Boundary Terms and Three-point Functions: An AdS/CFT Puzzle Resolved

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Supergravity, What's Next? GGI 26 October 2016

AdS/CFT correspondence applied to duality between

$\mathcal{N}=8,\ d=3$ A$BJM\ CFT_3 \leftrightarrow \mathcal{N}=8,\ D=4\ SG$

Ungaged $\mathcal{N}=8$ SG: E. Cremmer and B. Julia
Gauging of $SO(8)$: B. de Wit and H. Nicolai

i. An acute puzzle: ABJM contains $\Delta=1$ scalar operators $\mathcal{O}_{IJ}(x)$ in 35_v of SO(8).

$\langle\mathcal{O}_{IJ}(x)\mathcal{O}_{KL}(y)\mathcal{O}_{MN}(z)\rangle \neq 0$. Can be calculated exactly in the CFT because $\mathcal{O}_{IJ}(x)$ is in a short multiplet whose top component is $T_{\mu\nu}$.

ii. Many 3-pt correlators have been calculated in gravity duals by evaluation of a Witten diagram containing a cubic coupling from bulk Lagrangian.
Indeed, gauged $\mathcal{N}=8,\ D=4\ SG$ contains 35 fields A^{IJ} dual to the $\mathcal{O}_{IJ}$, but there is no cubic A^3 coupling!
Something new must be found to produce $\langle OOO\rangle$ from bulk SG!

iii. Resolution- SUSY requires that renormalized on-shell bulk action contains a *c*ubic BOUNDARY term in addition to standard counterterms from holographic renormalization.

$$\text{New bdy. term:} \quad S_3 = \frac{1}{8\pi G_4}\frac{1}{6}\int d^3x\sqrt{-h}\,A^{IJ}A^{JK}A^{KI}$$

This bdy term produces $\langle \mathcal{O}_{IJ}(x)\mathcal{O}_{KL}(y)\mathcal{O}_{MN}(z)\rangle$ that matches the CFT result.

iv. In $\mathcal{N}=4$ SYM, the 3-point correlators of chiral primary operators are protected, i.e. independent of $\lambda = g_{YM}^2 N$.

But in ABJM, $\langle \mathcal{O}_{IJ}(x)\mathcal{O}_{KL}(y)\mathcal{O}_{MN}(z)\rangle$ contains strong coupling effects calculated using supersymmetric localization. So the agreement between the gravity and gauge theory results is a *precision test of holography.*

v. Two branches of AdS/CFT mass formula:

$$\Delta = (d \pm \sqrt{d^2 + 4m^2L^2})/2$$

The bulk dual of a scalar operator with $\Delta < d/2$ requires *alternate quantization*. This includes $\Delta = 1$ in $d = 3$.
Usual role of "source" and "vev" terms in near bdy behavior are interchanged. One must use Legendre transform of on-shell action as generating functional of CFT correlators.

For $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle$ Klebanov and Witten, 1999
For $\langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z) \rangle$ in our work, 2016

II. First evidence for new $S_{\mathrm{b}dy}$ from consistent truncation of $\mathcal{N}=8$ gauged SG to $\mathcal{N}=1$. [DZF + S. Pufu, 1302.7310]

Truncation contains gravity multiplet $g_{\mu\nu},\ \psi_\mu$ +
3 chiral multiplets $z^\alpha = A^\alpha + iB^\alpha,\ \chi^\alpha$.

Scalars $A^\alpha(r,x)$ dual to $\Delta=1$ operators, $\implies$ alternate quant.
Pseudoscalars $B^\alpha(r,x)$ dual to $\Delta=2$ operators, $\implies$ std. quant.

Most of the ideas needed for our work are easier to describe in the $\mathcal{N}=1$ truncation. (DZF).
Extension to full $\mathcal{N}=8$ SG by (KP).

