THE C-MAP
BEYOND THE CLASSICAL LEVEL

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In supergravity one may encounter surprising phenomena without an obvious explanation, whose underlying origin can be understood in the context of string theory. One of them concerns the so-called c-map (c stands for Calabi) according to which certain 3D N=2 supergravity systems can be uplifted to 4D in two inequivalent ways.

Cecotti, Ferrara, Girardello, 1989

In string theory this phenomenon can be explained by noting that 10D string theory compactified on a circle has two different inequivalent decompactification limits corresponding to either the momentum modes or to the winding modes becoming massless. Upon a (supersymmetric) compactification on a CY manifold times a circle one thus recovers the IIA and IIB decompactification limits in the uplift to four dimensions!

This is therefore a manifestation of ‘string-string duality’ directly in supergravity.

The c-map has been studied extensively, but mainly for Lagrangians quadratic in derivatives. The map then acts between special-Kähler and (restricted) quaternion-Kähler manifolds, the target spaces of vector multiplets and hypermultiplets.

Cecotti, Ferrara, Girardello, 1989
Ferrara, Sabharwal, 1990
dW, Vanderseypen, Van Proeyen, 1992

It is usually defined as a map between vector multiplets and hypermultiplets. However, off-shell it is best defined as a map between vector and tensor multiplets. The latter can be dualized to hypers. *dW, Saueressig, 2006*

There have been few systematic attempts to explore the c-map beyond the classical level by considering effective actions with higher-derivative terms.

Antoniadis, Gava, Narain, Taylor, 1994
Berkovits, Siegel, 1996
Roček, Vafa, Vandoren, 2006

In order to study/classify higher-derivative couplings it is important to use off-shell methods and refrain from imposing gauge conditions and eliminating auxiliary fields.

To study the c-map for higher derivative interactions in a systematic fashion one needs a dictionary between 4D and 3D supermultiplets. This dictionary should be off-shell, so that it is applicable to any 4D/3D supersymmetric Lagrangian.

As we know from classical Kaluza-Klein theory, the dictionary must satisfy obvious covariance properties, as otherwise the resulting lower-dimensional theory will not exhibit any recognizable structure and will be useless in practical applications.

For such a covariant dictionary it is important to satisfy the following additional properties:
- infinitesimal transformations should form a closed algebra, independent of the equations of motion
- the R-symmetry should be realized fully (and in supergravity locally)
- preferably the multiplets should have a high degree of irreducibility and gauge choices must be avoided.

Note: lower-dimensional multiplets are in general smaller. For instance, a *D*-dimensional gauge field decomposes into a (*D-1*)-dimensional gauge field and a scalar. This phenomenon can also happen for supermultiplets.

4 → 3 Off-shell dimensional reduction

- *Kaluza-Klein Ansätze*: the fields should transform consistently under 3-dimensional diffeomorphisms.
- *R-symmetry basis of the spinor fields*: spinors in 4 dimensions transforming under a related R-symmetry, transform as a spinor in 3 dimensions with a different R-symmetry. The total number of spinor components remains of course the same.

The dimension of the Lorentz group decreases and of the R-symmetry group increases under dimensional reduction. However, the resulting R-symmetry group is usually not fully realized locally, but is obtained in a (partially) gauge-fixed form. The missing gauge degrees of freedom can be provided by appropriate local ‘phase factors’.

			phase factor
	$D=5$	$\mathrm{Spin}(4,1)\times\mathrm{SU}(2)_\mathrm{V}$	$\mathbf{1}$
	$D=4$	$\mathrm{Spin}(3,1)\times\mathrm{SU}(2)_\mathrm{V}\times\mathrm{U}(1)_\mathrm{A}$	$\mathrm{U}(1)$
Euclidean →		$\mathrm{Spin}(4)\times\mathrm{SU}(2)_\mathrm{V}\times\mathrm{SO}(1,1)$	$\mathrm{SO}(1,1)$
	$D=3$	$\mathrm{Spin}(2,1)\times\mathrm{SU}(2)_\mathrm{V}\times\mathrm{SU}(2)_\mathrm{A}$	$\mathrm{SU}(2)/\mathrm{U}(1)$

