Boundary Terms and Three-Point Functions: An AdS/CFT Puzzle Resolved

Krzysztof Pilch

USC University of Southern California

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Dan Freedman, Silviu Pufu and Nick Warner
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$\mathcal{N}=8,\ d=4$ Supergravity

[Cremmer-Julia '79][de Wit-Nicolai '82]

Bosonic sector

- **1** graviton, $e_\mu{}^\alpha$
- **28** gauge fields, $A_\mu{}^{IJ}$
- $\mathbf{35}_v \oplus \mathbf{35}_c$ scalar fields, ϕ_{ijkl}

Fermionic sector

- $\mathbf{8}_s$ gravitini, $\psi_\mu{}^i/\psi_{\mu i}$
- $\mathbf{56}_s$ spins 1/2 fields, χ^{ijk}/χ_{ijk}

  $\gamma^5\chi^{ijk}=\chi^{ijk}$, etc.

The scalar 56-bein in the symmetric gauge is

$$\mathcal{V}\equiv\begin{pmatrix} u_{ij}{}^{IJ} & v_{ijIJ} \\ v^{ijIJ} & u^{ij}{}_{IJ}\end{pmatrix} = \exp\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}\,\phi^{ijkl} \\ -\frac{1}{\sqrt{2}}\,\phi_{ijkl} & 0\end{pmatrix} \in \frac{E_{7(7)}}{SU(8)}$$

where

$$\phi_{ijkl} = (\phi^{ijkl})^*\,, \qquad \phi_{ijkl} = \frac{1}{24}\eta_{ijklmnpq}\phi^{mnpq}$$

We will work with the asymptotic expansion $(r\to\infty)$

$$\phi^{ijkl}(r,\vec{x}) = e^{-r/L}\phi_{(1)}{}^{ijkl}(\vec{x}) + e^{-2r/L}\phi_{(2)}{}^{ijkl}(\vec{x}) + \ldots\,,$$

$$\phi_{(n)}{}^{ijkl}(\vec{x}) = \alpha_{(n)}{}^{ijkl}(\vec{x}) - i\,\beta_{(n)}{}^{ijkl}(\vec{x})\,.$$

where α^{ijkl} / β^{ijkl} are the scalars / pseudoscalars.

The Scalar Potential

The scalar potential is

$$\mathcal{P}(\phi) \;=\; -\Big(\frac{3}{4}\big|A_1{}^{ij}\big|^2 - \frac{1}{24}\big|A_{2i}{}^{jkl}\big|^2\Big)$$

where the A-tensors have the following expansions [de Wit-Nicolai '82]

$$A_1{}^{(ij)} \;=\; \big(1 + \frac{1}{192}\,|\phi|^2\big)\,\delta^{ij} + \frac{\sqrt{2}}{96}\,\phi^{ikmn}\phi_{mnpq}\phi^{pqkj} + O(\phi^4)$$

$$A_{2\,l}{}^{[ijk]} \;=\; -\frac{\sqrt{2}}{2}\big(1 + \frac{1}{144}\,|\phi|^2\big)\,\phi^{ijkl} - \frac{3}{8}\,\phi_{mnl[i}\phi^{jk]mn} + \frac{\sqrt{2}}{16}\,\phi_{lpqr}\phi^{pqs[i}\phi^{jk]rs} + O(\phi^4)$$

and $|\phi|^2 \;=\; \phi_{ijkl}\phi^{ijkl}$. But then

$$|A_1{}^{ij}|^2 \;=\; 8 + \frac{1}{12}\,|\phi^2| - \frac{\sqrt{2}}{96}\,\big(\phi^{ijkl}\phi_{klmn}\phi^{mnij} + \text{c.c.}\big) + O(\phi^4)$$

$$|A_{2\,l}{}^{ijk}|^2 \;=\; \frac{1}{2}\,|\phi|^2 - \frac{3\sqrt{2}}{16}\big(\phi^{ijkl}\phi_{klmn}\phi^{mnij} + \text{c.c.}\big) + O(\phi^4)$$

Note that $4 \times 96 = 24 \times 16$, hence

$$\mathcal{P}(\phi) \;=\; -6 - \frac{1}{24}\,|\phi|^2 + O(\phi^4)$$

has no cubic terms in its expansion! Hence THE PUZZLE.