III. The bosonic action is

$$
\begin{aligned}
S &= \frac{1}{8\pi G_4}\int d^4x\sqrt{-g}\Bigg[\frac{1}{2}R - \sum_{\alpha=1}^{3}\frac{|\partial_\mu z^\alpha|^2}{(1-|z^\alpha|^2)^2} \\
&+ \frac{1}{L^2}\left(-3+\sum_{\alpha=1}^{3}\frac{2}{1-|z^\alpha|^2}\right)\Bigg].
\end{aligned}
$$

Simple Kähler metric– 3 *decoupled* copies of Poincaré disc.
Potential– Cos. const. $+$ 3 *decoupled* terms.
NO CUBIC TERMS!

Repeat potential:

$$V(z,\bar{z}) = \frac{1}{L^2}\left(-3 + \sum_{\alpha=1}^{3} \frac{2}{1-|z^\alpha|^2}\right).$$

Find holomorphic superpotential $W(z^\alpha)$ such that $V(z,\bar{z})$ takes standard form in $\mathcal{N}=1$ SG:

$$V = e^K\left(\nabla_\alpha W\, K^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\bar{W} - 3W\bar{W}\right) \qquad\qquad \nabla_\alpha W \equiv (\partial_\alpha + K_\alpha)W.$$

Result: $\quad W = (1 + z^1 z^2 z^3)/L.$

An algebraic miracle that a highly coupled $W(z)$ produces an uncoupled $V(z,\bar{z})$!

The cubic term in W is what we need in 2016, but how do we move it into the action?

IV. Bogomolny argument: (similar to Skenderis, Townsend, 1999)
a. Insert planar domain wall ansatz into $\mathcal{N}=1$ bosonic action.

$$ds^2 = dr^2 + e^{2A(r)}\eta_{ij}dx^i dx^j \qquad z^\alpha = z^\alpha(r) \qquad \bar{z}^{\bar{\beta}} = \bar{z}^{\bar{\beta}}(r).$$

b. Manipulate by partial integration and grouping of terms to obtain factored form: (r_0 is radial cutoff.)

$$\begin{aligned} S \;=\; \int^{r_0} d^3x dr \Big[& e^{3A}(\partial_r A - e^{K/2}|W|)^2 \\ & -K_{\alpha\bar{\beta}}(\partial_r z^\alpha + \sqrt{\frac{W}{\bar{W}}}K^{\alpha\bar{\gamma}}\partial_{\bar{\gamma}}\bar{W})(c.c.)^{\bar{\beta}} \\ & \frac{\partial}{\partial r}(2e^{3A}e^{K/2}|W|)\Big] \end{aligned}$$

The quadratic factors give the BPS eqtns for $A(r),\ z^\alpha(r),\ \bar{z}^{\bar{\beta}}(r)$.
The action then vanishes except for the boundary term.

c. The boundary term must be cancelled by an equal and opposite CT. Otherwise the vacuum energy of the BPS domain wall will not vanish, violating SUSY. Thus we must add to the action:

$$S_3 = -\frac{1}{4\pi G_4}\int d^3x\, e^{3r_0/L} e^{K/2}|W|.$$

For Kähler potential and superpotential:
$K = -\sum_\alpha \log(1-|z^\alpha|^2), \qquad W = (1+z^1z^2z^3)/L,$

$$S_3 \to \frac{-1}{4\pi G_4 L}\int d^3x\, e^{3r_0/L}\Big[(1+z^\alpha\bar{z}^{\bar{\alpha}}/2)[1+\frac{1}{2}(z^1z^2z^3+c.c.)]+...\Big].$$

AdS/CFT asymptotic behavior $z \sim e^{-r/L}$ as $r \to \infty$. Thus we have

i. cubic + linear divergences that match CT's of holog. ren.

ii. finite cubic CT with the right coefficient to obtain $\langle OOO \rangle$.

iii. ... indicates terms which vanish faster than $e^{-3r/L}$ and so can be dropped.