Banerjee, dW, Katmadas, Reys, 2012, 2016

Weyl multiplet and KK vector multiplet:

First Kaluza-Klein ansätze with gauge choices, e.g.

$$e_M{}^A = \begin{pmatrix} e_\mu{}^a & B_\mu \phi^{-1} \\ 0 & \phi^{-1} \end{pmatrix}, \qquad e_A{}^M = \begin{pmatrix} e_a{}^\mu & -e_a{}^\nu B_\nu \\ 0 & \phi \end{pmatrix}, \qquad b_M = \begin{pmatrix} b_\mu \\ 0 \end{pmatrix}$$

Include compensating gauge transformations to preserve this form.
There is no conflict between conformal invariance and dimensional reduction.

Decompose the 4D supersymmetry transformations:

$$\boxed{\delta_{\rm Q}(\epsilon)\big|_{4D}^{\rm reduced}\Psi = \delta_{\rm Q}(\epsilon)\big|_{3D}\Psi + \delta_{\rm S}(\tilde\eta)\big|_{3D}\Psi + \delta_{\rm SU(2)}(\tilde\Lambda)\big|_{3D}\Psi + \delta_{\rm SU(2)/U(1)}(\tilde\Sigma)\big|_{3D}\Psi}$$

local parameters with extra field-dependent terms

not a symmetry but a compensating R-symmetry transformation

By redefining the fields with a uniform SU(2)/U(1) phase factor depending on an extra complex field υ^0, one can extend the U(1) R-symmetry to SU(2) and decouple the KK vector multiplet from the Weyl multiplet.

This SU(2)/U(1) phase factor takes the following form

$$\Phi^p{}_q(x^0,\upsilon^0,\bar\upsilon^0) = \frac{1}{\sqrt{2\,L^0\,\big(L^0+\frac12 x^0\big)}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Lambda_\mathrm{A}/2}\,\big(L^0+\frac12 x^0\big) & -\mathrm{e}^{\mathrm{i}\Lambda_\mathrm{A}/2}\,\mathrm{i}\upsilon^0 \\ -\mathrm{e}^{-\mathrm{i}\Lambda_\mathrm{A}/2}\,\mathrm{i}\bar\upsilon^0 & \mathrm{e}^{\mathrm{i}\Lambda_\mathrm{A}/2}\,\big(L^0+\frac12 x^0\big) \end{pmatrix}$$

(Λ_A is an arbitrary function associated with the U(1)).

The field ϕ has now been extended to a triplet $L^p{}_q{}^0 = \Phi\begin{pmatrix} -\mathrm{i}\phi & 0 \\ 0 & \mathrm{i}\phi \end{pmatrix}\Phi^{-1}$

$$L^p{}_q{}^0(x,\upsilon,\bar\upsilon) = \begin{pmatrix} -\frac12\,\mathrm{i}\,x^0 & \upsilon^0 \\ -\bar\upsilon^0 & \frac12\,\mathrm{i}\,x^0 \end{pmatrix}$$ of length $\phi = L^0 = \sqrt{|\upsilon^0|^2+\frac14 (x^0)^2}$

Upon reduction the *4D* Weyl multiplet now decomposes into the *3D* Weyl multiplet and a Kaluza-Klein vector multiplet.

The full dictionary now follows from matching the supersymmetry transformations.

This leads to the covariant 4D/3D dictionary

Banerjee, dW, Katmadas, arXiv:1512.06686

Naturally this will lead to many of the known results about 3D supergravity!

Roček, Van Nieuwenhuizen,1986
Lindström, Roček, 1989
Howe, Izquierdo, G. Papdopoulos, Townsend, 1996
Bergshoeff, Hohm, Rosseel, Townsend, 2010
Kuzenko, Lindström, Tartaglino-Mazzucchelli, 2011
Gran, Greitz, Howe, Nilsson, 2012
Butter, Kuzenko, Novak, Tartaglino-Mazzucchelli, 2013

Subsequently we study the consequences of this dictionary for c-map for invariants without and with higher-derivative couplings.