Comment

For maximal supergravities in $d = 4$, 5 and 7, there is a truncation of the potential to the $\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N)$ sector with $N = 8$, 6 and 5, respectively,

$$\mathcal{P} = -\frac{1}{2}\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2 - 2\sum_{i=1}^{N} X_i^2\Big]$$

where [Cvetič-Gubser-Lü-Pope '99]

$$X_i = \exp(-\tfrac{1}{2}\vec{b}_i\cdot\vec{\varphi})\,, \qquad \vec{b}_i = \text{weights of } \boldsymbol{N} \text{ of } \mathrm{SL}(N,\mathbb{R})$$

and $\varphi^1,\ldots,\varphi^{N-1}$ are canonically normalized scalar fields. Then

$$\begin{aligned}\mathcal{P} \propto\ & (N^2-2N) + (2N-4)(x_1+\ldots x_N) \\ & + (N-4)(x_1^2+\ldots+x_N^2) + (x_1+\ldots x_N)^2 \\ & + \Big(\frac{N}{3}-\frac{8}{3}\Big)(x_1^3+\ldots+x_N^3) + (x_1+\ldots+x_N)(x_1^2+\ldots+x_N^2) \\ & + \ldots\end{aligned}$$

where

$$x_i \equiv -\tfrac{1}{2}\vec{b}_i\cdot\vec{\varphi}\,, \qquad x_1+\ldots+x_N = 0$$

The cubic term vanishes only for $d = 4$.

DZF's Bogomolny Type Argument

In DZF's talk, the supersymmetric boundary counterterm was given by the superpotential, W, of $\mathcal{N} = 1$ supergravity

$$S_{\text{s-ct}} = -\frac{1}{4\pi G_4} \int d^3x\, e^{3r_0}\, e^{K/2} |W|$$

[Freedman-Pufu '13]

It can be derived by a Bogomolny type argument in $\mathcal{N} = 1$ supergravity.

[Skenderis-Townsend '99]

- Assume a domain wall background metric

$$ds^2 = e^{2\mathcal{A}(r)}(dx_m dx^m) + dr^2, \qquad z^\alpha = z^\alpha(r), \quad \bar{z}^\alpha = \bar{z}^\alpha(r)$$

- Rewrite the supergravity action as a sum of squares + boundary terms:

$$\left| \frac{dz^\alpha}{dr} - e^{K/2} \sqrt{\frac{W}{\overline{W}}}\, K^{\alpha\bar{\gamma}} \nabla_{\bar{\gamma}} \overline{W} \right|^2, \qquad \left| \frac{dA}{dr} - e^{K/2} |W| \right|^2$$

  and $S_{\text{boundary}} = -S_{\text{s-ct}}$.

- Can we apply the same type argument to the full $\mathcal{N} = 8$ supergravity?
- How would it work with no W?

$\mathcal{N} = 8$ "Bogomolny Argument"

- Take the Poincaré invariant domain wall metric

$$ds^2 \;=\; e^{2\mathcal{A}(r)}(-dx_0^2 + dx_1^2 + dx_2^2) + dr^2$$

- Set the vector fields, ${A_\mu}^{IJ} = 0$.
- But, keep the scalar fields, $\phi^{ijkl}(\vec{x}, r)$, arbitrary.

The bosonic action, modulo the [Gibbons-Hawking] boundary term, is

$$S_{\rm B} = \int d^4x\, e^{3\mathcal{A}} \Big[3(\mathcal{A}')^2 - \frac{1}{96}\, {\mathcal{A}_\mu}^{ijkl} {\mathcal{A}^\mu}_{ijkl} + \frac{3}{4}\, g^2 \, |{A_1}^{ij}|^2 - \frac{1}{24}\, g^2 \, |{A_{2i}}^{jkl}|^2 \Big]$$

where ${\mathcal{A}_\mu}^{ijkl} \;=\; \partial_\mu \phi^{ijkl} + O(\phi^3)$.

Hints:

- In $\mathcal{N} = 1$ truncations, $e^K|W|^2$ is an eigenvalue of $({A_1}^{ik} A_{1kj})$.
- For $\mathcal{N} = 1, 2, 4$ domain wall solutions, the BPS equations are

$$\begin{aligned} \delta{\psi_a}^i \;&=\; \mathcal{A}'\,\gamma^3\,\epsilon^i + \sqrt{2}\,g\,{A_1}^{ij}\epsilon_j \;=\; 0 \\ \delta\chi^{ijk} \;&=\; -{\mathcal{A}_r}^{ijkl}\,\gamma^3\,\epsilon_l \;-\; 2\,g\,{A_{2\,l}}^{ijk}\,\epsilon^l \;=\; 0 \end{aligned} \qquad \boxed{\epsilon^i \;=\; {\Pi^i}_j\,\epsilon^j}$$

and imply an algebraic constraint, $\gamma^3\epsilon^i = X^{ij}\epsilon_j$, $X^{ik}X_{jk} = {\Pi^i}_j$.