V. **Find CT's by extension of local SUSY to the boundary.**

i. In usual proofs of invariance in SG, one is happy to achieve invariance up to total derivatives, i.e. $\delta S = \int d^4x \partial_\mu [\sqrt{-g}\bar{\epsilon}(x) X^\mu]$. Correct because the $\epsilon(x)$ are arbitrary functions which can be assumed to vanish as $r \to \infty$.

ii. However, in AdS/CFT the behavior as $r \to \infty$ is crucial. The $\epsilon(r, x)$ are Killing spinors and the fields vanish at rates fixed by field eqtns. So we collect bdy terms and write

$$\int d^4x \partial_\mu [\sqrt{-g}\,\bar{\epsilon}(x) X^\mu] = \int_{r=r_0} d^3x \sqrt{-h}\,\bar{\epsilon}\, X^r \equiv \delta S_{bdy}.$$

iii. Find set of CTs: $S_{CT} = \int d^3x \sqrt{-h}\,\mathcal{L}_{CT}$, such that $\delta_{SUSY} S_{CT} = -\delta S_{bdy}$.

VI. First work out bdy. terms and CT's in global limit of a general $\mathcal{N} = 1$ SG model. A limit in which the back reaction of the matter fields is consistently suppressed, so the gravitino can be dropped. Result is an action that has global SUSY on AdS_4.
Similar to construction of Festuccia and Seiberg, 1105.0689

i. In this global limit, the SUSY parameters are AdS Killing spinors. Killing spinors satisfy

$$(D_\mu + \frac{1}{2L}\gamma_\mu)\epsilon(r,x) = 0$$

They can be found explicitly for the AdS_4 metric $ds^2 = dr^2 + e^{2r/L}\eta_{ij}dx^i dx^j$.
Their leading components grow at the bdy. as $\epsilon(r,x) \sim e^{r/2L}$.

ii. This limiting procedure works for any Kähler metric and any supot. of the form $W_{SG} = (1 + W(z^\alpha))$ with cubic $W(z^\alpha)$. This guarantees that the SG model has an AdS stationary point with cos. const. $\Lambda = -3/L^2$, the SUSY value.

iii. Further simplifications: info on CT's that we need is captured by case of one chiral multiplet $z, \; \chi$ with a flat Kähler potential $K = z\bar{z}$ and cubic $W = z^3/3 \quad$ or $\quad z^1 z^2 z^3$.

iv. Result is a simple (off-shell) action.

$$
\begin{aligned}
S &= S_{kin} + S_F + S_{\bar{F}} \\
S_{kin} &= \int d^4x\sqrt{-g}\Big[-\partial_\mu z\partial^\mu \bar{z} - \frac{1}{2}\bar{\chi}\gamma^\mu D_\mu \chi \\
&\qquad + (F + z/L)(\bar{F} + \bar{z}/L) + 2z\bar{z}/L^2\Big] \\
S_F &= \int d^4x\sqrt{-g}[FW' - \frac{1}{2}W'' \bar{\chi}P_L\chi + 3W/L] \\
S_{\bar{F}} &= (S_F)^* \,.
\end{aligned}
$$

The 3 terms $S_{kin}, \; S_F, \; S_{\bar{F}}$ are *separately* invariant under:

$$
\delta z = \bar{\epsilon}P_L\chi \qquad \delta P_L\chi = P_L(\gamma^\mu\partial_\mu z + F)\epsilon \qquad \delta F = \bar{\epsilon}(\gamma^\mu D_\mu - 1/L)P_L\chi.
$$

S_F is very simple and so is its SUSY variation:

$$\delta S_F = \int d^4x\sqrt{-g}[\nabla_\mu(\bar{\epsilon}\gamma^\mu W' P_L\chi) - \bar{\epsilon}(\overleftarrow{D}_\mu\, \gamma^\mu - 2/L)W' P_L\chi].$$

Last term vanishes by adjoint of Killing spinor eqtn.
First term is the bdy term we are looking for! It is cancelled by CT

$$S_{cubic} = -\int d^3x\sqrt{-g}[W(z) + \bar{W}(\bar{z})].$$

After change to previous normalization, one reproduces the CT S_3 from Bogomolny argument. The $W + \bar{W}$ CT is very important, but there are other bdy terms from $\delta S_{\rm kin}$.