Off-shell c-map: 4D ➡ 3D

- R-symmetry: $\mathrm{SU(2)_V \times U(1)_A \longrightarrow SU(2)_V \times SU(2)_A}$
- New gauge connections: $A_\mu^{\pm} \propto T_a{}^{\pm} + \cdots$
- Uniform decomposition rule of local supersymmetry.
- Spinor conversion: $\psi^i\big|_{4D} \to \psi^{i\,p}\big|_{3D}$ involves phase factor

  subject to the condition $C^{-1}\bar{\psi}_{i\,p} = \varepsilon_{ij}\,\varepsilon_{pq}\,\psi^{j\,q}$
- The Weyl multiplet decomposes into the *3D* Weyl multiplet and a vector multiplet.

The dictionary should be consistent with both $\mathrm{U(1)_A}$ and $\mathrm{SU(2)_A}$!

!

3D Multiplets

c-map

Weyl multiplet
$(14+2)\oplus(8+8)$

$$e_\mu{}^a \quad \psi_\mu{}^{i\,p} \quad V_\mu{}^i{}_j \longleftrightarrow A_\mu{}^p{}_q$$
$$A \quad \chi^{i\,p} \quad D$$

$$\boxed{\times(-1)}$$

Vector multiplet
$8\oplus 8$

$$X^p{}_q \quad \Omega^{i\,p} \quad F^\mu \quad Y^i{}_j \qquad (D_\mu F^\mu = 0)$$

$$\updownarrow$$

Tensor multiplet
$8\oplus 8$

$$L^i{}_j \quad \varphi^{i\,p} \quad E^\mu \quad G^p{}_q \qquad (D_\mu E^\mu = 0)$$

Hypermultiplets follow by dualizing the vector of the tensor and the vector multiplets to a scalar. There exist two types of hypermultiplets!

Henceforth: $X^p{}_q \longrightarrow L^p{}_q\,;\quad G^p{}_q \longrightarrow Y^p{}_q$

The dictionary: *4D Weyl multiplet and the KK multiplet*

KK vector

$$e_M{}^A = \begin{cases} e_\mu{}^a = e_\mu{}^a \\ e_\mu{}^4 = \boxed{W_\mu{}^0}\,(L^0)^{-1} \\ e_{\hat 4}{}^a = 0 \\ e_{\hat 4}{}^4 = \boxed{(L^0)}^{-1} \end{cases}$$

$$L^0 = \sqrt{|v^0|^2 + \tfrac{1}{4}\boxed{(x^0)}^2}$$

generalized phase factors

SU(2)/U(1)

$$\mathcal{V}_M{}^i{}_j|_{4D} = \begin{cases} \mathcal{V}_\mu{}^i{}_j = \mathcal{V}_\mu{}^i{}_j + W_\mu{}^0\, Y^0{}^i{}_j\,(L^0)^{-2} \\ \mathcal{V}_{\hat 4}{}^i{}_j = \boxed{Y^0{}^i{}_j}\,(L^0)^{-2} \end{cases}$$

contains $\mathcal{A}_\mu{}^p{}_q$

$$A_M = \begin{cases} A_\mu = \dfrac{\mathrm{i}v^0\,\overleftrightarrow{\mathcal{D}}_\mu \bar v^0}{2\,L^0(L^0 + \frac{1}{2}x^0)} + \dfrac{1}{L^0}\left[\tfrac{1}{4}F(W^0)_\mu + W_\mu{}^0\, C\right] \\ A_{\hat 4} = (L^0)^{-1}\, C \end{cases}$$