[Ahn-Woo '00, Pope-Warner '04, Bobev-KP-Warner '14, ...]

Some Elementary Algebra

We all know that

- A hermitian matrix H can be diagonalized by a unitary transformation, U,
$$H = U\Lambda U^\dagger$$
- A real, symmetric matrix, A, can be diagonalized by an orthogonal congruence, O,
$$A = O\Lambda O^T$$

What if A is symmetric but complex?

Some Elementary Algebra

We all know that

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What if A is symmetric but complex?

- A complex, symmetric matrix, A, can be diagonalized by a unitary congruence, S,
$$A = SDS^{T}, \qquad D \geq 0$$
where
$$AA^{\dagger} = SD^{2}S^{\dagger}$$
[Autonne '1915, Takagi '25]

Now, let's apply this to the symmetric matrix $(A_1{}^{ij})$ of $\mathcal{N} = 8$ supergravity.

* For a different use of AT-factorization in supergravity, see [Kodama-Nozawa '15].

$\mathcal{N} = 8$ "Bogomolny Argument"

Start with the AT-factorization and define

$$A_1{}^{ij} = (SDS^T)^{ij}\,, \qquad (S^i{}_j) \in \mathrm{SU}(8)$$

$$X^{ij} = (SS^T)^{ij} \qquad \Longrightarrow \qquad X^{ij} = X^{ji}\,, \qquad (X^{ij}) \in \mathrm{SU}(8)$$

Then

$$e^{3A}\left[3(\mathcal{A}')^2 + \frac{3}{4}\,g^2\,|A_1{}^{ij}|^2\right] = \frac{3}{8}\,e^{3A}\left|\mathcal{A}'\,X_{ij} - \sqrt{2}\,g\,A_{1\,ij}\right|^2 + \frac{3}{4\sqrt{2}}\,g\,\mathcal{A}'\,e^{3A}\left[X_{ij}\,A_1{}^{ij} + X^{ij}\,A_{1\,ij}\right]$$

$$e^{3A}\left[-\frac{1}{96}\,\mathcal{A}_r{}^{ijkl}\mathcal{A}_{r\,ijkl} - \frac{g^2}{24}|A_{2i}{}^{jkl}|^2\right] = -\frac{1}{96}\,e^{3A}\left|\mathcal{A}_r{}^{ijkl} + 2\,g\,X^{im}\,A_{2m}{}^{jkl}\right|^2 + \frac{g}{48}\,e^{3A}\left[\mathcal{A}_{r\,ijkl}\,X^{im}\,A_{2m}{}^{jkl} + \mathcal{A}_r{}^{ijkl}\,X_{im}\,A_2{}^m{}_{jkl}\right]$$

Using

$$D_\mu A_1{}^{ij} = \frac{1}{12\sqrt{2}}\left(A_2{}^i{}_{klm}\mathcal{A}_\mu{}^{jklm} + A_2{}^j{}_{klm}\mathcal{A}_\mu{}^{iklm}\right) \qquad \text{[de Wit-Nicolai '82]}$$

the cross-terms can be rewritten as a boundary term

$$(\,\ldots\,) = \frac{g}{2\sqrt{2}}\,\frac{\partial}{\partial_r}\mathrm{Tr}\left[e^{3A}\,D\right] = \frac{g}{2\sqrt{2}}\,\frac{\partial}{\partial_r}\mathrm{Tr}\left[e^{3A}\,\sqrt{A_1 A_1^\dagger}\right]$$

The $\mathcal{N} = 8$ Boundary Counterterm

$$S_{\text{s-ct}} = -\frac{1}{4L}\int d^3x\, e^{3r_0/L}\, \text{Tr}\sqrt{A_1 A_1^\dagger}$$

$$\boxed{L = \frac{1}{\sqrt{2}\,g}}$$

$$= \int d^3x\, e^{3r_0/L}\Big[-\frac{2}{L} - \frac{1}{96\,L}\,\phi_{ijkl}\phi^{ijkl} + \frac{1}{384\sqrt{2}\,L}\left(\phi_{ijkl}\phi_{ijmn}\phi^{klmn} + \text{c.c.}\right) + \dots\Big].$$

