (N being the number of particles in the open shell). The states of one particle in the **p** shell correspond to the representation (p,0,0). Given the constancy of N**p** the *irreps* can be labeled with only two numbers. Elliott's choice was $\lambda=n_1-n_3$ and $\mu=n_2-n_3$. In the cartesian basis we have; $n_x=a+\mu$, $n_y=a$, and $n_z=a+\lambda+\mu$, with $3a+\lambda+2\mu=N$**p**.

The SU3 symmetry of the harmonic oscillator

The quadratic Casimir operator of SU(3) is built from the generators

$$\vec{L} = \sum_{i=1}^{N} \vec{l}(i) \qquad Q^{(2)}_{\alpha} = \sum_{i=1}^{N} q^{(2)}_{\alpha}(i)$$

as:

$$C_{SU(3)} = \frac{3}{4}(\vec{L} \cdot \vec{L}) + \frac{1}{4}(Q^{(2)} \cdot Q^{(2)})$$

and commutes with them. With the usual group theoretical techniques, it can be shown that the eigenvalues of the Casimir operator in a given representation (λ, μ) are:

$$C(\lambda, \mu) = \lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)$$

Elliott's Model

Once these tools ready we come back to the physics problem as posed by Elliott's hamiltonian

$$H = H_0 + \chi(Q^{(2)} \cdot Q^{(2)})$$

which can be rewritten as:

$$H = H_0 + 4\chi C_{SU(3)} - 3\chi(\vec{L} \cdot \vec{L})$$

The eigenvectors of this problem are thus characterized by the quantum numbers λ, μ, and L. We can choose to label our states with these quantum numbers because O(3) is a subgroup of SU(3) and therefore the problem has an analytical solution:

Elliott's Model

$$E(\lambda,\mu,L)=\hbar\omega(p+\frac{3}{2})+4\chi(\lambda^2+\lambda\mu+\mu^2+3(\lambda+\mu))-3\chi L(L+1)$$

This final result can be interpreted as follows: For an attractive quadrupole quadrupole interaction ($\chi<0$) the ground state of the problem pertains to the representation which maximizes the value of the Casimir operator, and this corresponds to maximizing λ or μ (the choice is arbitrary). If we look at that in the cartesian basis, this state is the one which has the maximum number of oscillator quanta in the Z-direction, thus breaking the symmetry at the intrinsic level. We can then speak of a deformed solution even if its wave function conserves the good quantum numbers of the rotation group, i.e. L and L_z.

Elliott's Model

$$E(\lambda, \mu, L) = \hbar\omega(p + \frac{3}{2}) + 4\chi(\lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)) - 3\chi L(L+1)$$

For this one (and for every) (λ, μ) representation, there are different values of L which are permitted, for instance for the representation $(\lambda, 0)$ L=0,2,4$\ldots\lambda$. And their energies satisfy the L(L+1) law, thus giving the spectrum of a rigid rotor. The problem of the description of deformed nuclear rotors is thus formally solved.

^{24}Mg, experimental data

E_{level}#	J^{π}	$T_{1/2}$	XREF	Commen
0.0	0+	stable	ABCDEFGHIJ KLMNOPQR	
1368.672 *5*	2+	1.33 ps *6*	A CDE GHIJ KLMNO Q	μ=+1.02 *4*; Q=-0.166 *6* **J^{π}:** E2 to 0+.
4122.889 *12*	4+	22 fs *2*	A C E HIJ KLM P R	μ=+1.6 *12* **J^{π}:** E2 to 2+, L=4 in (p,p'
4238.24 *3*	2+	41 fs *4*	A CDE HIJ KLM OP R	μ=+1.2 *4* **J^{π}:** E2 to 0+.
5235.12 *4*	3+	61 fs *7*	A C HIJ KLM R	**J^{π}:** M1+E2 to 2+, L=3 in
6010.84 *4*	4+	49 fs *5*	C HIJ KLM OP R	μ=+2.0 *16* **J^{π}:** γ's to 0+ and 4+.
6432.30 *11*	0+	53 fs *8*	HIJ K O R	**J^{π}:** L=0 in (p,p').
7349.00 *3*	2+	6 fs *2*	C HIJ KLM O	**J^{π}:** E2 to 0+.
7555.04 *15*	1-	270 fs *55*	HIJ KLM	**J^{π}:** E1 to 0+.
7616.47 *4*	3-	890 fs *140*	C HIJ KL O	**J^{π}:** L=3 in (p,p'), (α,γ).
7747.51 *9*	1+	10 fs *3*	HIJ L	**J^{π}:** M1 to 0+.
7812.35 *11*	5+	20 fs *4*	C HIJ M R	**J^{π}:** From (α,γ).
8114.2 *20*	6+	3.6 fs *10*	HI M R	**J^{π}:** L=6 in (p,p').
8357.98 *13*	3-	56 fs *8*	HIJ LM O R	**J^{π}:** L=0 in (p,p').
8437.31 *15*	1-	10 fs *2*	HIJ	**J^{π}:** E1 to 0+.
8439.36 *4*	4+	3.8 fs *11*	C LM	**J^{π}:** log ft=3.93 from 4+, (
8654.53 *15*	2+	8.2 fs *21*	HIJ LM R	**J^{π}:** L=2 in (p,p'), γ to 0+.
8864.29 *9*	2-	4.4 fs *15*	HIJ LM	**J^{π}:** L=1 in (^{3}He,d) and (^{3}
9003.34 *9*	2+	7.6 fs *14*	HIJ M O	**J^{π}:** γ's to 0+ and 4+, L=2
9145.99 *15*	1-		HI	**J^{π}:** L=1 in (p,p').