SU(2) invariant up to a local field-dependent U(1) transformation

4D vector multiplet

SU(2) invariant up to a local U(1) transformation

$$X = -\tfrac{1}{4}\mathrm{i}\Big[\frac{x\,\bar{v}^0 - \bar{v}\,x^0}{L^0} - \frac{\bar{v}\,v^0 - v\,\bar{v}^0}{L^0(L^0+\tfrac{1}{2}x^0)}\,\bar{v}^0\Big]$$

$$W_M = \begin{cases} W_\mu = & W_\mu - \tfrac{1}{4}W_\mu{}^0\,(L^0)^{-2}\,\big(x\,x^0 + 2\,v\,\bar{v}^0 + 2\,\bar{v}\,v^0\big) \\ W_{\hat{4}} = & -\tfrac{1}{4}(L^0)^{-2}\,\big(x\,x^0 + 2\,v\,\bar{v}^0 + 2\,\bar{v}\,v^0\big) \end{cases}$$

SU(2) invariant

These results are invariant under the typical KK shift symmetry:

$$\begin{aligned} \delta L^p{}_q &= \alpha\, L^p{}_q{}^0\,, & \qquad \delta L^p{}_q{}^0 &= 0\,,\\ \delta W_\mu &= \alpha\, W_\mu{}^0\,, & \delta W_\mu{}^0 &= 0\,,\\ \delta Y^i{}_j &= \alpha\, Y^i{}_j{}^0\,, & \delta Y^i{}_j{}^0 &= 0\,, \end{aligned}$$

with the exception of $W_{\hat{4}}$ which is shifted by a constant!

4D tensor multiplet

$$L_{ij} = -\varepsilon_{ik}\, L^k{}_j\, L^0$$

$$E^A = \begin{cases} E^a = & \tfrac{1}{2}\mathrm{i}\, L^0\, \varepsilon^{abc}\, F(E)_{bc}\,, \\[2ex] E^4 = & \tfrac{1}{2} Y^p{}_q\, L^q{}_p{}^0 + \tfrac{1}{2} L^i{}_j\, Y^j{}_i{}^0\,, \end{cases}$$

$$E_{\hat{4}\mu} = E_\mu$$

SU(2) invariant up to a local U(1) transformation

$$G = \tfrac{1}{2}\mathrm{i}\Big[-y\, v^0 + w\, x^0 - \frac{\bar{w}\, v^0 - w\, \bar{v}^0}{L^0 + \frac{1}{2} x^0}\, v^0 \Big]$$

In view of the fact that the vectors are in the c-map image of the tensors, we first remind you of the subtleties of the 4D and 3D tensor multiplet Lagrangians.

The vector and tensor multiplet take (almost) the same form

$$\mathcal{L}_{\rm tensor}\Big|_{3D} =$$

$$\tfrac{1}{2}e\, F_{IJ}\, \mathcal{D}_\mu L^i{}_j{}^J\, \mathcal{D}^\mu L^j{}_i{}^J - \tfrac{1}{2}\, e\, F_{IJ}\, L^i{}_j{}^I\, L^j{}_i{}^J \left(\tfrac{1}{2}\, R + D - C^2\right)$$

$$- \tfrac{1}{2} e\, F_{IJ}(L)\Big[F(E)_{\mu\nu}{}^I\, F(E)^{\mu\nu J} + Y^p{}_q{}^I\, Y^q{}_p{}^J\Big]$$

$$\boxed{\begin{aligned} &- \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho}\, F_{IJ}(L)\, F(E)_{\mu\nu}{}^I\, L^i{}_j{}^J\, \mathcal{V}_\rho{}^j{}_i \\ &+ \mathrm{i}\varepsilon^{\mu\nu\rho} F_{IJK i}{}^j(L)\, E_\mu{}^I\, \partial_\nu L^i{}_k{}^J\, \partial_\rho L^k{}_j{}^K \end{aligned}}$$

The $F_{IJ}(L)$ are SU(2) invariant and homogeneous functions of degree -1

The Lagrangian is not *manifestly* gauge invariant and SU(2) invariant!
But the Lagrangian is invariant because

$$\varepsilon^{\mu\nu\rho}\partial_\mu\Big[F(L)_{IJKi}{}^j\ \partial_\nu L^i{}_k{}^J\, \partial_\rho L^k{}_j{}^K\Big] = 0$$

$$\delta_{\rm SU(2)}\Big[F(L)_{IJKi}{}^j\ \partial_\mu L^i{}_k{}^J\, \partial_\nu L^k{}_j{}^K\Big] = \partial_{[\mu}\Big[\partial_{\nu]}\Lambda_i{}^j\, F(L)_{IJ}\, L^i{}_j{}^J\Big]$$

Note: cohomological issues!