- Both the divergent and finite terms in S_B are cancelled at the boundary:

$$S_{\text{B}} + S_{\text{s-ct}} = \int d^3x dr\, e^{3A}\Big[$$

$$\frac{3}{8}\left|A'\,X_{ij} - \frac{1}{L}\,A_{1\,ij}\right|^2 - \frac{1}{96}\left|\mathcal{A}_r{}^{ijkl} + \frac{\sqrt{2}}{L}\,X^{im}\,A_{2m}{}^{jkl}\right|^2 - \frac{1}{96}\,g^{ab}\,\mathcal{A}_a{}^{ijkl}\mathcal{A}_{b\,ijkl}\Big]$$

where

$$A \sim \frac{r}{L} + O(e^{-2r/L}), \qquad |\,\dots\,|^2 \sim O(e^{-4r/L}) \qquad \Longrightarrow \qquad [\,\dots\,] \sim O(e^{-4r/L})$$

$$g^{ab} \sim O(e^{-2r/L}), \qquad \phi^{ijkl} \sim O(e^{-r/L})$$

The $\mathcal{N} = 8$ Boundary Counterterm

- The cubic counterterm is purely scalar ($\phi = \alpha + i\beta$)

$$\frac{1}{384\sqrt{2}\,L}\left(\phi_{ijkl}\phi_{ijmn}\phi^{klmn} + \text{c.c.}\right) = \frac{\sqrt{2}}{384\,L}\,\alpha^{ijkl}\alpha^{klmn}\alpha^{mnij}$$

which is a consequence of SO(8) identities, e.g., [de Wit '79]

$$\underbrace{\alpha^{mn[ij}\alpha^{kl]mn}}_{\text{self-dual}}\,, \qquad \underbrace{\beta^{mn[ij}\beta^{kl]mn}}_{\text{anti-self-dual}}$$

and/or the SO(8) branching rules

$$\mathbf{35}_i \otimes \mathbf{35}_i \longrightarrow \mathbf{1} + \mathbf{35}_i + \ldots\,, \qquad \mathbf{35}_i \otimes \mathbf{35}_j \longrightarrow \mathbf{35}_k + \ldots$$

- In the $\mathcal{N} = 1$ theories in DZF's talk, that can be obtained by a consistent truncation to either the $U(1)^3$ or $SU(3)\times U(1)^2$ invariant sectors, the natural counterterm (single field $z = A + iB$)

$$e^{K/2}(W + \overline{W}) \propto \ldots + (A^3 - 3\,AB^2)$$

is a representative of a family of counterterms allowed by the lower symmetry. They are all supersymmetric and have *the same scalar cubic term*!

Boundary Sources and $\mathcal{N} = 8$ Supersymmetry

$$\widetilde{S}_{\rm ren} \;=\; S_{\rm bulk} + S_{\rm s\text{-}ct} + S_{\chi\text{-}ct} + S_L$$

- $S_{\rm bulk}$ is the bulk action of $\mathcal{N} = 8$ gauged supergravity.
- $S_{\rm s\text{-}ct}$ is the scalar counterterm introduced above.
- $$S_{\chi\text{-}ct} \;=\; \frac{1}{24} \int d^3x\, e^{-3r_0} \left[\bar{\chi}^{ijk}\chi^{ijk} + \text{c.c.}\right]$$
  is the spin-1/2 counterterm.
- $$S_L \;=\; \frac{1}{48} \int d^3x\, \mathfrak{A}^{ijkl}(\vec{x})\alpha_{(1)}{}^{ijkl}(\vec{x})$$

$$\begin{aligned}\mathfrak{A}^{ijkl}(\vec{x}) \;&=\; -\lim_{r\to\infty} e^{-r/L}\,\Pi^{ijkl}(\vec{x},r) \\ &=\; -\frac{1}{L}\left[\alpha_{(2)}{}^{ijkl}(\vec{x}) + \frac{3}{4\sqrt{2}}\,\alpha_{(1)}{}^{mn[ij}(\vec{x})\alpha_{(1)}{}^{k]lmn}(\vec{x})\right]\end{aligned}$$

  is the conjugate of the scalar source, $\alpha_{(1)}{}^{ijkl}$.