Intrinsic States

We can describe the intrinsic states and its relationship with the physical ones using another chain of subgroups of SU(3). The one we have used until now is; SU(3)$\supset$O(3)$\supset$U(1) which corresponds to labeling the states as $\Psi([\tilde{f}](\lambda\mu)LM)$. $[\tilde{f}]$ is the representation of U(Ω) conjugate of the U(4) spin-isospin representation which guarantees the antisymmetry of the total wave function. For instance, in the case of ^{20}Ne, the fundamental representation (8,0) (four particles in p=2) is fully symmetric, $[\tilde{f}]$=[4], and its conjugate representation in the U(4) of Wigner $[1, 1, 1, 1]$, fully antisymmetric.

Intrinsic States

The other chain of subgroups, SU(3)⊃SU(2)⊃U(1), does not contain O(3) and therefore the total orbital angular momentum is not a good quantum number anymore. Instead we label the wave functions as; $\Phi([\tilde{f}](\lambda\mu)q_0\Lambda K)$, where q_0 is a quadrupole moment whose maximum value is $q_0 = 2\lambda + \mu$ related to the intrinsic quadrupole moment, Q_0=q_0+3. K is the projection of the angular momentum on the Z-axis and Λ is an angular momentum without physical meaning. Both representations provide a complete basis, therefore it is possible to write the physical states in the basis of the intrinsic ones.

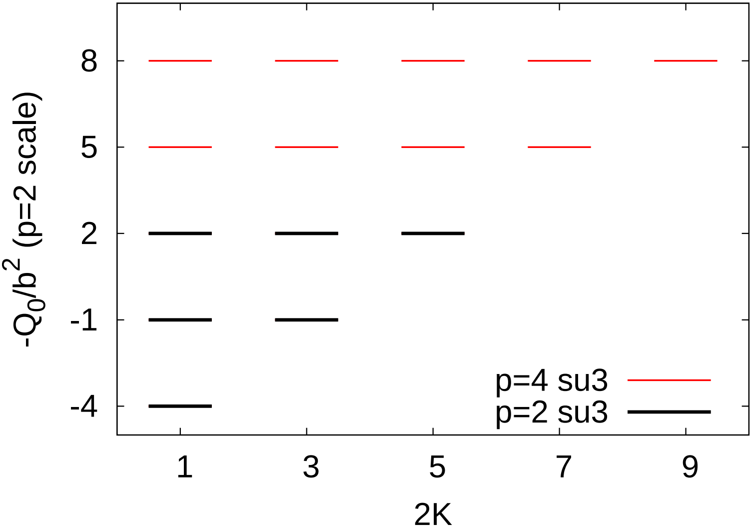
Intrinsic States

Actually, the physical states can be projected out of the intrinsic states with maximum quadrupole moment as:

$$\Psi([\tilde{f}](\lambda\mu)LM) = \frac{2L+1}{a(\lambda\mu KL)} \int D^{L}_{MK}(\omega)\Phi_{\omega}([\tilde{f}](\lambda\mu)(q_0)_{max}\Lambda K)d\omega$$

Remarkably, this is the same kind of expression used in the unified model; the Wigner functions D being the eigenfunctions of the rigid rotor and the intrinsic functions the solutions of the Nilsson model.