Properties of 4D and 3D tensor Lagrangians

$$F_{IJ} = \frac{\partial^2 F(x,v,\bar v)}{\partial x^I\,\partial x^J} = -\frac{\partial^2 F(x,v,\bar v)}{\partial v^I\,\partial \bar v^J} \qquad \frac{\partial^2 F(x,v,\bar v)}{\partial x^I\,\partial v^J} = \frac{\partial^2 F(x,v,\bar v)}{\partial x^J\,\partial v^I}$$

but $F(x,v,\bar v)$ is never SU(2) invariant!

Lindström, Roček, 1983
dW, Roček, Vandoren, 2001
dW, Saueressig, 2006

However, there exists a so-called tensor potential which is SU(2) invariant (for superconformal couplings):

$$\chi_{\rm tensor}(L) = F_{IJ}\,(x^I x^J + 4\,v^I\bar v^J) = -F(v,\bar v,x) + x^I\,\frac{\partial F(x,v,\bar v)}{\partial x^I}$$

The two-form associated with $\varepsilon^{\mu\nu\rho}\partial_\mu\Big[F(L)_{IJK i}{}^j\;\partial_\nu L^i{}_k{}^J\,\partial_\rho L^k{}_j{}^K\Big]=0$ can be solved in terms of a one-form, but in general this one-form is not SU(2) invariant either.

dW, Philippe, Van Proeyen, 1983

This affects the SU(2) properties of the hypermultiplet system that arises upon tensor-scalar duality!

The vector Lagrangian

$$\mathcal{L}_{\rm vector}\Big|_{3D} = -\tfrac{1}{2} e\, \mathcal{F}_{AB}(L)\, L^p{}_q{}^A\, L^q{}_p{}^B \big[\tfrac{1}{2} R - D - C^2\big]$$
$$\qquad -\tfrac{1}{2} e\, \mathcal{F}_{AB}(L) \big[F(W)_{\mu\nu}{}^A\, F(W)^{\mu\nu B} + Y^i{}_j{}^A\, Y^j{}_i{}^B\big] + \cdots$$

which includes the KK vector multiplet, with

$$\boxed{A, B = \Lambda, 0}$$

$$\mathcal{F}_{\Lambda\Sigma} = \frac{1}{4\, L^0}\, N_{\Lambda\Sigma}$$
$$\mathcal{F}_{\Lambda 0} = \mathcal{F}_{0\Lambda} = \frac{1}{8\,(L^0)^3}\, N_{\Lambda\Sigma}\, L^p{}_q{}^\Sigma\, L^q{}_p{}^0$$
$$\mathcal{F}_{00} = \frac{1}{16\,(L^0)^3}\, N_{\Lambda\Sigma}\bigg[L^p{}_q{}^\Lambda\, L^q{}_p{}^\Sigma + \frac{3\, L^p{}_q{}^\Lambda\, L^q{}_p{}^0\;\, L^r{}_s{}^\Sigma\, L^s{}_r{}^0}{2\,(L^0)^2}\bigg]$$

The corresponding vector potential equals

$$\chi_{\rm vector} \equiv -\,2\, \mathcal{F}_{AB}\, L^p{}_q{}^A\, L^q{}_p{}^B$$
$$= -\,\frac{N_{\Lambda\Sigma}}{4\, L^0}\, \bigg[L^p{}_q{}^\Lambda\, L^q{}_p{}^\Sigma + \frac{L^p{}_q{}^\Lambda\, L^q{}_p{}^0\;\, L^r{}_s{}^\Sigma\, L^s{}_r{}^0}{2\,(L^0)^2}\bigg]$$
$$= \frac{2\, N_{\Lambda\Sigma}\, X^\Lambda \bar X^\Sigma}{L^0}\,,$$