The Legendre trasformed action $\widetilde{S}_{\rm ren}$ is on-shell invariant under the $\mathcal{N} = 8$ superconformal symmetry generated by the AdS$_4$ Killing spinors, ϵ^i/ϵ_i,

$$\epsilon^i(r,\vec{x}) \;=\; e^{r/2L}\zeta_+{}^i(\vec{x}) + e^{-r/2L}\zeta_-{}^i$$

$$\gamma^5\zeta_\pm{}^i \;=\; \zeta_\pm{}^i, \qquad \gamma^3\zeta_\pm{}^i \;=\; \mp\zeta_{\pm\,i}\,, \qquad \partial\!\!\!/\zeta_+{}^i \;=\; -\frac{3}{L}\,\zeta_{-\,i}$$

Highlights

Define

$$\Xi^{ijk} = \frac{1}{2}\left(\chi^{ijk} - \gamma^3 \chi_{ijk}\right), \qquad \Upsilon^{ijk} = \frac{1}{2}\left(\chi^{ijk} + \gamma^3 \chi_{ijk}\right)$$

$$\gamma^3\,\Xi^{ijk} = -\Xi_{ijk}, \qquad \gamma^3\,\Upsilon^{ijk} = \Upsilon_{ijk}$$

$$\Xi^{ijk} = e^{-3r/2L}\,\Xi_{(3/2)}{}^{ijk} + \ldots, \qquad \Upsilon^{ijk} = e^{-3r/2L}\,\Upsilon_{(3/2)}{}^{ijk} + \ldots$$

and rewrite the supersymmetry transformations for the boundary fields

$$\begin{aligned}
\delta\alpha_{(1)}{}^{ijkl} &= 8\,\bar{\zeta}_+{}^{[i}\Upsilon_{(3/2)}{}^{jkl]} + \ldots \\
\delta\beta_{(1)}{}^{ijkl} &= -8i\,\bar{\zeta}_+{}^{[i}\Xi_{(3/2)}{}^{jkl]} + \ldots \\
\delta\alpha_{(2)}{}^{ijkl} &= 8\left(\bar{\zeta}_-{}^{[i}\Xi_{(3/2)}{}^{jkl]} + \bar{\zeta}_+{}^{[i}\Upsilon_{(5/2)}{}^{jkl]}\right) + \ldots \\
\delta\beta_{(2)}{}^{ijkl} &= -8i\left(\bar{\zeta}_-{}^{[i}\Upsilon_{(3/2)}{}^{jkl]} + \zeta_+{}^{[i}\Xi_{(5/2)}{}^{jkl]}\right) + \ldots
\end{aligned}$$

$$\begin{aligned}
\delta\Xi_{(3/2)}{}^{ijk} = -\frac{2i}{L}\,\beta_{(1)}{}^{ijkl}\,\zeta_-{}^{l} - \frac{1}{L}\Big[\alpha_{(2)}{}^{ijkl} + \frac{3}{4\sqrt{2}}\,\alpha_{(1)}{}^{mn[ij}\alpha_{(1)}{}^{k]lmn} \\
+ \frac{3}{4\sqrt{2}}\,\beta_{(1)}{}^{mn[ij}\beta_{(1)}{}^{k]lmn} - iL\gamma^3\partial\!\!\!/\beta_{(1)}{}^{ijkl}\Big]\,\zeta_+{}^{l}
\end{aligned}$$

$$\begin{aligned}
\delta\Upsilon_{(3/2)}{}^{ijk} = \frac{2}{L}\,\alpha_{(1)}{}^{ijkl}\,\zeta_-{}^{l} - \frac{i}{L}\Big[-\beta_{(2)}{}^{ijkl} + \frac{3}{4\sqrt{2}}\,\alpha_{(1)}{}^{mn[ij}\beta_{(1)}{}^{k]lmn} \\
- \frac{3}{4\sqrt{2}}\,\beta_{(1)}{}^{mn[ij}\alpha_{(1)}{}^{k]lmn} - iL\gamma^3\partial\!\!\!/\alpha_{(1)}{}^{ijkl}\Big]\,\zeta_+{}^{l}
\end{aligned}$$

Highlights

- The sources $(\mathfrak{A}^{ijkl}(\vec{x}),\ \beta_{(1)}{}^{ijkl}(\vec{x}),\ \Xi_{(3/2)}{}^{ijk}(\vec{x}))$ form a closed multiplet on-shell. We need to use the spin-1/2 EOMs, e.g.,

$$\Upsilon_{(5/2)}{}^{ijk} = L\partial\!\!\!/\,\Xi_{(3/2)\,ijk} - \frac{1}{12\sqrt{2}}\eta_{ijkpqrlm}\left(\alpha_{(1)}{}^{npqr}\,\Upsilon_{(3/2)}{}^{lmn} - i\,\beta_{(1)}{}^{npqr}\,\Xi_{(3/2)}{}^{lmn}\right)$$