VII. Systematic study based on principle:

The generating functional of CFT correlators must be supersymmetric under trf. rules of the sources for the operators dual to $A,\ B,\ \chi$.. The sources are the "leading" terms in large r behavior of the fields. This determines their trf. rules.

Usually, the generating functional is $S_{\text{on-shell}}$. But we need alternate quant. for $A(r,x)$, so the gen. fnl. is the Legendre transform.

To get started on this program, we need the large r behavior of Killing spinors $\epsilon(r,x)$ and the fields determined from the EOM's.

$$\epsilon(r,x) = e^{r/2L}\eta_-(x) + e^{-r/2L}\eta_+(x)$$

The subscripts on $\eta_\pm(x)$ indicate "radiality" projections:

$$\gamma^3\eta_\pm = \pm\eta_\pm, \qquad \bar{\eta}_\pm\gamma^3 = \mp\bar{\eta}_\pm\,.$$

Similar for spinor field $\gamma^3\chi_\pm(r,x) = \pm\chi_\pm(r,x)$.

Large r expansions of the fields:

$$A(r,x) = e^{-r/L}A_1(x) + e^{-2r/L}A_2(x) + \dots\,,$$

$$B(r,x) = e^{-r/L}B_1(x) + e^{-2r/L}B_2(x) + \dots\,,$$

$$\chi(r,x) = e^{-3r/2L}\chi_{3/2}(x) + e^{-5r/2L}\chi_{5/2}(x) + \dots\,.$$

Substitute in the trf. rules to obtain

$$\begin{aligned}
\delta A_1 &= \tfrac{1}{2}\bar\eta_-\chi_{3/2+}, & \delta A_2 &= \tfrac{1}{2}\left(\bar\eta_-\chi_{5/2+} + \bar\eta_+\chi_{3/2-}\right)\\
\delta B_1 &= -\tfrac{i}{2}\bar\eta_-\gamma_5\chi_{3/2-}\,, & \delta B_2 &= -\tfrac{i}{2}\left(\bar\eta_-\gamma_5\chi_{5/2-} + \bar\eta_+\gamma_5\chi_{3/2+}\right)\\
\delta\chi_{3/2-} &= \left(\tfrac{1}{L}A_2 - \tfrac{\kappa}{L}(A_1^2 - B_1^2) + i\gamma_5\partial\!\!\!/ B_1\right)\eta_- - \tfrac{2i}{L}B_1\gamma_5\eta_+ \\
\delta\chi_{3/2+} &= i\gamma_5\left(\tfrac{1}{L}B_2 + \tfrac{2\kappa}{L}A_1B_1\right)\eta_- + \partial\!\!\!/ A_1\eta_- - \tfrac{2}{L}A_1\eta_+\,.
\end{aligned}$$

These are the trf. rules of the asymptotic coefficients.

VIII. What are the sources?

i. B has std. quant., so its source is the leading coeff $B_1(x)$.

ii. δB_1 trfs. into $\chi_{3/2-}(x)$, so this is the spinor source.

iii. The scalar part of $\delta\chi_{3/2-}$ is the source of A, namely

$$A_2 + \kappa(A_1^2 - B_1^2).$$

This consists of vev rate coefficient A_2 plus nonlinear terms req'd by SUSY and Legendre trf.

Conclusion: $B_1, \quad \chi_{3/2-}, \quad A_2 + \kappa(A_1^2 - B_1^2)$ are the sources of
CFT ops. $\mathcal{O}_2, \quad \mathcal{O}_{3/2}, \quad \mathcal{O}_1$,
(The subscripts indicate scale dimension Δ).