Remaining terms in the bosonic action

$$-\tfrac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho}\,\mathcal{F}(L)_{AB}\,F(W)_{\mu\nu}{}^{A}\,L^{p}{}_{q}{}^{B}\,\mathcal{A}_{\rho}{}^{q}{}_{p}$$
$$+\,\mathrm{i}\,\varepsilon^{\mu\nu\rho}\,\mathcal{F}(L)_{ABC}{}^{p}{}_{q}\,\partial_\mu L^{q}{}_{r}{}^{A}\,\partial_\nu L^{r}{}_{p}{}^{B}\,W_{\rho}{}^{C}$$

originate from an expression proportional to

$$\mathrm{i}\varepsilon^{\mu\nu\rho}\,F(W)_{\mu\nu}{}^{A}\,\mathcal{D}_\rho L^{p}{}_{q}{}^{B}\,f(L)^{q}{}_{p\,AB}$$

The cohomology situation is quite different in this case!

Conclusion

The reduction of the vector multiplet sector leads to a restricted set of 3D theories:

- They satisfy a shift symmetry of the KK type
- Trivial cohomology for the terms linear in the gauge fields
- Upon vector-scalar duality the resulting hypermultiplet system will have a restricted structure with respect to SU(2).

NB: The analysis of the reduction of the vector multiplet action is quite involved!

Generic 3D Lagrangian (some characteristic terms)

$$
\begin{aligned}
e^{-1}\mathcal{L} = {} & \tfrac{1}{4}\left(\chi_{\text{hyper}} + \chi_{\text{tensor}} + \tilde{\chi}_{\text{hyper}} + \chi_{\text{vector}}\right)\left(\tfrac{1}{2}R - C^2\right) \\
& + \tfrac{1}{4}\left(\chi_{\text{hyper}} + \chi_{\text{tensor}} - \tilde{\chi}_{\text{hyper}} - \chi_{\text{vector}}\right) D \\
& - \tfrac{1}{2}\,\Omega_{\alpha\beta}\,\varepsilon^{ij}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta - \tfrac{1}{2}\,\tilde{\Omega}_{\alpha\beta}\,\varepsilon^{pq}\,\mathcal{D}_\mu \tilde{A}_p{}^\alpha\,\mathcal{D}^\mu \tilde{A}_q{}^\beta \\
& + \tfrac{1}{2}F_{IJ}\,\mathcal{D}_\mu L^i{}_j{}^J\,\mathcal{D}^\mu L^j{}_i{}^J + \tfrac{1}{2}\mathcal{F}_{AB}\,\mathcal{D}_\mu L^p{}_q{}^A\,\mathcal{D}^\mu L^q{}_p{}^B
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_{\text{hyper}} &= \tfrac{1}{2}\,\Omega_{\alpha\beta}\,\varepsilon^{ij}\,A_i{}^\alpha\,A_j{}^\beta \\
\tilde{\chi}_{\text{hyper}} &= \tfrac{1}{2}\,\tilde{\Omega}_{\alpha\beta}\,\varepsilon^{pq}\,\tilde{A}_p{}^\alpha\,\tilde{A}_q{}^\beta \\
\chi_{\text{tensor}} &= -2\,F_{IJ}\,L^i{}_j{}^I\,L^j{}_i{}^J \\
\chi_{\text{vector}} &= -2\,\mathcal{F}_{AB}\,L^p{}_q{}^A\,L^q{}_p{}^B
\end{aligned}
$$

In view of the c-map you may wonder which of these actions can be uplifted to two different *4D* supergravities!

It is clear that this can only be the case when the vector and tensor systems are of the same restricted type: they should both be manifestly gauge invariant and SU(2) invariant.

Furthermore the hypermultiplet sector should be related to the (restricted) vector and tensor multiplets by scalar-vector duality. Hence their target spaces should be hyperkähler cones with one more abelian triholomorphic isometry as their quaternionic dimension.