- The boundary terms in δS_{bulk} in can be quickly determined from

$$\delta\mathcal{L}_{\text{bulk}} = \bar{V}_i\epsilon^i + \bar{X}^{\mu}{}_i D_\mu\epsilon^i + \text{c.c.}\,.$$

and then using the bulk invariance. The result is

$$\delta S_{\text{bulk}} \;=\; \int d^3x\; e^{3r_0/L}\left[-\frac{1}{6}\mathcal{A}^{3\,ijkl}\bar{\epsilon}_i\chi_{jkl} - \frac{1}{12}\delta\bar{\chi}_{jkl}\gamma^3\chi^{jkl} + \text{c.c.}\right]$$

  - Using radiality

$$\int d^3x\; e^{3r_0/L}\left[\delta\bar{\chi}_{jkl}\gamma^3\chi^{jkl}\right] \;=\; \int d^3x\left[\delta\bar{\Xi}^{ijk}_{(3/2)}\Upsilon^{ijk}_{(3/2)} + \delta\bar{\Upsilon}^{ijk}_{(3/2)}\Xi^{ijk}_{(3/2)} + O(e^{-r0/L})\right]$$

    It combines with $\delta S_{\chi\text{-ct}}$. Both vanish when SOURCES $= 0$.

- For SOURCES $= 0$, using the "Bogomolny estimate"

$$\delta\widetilde{S}_{\text{ren}} = \delta S_{\text{bulk}} + \delta S_{\text{s-ct}} \;=\; \int d^3x\; O(e^{-r_0/L}) \quad\rightarrow\quad 0$$

- For SOURCES $\neq 0$,

$$\delta\widetilde{S}_{\text{ren}} \;=\; \int d^3x\left[-\frac{1}{3}\frac{\partial}{\partial x^a}\left(\alpha_{(1)}{}^{ijkl}\zeta_{+i}\gamma^a\Xi_{(3/2)}{}^{jkl}\right) + O(e^{-r_0/L})\right] \quad\rightarrow\quad 0$$

The Correlators

We want to use the AdS/CFT to compute the 2- and 3-point functions for $\Delta = 1$ operators $\mathcal{O}_{IJ}(\vec{x})$ in ABJM theory. Heuristically,

$$\mathcal{O}_{IJ} = \mathrm{Tr}\left[X_I X_J - \frac{1}{8}\delta_{IJ} X_K X_K\right]$$

It is more natural to work with the symmetric tensor representation of $\mathbf{35}_v$ – change from the SU(8) to SL$(8,\mathbb{R})$ basis

$$A^{IJ} = \frac{1}{96}(\Gamma_{IK})^{ij}(\Gamma_{JK})^{kl}\alpha^{ijkl}$$

The renormalized action for the scalars continued to the Euclidean signature reads

$$S_{\text{ren}} = \frac{1}{\kappa^2}\int d^4x\sqrt{g}\left[\frac{1}{4}\partial_\mu A^{IJ}\partial^\mu A^{IJ} - \frac{1}{2}A^{IJ}A^{IJ}\right]$$

$$+\frac{1}{\kappa^2}\int d^3x\, e^{3r_0}\left[\frac{1}{4}A^{IJ}A^{IJ} - \frac{1}{6\sqrt{2}}A^{IJ}A^{JK}A^{KI}\right] + O(A^4)$$

where $\kappa^2 = 1/8\pi G_4$ and $L = 1$. Near the boundary

$$A^{IJ}(r,\vec{x}) = e^{-r}A^{IJ}_{(1)}(\vec{x}) + e^{-2r}A^{IJ}_{(2)}(\vec{x}) + \cdots$$

The bulk fields with Dirichlet boundary data $A^{IJ}_{(1)}(\vec{x})$ are constructed using the usual bulk-boundary propagator

$$A^{IJ}(r,\vec{x}) = \int d^3y\, K_2(r,\vec{x};\vec{y}) A^{IJ}_{(1)}(\vec{y})\,, \qquad K_2(r,\vec{x};\vec{y}) \equiv \frac{1}{\pi^2}\frac{e^{-2r}}{\left(e^{-2r}+|\vec{x}-\vec{y}|^2\right)^2}$$