IX. The renormalized action is $Sren = S_{\text{bulk}} + S_{\text{bdy}}$, where:

$$S_{\text{bulk}} = S_{\text{kin}} + S_F + S_{\bar{F}}$$

$$S_{\text{bdy}} = S_3 + S_2 + S_\chi\,,$$

and

$$S_3 = -\int d^3x\sqrt{-g}[W + \bar{W} = 2\kappa(A^3 - 3AB^2)/L] \qquad z = A + iB$$

$$S_2 = -\frac{1}{L}\int d^3x\sqrt{-g}\,z\bar{z}\,,$$

$$S_\chi = -\frac{c}{4}\int d^3x\sqrt{-g}\,\bar{\chi}\chi\,.$$

S_2 comes from holog. renormalization.
S_χ is std spinor CT introduced by Henningson and Sfetsos, 1998

X. Legendre transform of $S_{\text{on-shell}}$:

Trade $A_1(x) \to \mathcal{A}(x)$, which is bdy limit of $\Pi_A(x) = -\frac{\delta L}{\delta\, \partial_r A(x)}$.

$$\tilde{S}_{\text{on-shell}}[\mathcal{A}, ...] \equiv S_{\text{on-shell}} + \int d^3x\, \mathcal{A}(x) A_1(x)$$

To be evaluated at the extremum $\frac{\delta \tilde{S}}{\delta A_1} = 0$. This gives

$$\mathcal{A}(x) = -2[A_2(x) - \frac{\kappa}{L}(A_1^2(x) - B_1^2(x))] \quad *$$

This confirms previous identification of the source of $A(r, x)$.

It is $\tilde{S}[\mathcal{A}, \chi_{3/2-}, B_1]$ that must be invariant under trf rules, in which $\delta\mathcal{A} \equiv$ SUSY trf of * using fermion EOMs to relate $\chi_{5/2+} = \partial\!\!\!/ \chi_{3/2-}$.

A mechanical calculation then gives

$$
\begin{aligned}
\delta\tilde{S}_{\text{on-shell}} &= \int d^3x[\frac{\delta\tilde{S}}{\delta B_1}\delta B_1 + \frac{\delta\tilde{S}}{\delta\chi_{3/2-}}\delta\chi_{3/2-} + \frac{\delta\tilde{S}}{\delta\mathcal{A}}\delta\mathcal{A}] \\
&= \int d^3x\,\partial_a V^a(x) \equiv 0\,.
\end{aligned}
$$

This calculation also fixes $c = 1$ for S_χ.
This establishes SUSY invariance of $\tilde{S}$!

XI. Summarize SUSY trfs of sources:

$$\begin{aligned}
\delta B_1 &= -\frac{i}{2}\bar{\eta}_-\gamma_5\chi_{3/2-} \\
\delta\chi_{3/2-} &= \left(-i\partial\!\!\!/ B_1\gamma_5 + \frac{1}{2}\mathcal{A}\right)\eta_- - \frac{2i}{L}B_1\gamma_5\eta_+ \\
\delta\mathcal{A} &= -\left(\bar{\eta}_-\partial\!\!\!/\chi_{3/2-} + \bar{\eta}_+\frac{1}{L}\chi_{3/2-}\right).
\end{aligned}$$

Roughly resemble std superconformal trfs of $\mathcal{N}=1,\ d=3$ scalar multiplet, but with artefacts of the bulk theory.
To establish the physics, one can check the algebra of Poincaré SUSY trfs, which gives the std. result:

$$[\delta_1,\delta_2]\Phi(x) = -(\bar{\epsilon}_1\gamma^a\epsilon_2)\partial_a\Phi(x)$$

with $\epsilon = i\gamma_5\eta_-$ and $\Phi \to B_1,\ \chi_{3/2-},\ \mathcal{A}$.

CONCLUSIONS:

0. Background information.

1. We have shown in the $\mathcal{N} = 1$ toy model with $W = \kappa z^3/3L$ that the Legendre transform of the renormalized on-shell action is a supersymmetric functional of the sources. The same argument works for the consistent truncation of $\mathcal{N} = 8$ to the case $W = \kappa z^1 z^2 z^3/L$.

2. KP will extend this argument to full $\mathcal{N} = 8$ SG and also show how to use the Legendre transform to calculate the 2- and 3-point correlators of the $\mathcal{O}_{IJ}$ operators.