Upon dualization one has only vector and tensor multiplets. Therefore one may assume that the 4D Lagrangian has only (restricted off-shell) vector and tensor multiplets and no hypers.

$$\begin{tikzcd}
\boxed{(n_V, n_T)_4} & & \boxed{(\tilde{n}_V, \tilde{n}_T)_4} \\
\boxed{(n_V + 1, n_T)_3} \arrow[u] \arrow[rr, leftrightarrow, "\text{c-map}"] & & \boxed{(\tilde{n}_V + 1, \tilde{n}_T)_3} \arrow[u]
\end{tikzcd}$$

$$(\tilde{n}_V + 1)_3 = (n_T)_3$$
$$(n_V + 1)_3 = (\tilde{n}_T)_3$$

The c-map for higher-derivative interactions; an example

In general it is difficult to realize 3D higher-derivative couplings that can be uplifted to 4D in two inequivalent ways.

We have been able to construct one class of higher-derivative couplings for which this was possible, using the dictionary that we have derived. These couplings are based on defining composite vector and tensor multiplets with higher derivatives. They can be used in a full superconformal background.

dW, Saueressig, 2006

First in 4D: a composite vector multiplet

$$X^{\rm comp} = f(L)_I\,\bar G^I + f(L)_{IJ}{}^{ij}\,\bar\varphi_i^I\varphi_j^J$$

with $\quad f_{IJ}{}^{ij} = f_{JI}{}^{ij}\,, \qquad \varepsilon^{jk}\,\dfrac{\partial f_{IJij}}{\partial L^{klK}} = 0$

defines the beginning of a full vector multiplet

Likewise: a composite tensor multiplet

$$L_{ij}^{\rm comp} = g(X)_\Lambda\, Y_{ij}{}^\Lambda - \tfrac{1}{2} g(X)_{\Lambda\Sigma}\, \bar\Omega_{(i}{}^\Lambda \Omega_{j)}{}^\Sigma$$
$$+ \varepsilon_{ik}\varepsilon_{jl}\left[\bar g_\Lambda(\bar X)\, Y^{kl\,\Lambda} - \tfrac{1}{2}\bar g(\bar X)_{\Lambda\Sigma}\, \bar\Omega^{(k\,\Lambda}\Omega^{l)\,\Sigma}\right]$$

with $\dfrac{g_\Lambda}{X^\Sigma} = \dfrac{g_\Sigma}{X^\Lambda}$ defines the beginning of a full tensor multiplet

Now use the dictionary and convert these results to 3D.
Via the dictionary also the KK vector multiplet will appear.
The expressions will remain invariant under the KK shift transformations.

Furthermore note that products of composite or elementary tensors with vectors define invariant actions, both for 3D and 4D.

This (large) class of higher-derivative interactions can indeed be made consistent with the c-map!

So far we have not been able to identify other solutions! In particular we have not (yet) found a way to incorporate the so-called F-terms that contain the square of the Weyl tensor multiplied by a function of

$$\left(T_{AB}{}^{ij}\varepsilon_{ij}\right)^2 = -\frac{4}{(L^0)^4}\left[(\bar v^0 \overleftrightarrow{\mathcal{D}}_a\, x^0) - \frac{\bar v^0}{L^0 + \frac{1}{2}x^0}\,(v^0 \overleftrightarrow{\mathcal{D}}_a\, \bar v^0)\right]^2$$

Conclusions

Off-shell dimensional reduction is a useful and powerful tool for relating and studying higher-derivative couplings in various space-time dimension.

Its off-shell nature often enables a direct identification of the lower-dimensional couplings by considering just the bosonic terms

The off-shell c-map relates vector and tensor multiplets. The tensor multiplets should be restricted to a certain subclass. Hypermultiplets can be obtained by vector-scalar duality. As demonstrated it can also be used for higher-derivative couplings.

In that case one obtains hyperkähler cones, which, upon taking a superconformal quotient, leads to the quaternion-Kähler target spaces of the on-shell formulation. It should be of interest to analyze the symmetry structure of these hyperkähler cones in the same way as it was done long ago for the quaternion-Kähler spaces in the image of the c-map.

So far we have only been able to identify one class of higher-derivative couplings that are consistent with the c-map. There should be many more solutions!