Substitute into the action

$$S_{\text{on-shell}}[A^{IJ}_{(1)}] = -\frac{1}{4\kappa^2}\int d^3x\, d^3y\, \frac{A^{IJ}_{(1)}(\vec{x})A^{IJ}_{(1)}(\vec{y})}{\pi^2|\vec{x}-\vec{y}|^4} - \frac{1}{6\sqrt{2}\kappa^2}\int d^3x\, A^{IJ}_{(1)}(\vec{x})A^{JK}_{(1)}(\vec{x})A^{KL}_{(1)}(\vec{x}) + O(A^4_{(1)})$$

This would suffice if $\mathcal{O}_{IJ}(\vec{x})$ had $\Delta = 2$. For $\Delta = 1$ and alternate quantization we must perform the Legendre transform [Klebanov-Witten '99]

$$\widetilde{S}_{\text{on-shell}}[\mathfrak{A}^{IJ}] = S_{\text{on-shell}}[A^{IJ}_{(1)}] + \frac{1}{2\kappa^2}\int d^3x\, \mathfrak{A}^{IJ}(\vec{x})A^{IJ}_{(1)}(\vec{x})$$

computed after extremizing the right hand side with respect to $A^{IJ}_{(1)}(\vec{x})$,

$$\mathfrak{A}^{IJ}(\vec{x}) \;=\; -\frac{\delta S_{\text{on-shell}}[A_1]}{\delta A_1(\vec{x})} = \frac{1}{\pi^2}\int d^3y\, \frac{A^{IJ}_{(1)}(\vec{y})}{|\vec{x}-\vec{y}|^4} - \frac{1}{\sqrt{2}} A^{K(J}_{(1)}(\vec{x})A^{I)K}_{(1)}(\vec{x}) + O(A^3_{(1)})$$

This must be solved for $A^{IJ}_{(1)}$ in terms of $\mathfrak{A}^{IJ}(\vec{x})$.

$$\mathfrak{A}^{IJ}(\vec{x}) \;=\; \frac{1}{\pi^2}\int d^3y\,\frac{A^{IJ}_{(1)}(\vec{y})}{|\vec{x}-\vec{y}|^4} - \frac{1}{\sqrt{2}}A^{K(J}_{(1)}(\vec{x})A^{I)K}_{(1)}(\vec{x}) + O(A^3_{(1)})$$

Convolute with $1/2\pi^2|\vec{z}-\vec{x}|^2$ and use

$$\int d^3x\,\frac{1}{2\pi^2|\vec{z}-\vec{x}|^2}\,\frac{1}{\pi^2|\vec{x}-\vec{y}|^4} = -\delta^{(3)}(\vec{z}-\vec{y})$$

shown by formal Fourier transform or better by holographic regularization. Then

$$A^{IJ}_{(1)}(\vec{x}) = -\int d^3y\,\frac{\mathfrak{A}^{IJ}(\vec{y})}{2\pi^2|\vec{x}-\vec{y}|^2} - \frac{1}{(2\pi)^3}\int d^3y\,d^3z\,\frac{\mathfrak{A}^{K(I}(\vec{y})\mathfrak{A}^{J)K}(\vec{z})}{|\vec{x}-\vec{y}||\vec{y}-\vec{z}||\vec{x}-vz|} + O(\mathfrak{A}^3)$$

and

$$\tilde{S}_{\text{on-shell}}[\mathfrak{A}^{IJ}] = -\frac{1}{8\pi^2\kappa^2}\int d^3x\,d^3y\,\frac{\mathfrak{A}^{IJ}(\vec{x})\mathfrak{A}^{IJ}(\vec{y})}{|\vec{x}-\vec{y}|^2}$$

$$+\frac{1}{48\sqrt{2}\,\pi^3\kappa^2 L}\int d^3x\,d^3y\,d^3z\,\frac{\mathfrak{A}^{IJ}(\vec{x})\mathfrak{A}^{JK}(\vec{y})\mathfrak{A}^{KI}(\vec{z})}{|\vec{x}-\vec{y}||\vec{y}-\vec{z}||\vec{x}-\vec{z}|} + O(\mathfrak{A}^4)$$

Use $-\tilde{S}_{\text{on-shell}}[\mathfrak{A}^{IJ}]$ to compute connected correlators of $\mathcal{O}_{IJ}(\vec{x})$.