SU3 intrinsic states



Elliott's Model

Elliott's model was initially applied to nuclei belonging to the *sd*-shell that show rotational features like ^{20}Ne and ^{24}Mg. The fundamental representation for ^{20}Ne is (8,0) and its intrinsic quadrupole moment 19 b$^2 \approx 60$ efm^2. For ^{24}Mg we have (8,4) and 23 b$^2 \approx 70$ efm^2. To compare these figures with the experimental values we need to know the transformation rules from intrinsic to laboratory frame quantities and vice versa. In the Bohr Mottelson model these are:

$$Q_0(s) = \frac{(J+1)\,(2J+3)}{3K^2 - J(J+1)}\,Q_{spec}(J), \quad K \neq 1$$

$$B(E2, J \rightarrow J{-}2) = \frac{5}{16\pi}\, e^2 |\langle JK20|J{-}2, K\rangle|^2\, Q_0(t)^2 \quad K \neq 1/2,\, 1;$$

Elliott's Model

The expression for the quadrupole moments is also valid in the Elliott's model. However the one for the B(E2)'s is only approximately valid for very low spins. Using them it can be easily verified that the SU(3) predictions compare nicely with the experimental results

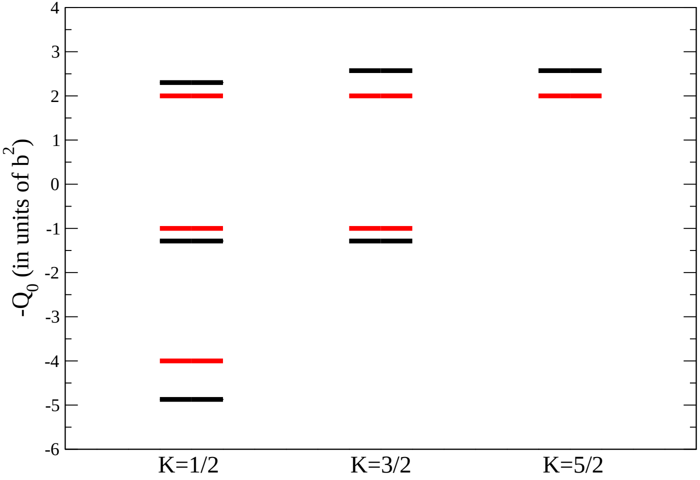
$Q_{spec}(2^+)$=–23(3) efm^2 and B(E2)($2^+ \rightarrow 0^+$)=66(3) e^{2}fm^4 for ^{20}Ne

$Q_{spec}(2^+)$=–17(1) efm^2 and B(E2)($2^+ \rightarrow 0^+$)=70(3) e^{2}fm^4 for ^{24}Mg.

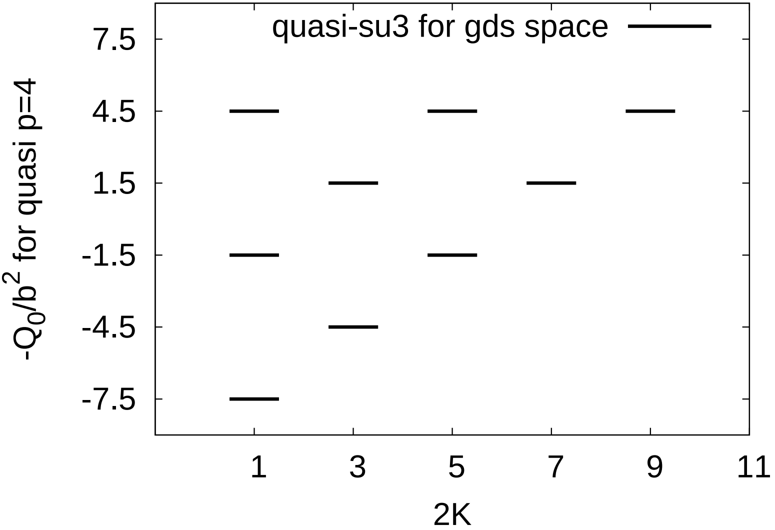
SU3 variants, Pseudo and Quasi-SU3

Besides Elliott's SU(3) there are other approximate symmetries related to the quadrupole quadrupole interaction which are of great interest. Pseudo-SU3 applies when the valence space consists of a quasi-degenerate harmonic oscillator shell except for the orbit with maximum j, we had denoted this space by r_p before. Its quadrupole properties are close to those of SU(3) in the shell with (p-1). Quasi-SU3 applies in a regime of large spin orbit splitting, when the valence space contains the intruder orbit and the Δj=2, Δl=2 ;Δj=4, Δl=4; etc, orbits obtained from it. Its quadrupole properties are similar to those of SU3 as well.

Pseudo-SU3 intrinsic states



Quasi-SU3 intrinsic states



Coexistence: Spherical, Deformed and Superdeformed states in ^{40}Ca

In the valence space of two major shells

$0f_{5/2}$

$1p_{1/2}$

$1p_{3/2}$

$0f_{7/2}$

pf-shell

$0d_{3/2}$

$1s_{1/2}$

$0d_{5/2}$

sd-shell

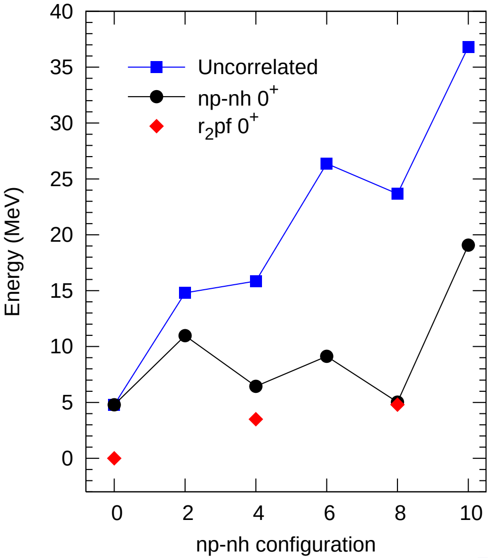
The relevant configurations are:

$[sd]^{24}$ 0p-0h in ^{40}Ca, SPHERICAL

$[sd]^{20}$ $[pf]^{4}$ 4p-4h in ^{40}Ca, DEFORMED

$[sd]^{16}$ $[pf]^{8}$ 8p-8h in ^{40}Ca, SUPERDEFORMED

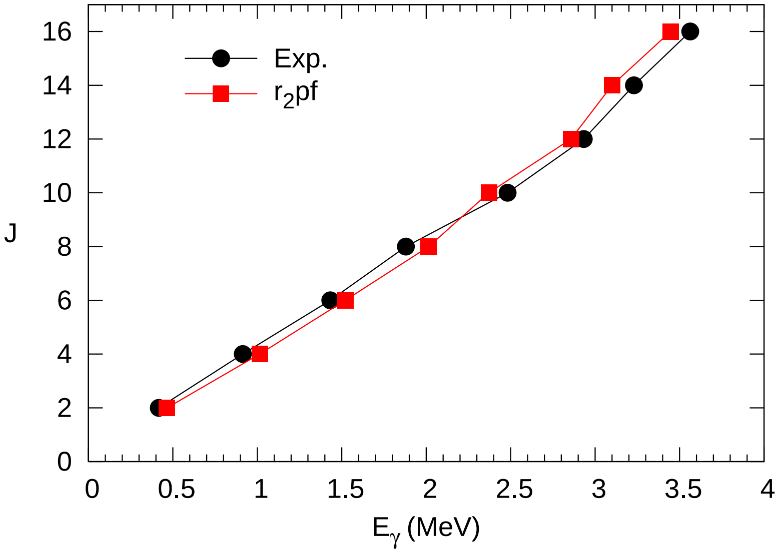
The correlation energies



The correlation energies

- In the 8p-8h configuration the correlations amount to 18.5 MeV. 5.5 MeV are due to T=1 pairing and 0.5 MeV to T=0 pairing, thus the neutron-proton pairing contribution is 2.33 MeV. The remaining 12.5 MeV are most likely of quadrupole origin.
- In the 4p-4h configuration, the pairing contributions are the same, but the quadrupole is just 3.5 MeV.
- The physical gound state gains 5 MeV of pairing energy by mixing with the other np-nh states, the ND bandhead 2 MeV, and the SD bandhead nothing

The Superdeformed band



SU(3) predictions

In the 4p-4h intrinsic state of ^{40}Ca, the two neutrons and two protons in the *pf*-shell can be placed in the lowest K=1/2 quasi-SU3 level of the p=3 shell. This gives a contribution Q_0=25 b^2. In the pseudo-sd shell, p=1 we are left with eight particles, that contribute with Q_0=7 b^2. In the 8p-8h the values are Q_0=35 b^2 and Q_0=11 b^2

Using the proper values of the oscillator length it obtains:

^{40}Ca 4p-4h band Q_0=125 e fm^2 (Q_0=148 e fm^2)

^{40}Ca 8p-8h band Q_0=180 e fm^2 (Q_0=226 e fm^2)

In very good accord with the data (Q_0=120 e fm^2 and Q_0=180 e fm^2). The values in blue assume strict SU3 symmetry in both shells. The SM results almost saturate the quasi-SU3 bounds. The SU3 values are a 25% larger.

Comparing with experiment