The Result

$$\tilde{S}_{\text{on-shell}}[\mathfrak{A}^{IJ}] = -\frac{1}{8\pi^2\kappa^2}\int d^3x\, d^3y\, \frac{\mathfrak{A}^{IJ}(\vec{x})\mathfrak{A}^{IJ}(\vec{y})}{|\vec{x}-\vec{y}|^2}$$
$$+\frac{1}{48\sqrt{2}\,\pi^3\kappa^2 L}\int d^3x\, d^3y\, d^3z\, \frac{\mathfrak{A}^{IJ}(\vec{x})\mathfrak{A}^{JK}(\vec{y})\mathfrak{A}^{KI}(\vec{z})}{|\vec{x}-\vec{y}||\vec{y}-\vec{z}||\vec{x}-\vec{z}|} + O(\mathfrak{A}^4)$$

The normalization of $\mathfrak{A}^{IJ}(\vec{x})$ vs the field theory sources is

$$\text{source for } \mathcal{O}_{IJ}(\vec{x}) \;=\; \frac{\mathcal{C}}{L}\mathfrak{A}^{IJ}(\vec{x})$$

Adjusting for this normalization (no sum)

$$\langle \mathcal{O}_{IJ}(\vec{x}_1)\mathcal{O}_{IJ}(\vec{x}_2)\rangle = \frac{C_2}{|\vec{x}_1-\vec{x}_2|^2},$$

$$\langle \mathcal{O}_{IJ}(\vec{x}_1)\mathcal{O}_{JK}(\vec{x}_2)\mathcal{O}_{KI}(\vec{x}_3)\rangle = \frac{C_3}{|\vec{x}_1-\vec{x}_2||\vec{x}_1-\vec{x}_3||\vec{x}_2-\vec{x}_3|}$$

$$C_2 \;=\; \frac{L^2}{16\pi^3 G_4 \mathcal{C}^2}, \qquad C_3 \;=\; -\frac{L^2}{64\sqrt{2}\pi^4 G_4 \mathcal{C}^3}$$

The normalization independent ratio

$$\frac{C_3^2}{C_2^3} \;=\; \frac{\pi G_4}{2L^2}$$

This should be reproduced by a field theory calculation in ABJM.

The Field Theory Calculation

[Jafferis '10], [Jafferis-Klebanov-Pufu-Safdi '11]
[Closset-Dumitrescu-Festuccia-Komargodski-Seiberg '12]

- In any $\mathcal{N} = 8$ SCFT in three dimensions, we have two point functions for canonically normalized energy momentum tensor

$$\langle T_{\mu\nu}(\vec{x}) T_{\rho\sigma}(0)\rangle = \frac{c_T}{64}(P_{\mu\rho}P_{\nu\sigma} + P_{\nu\rho}P_{\mu\sigma} - P_{\mu\nu}P_{\rho\sigma})\frac{1}{16\pi^2|\vec{x}|^2}$$

where $P_{\mu\nu} \equiv \eta_{\mu\nu}\partial^\lambda\partial_\lambda - \partial_\mu\partial_\nu$ and the $SO(8)$ R-symmetry current

$$\langle j^\mu_{IJ}(\vec{x}) j^\nu_{KL}(0)\rangle = \frac{c_T}{64}(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK})P^{\mu\nu}\frac{1}{16\pi^2|\vec{x}|^2}$$

where c_T is a constant that depends on the theory.

- By conformal and $SO(8)$ invariance, for $\Delta = 1$ scalar operators $\mathcal{O}_{IJ}(\vec{x})$ in $\mathbf{35}_v$ of $SO(8)$

$$\langle \mathcal{O}_{IJ}(\vec{x}_1)\mathcal{O}_{IJ}(\vec{x}_2)\rangle = \frac{c_2}{|\vec{x}_1 - \vec{x}_2|^2},$$

$$\langle \mathcal{O}_{IJ}(\vec{x}_1)\mathcal{O}_{JK}(\vec{x}_2)\mathcal{O}_{KI}(\vec{x}_3)\rangle = \frac{c_3}{|\vec{x}_1 - \vec{x}_2||\vec{x}_1 - \vec{x}_3||\vec{x}_2 - \vec{x}_3|}$$

By evaluating those correlators using supersymmetric localization for special choices of operators determined by the branching from $\mathcal{N} = 8$ to $\mathcal{N} = 2$, one finds

$$c_2 = \frac{c_T}{16(4\pi)^2}, \qquad c_3 = \frac{c_T}{16}\frac{1}{(4\pi)^3}, \qquad \frac{c_3^2}{c_2^3} = \frac{16}{c_T}$$

The Comparison

- From $\mathcal{N} = 8$ supergravity

$$\frac{C_3^2}{C_2^3} = \frac{\pi G_4}{2L^2}$$

- From $\mathcal{N} = 8$ SCFT

$$\frac{c_3^2}{c_2^3} = \frac{16}{c_T}$$

- For an $\mathcal{N} = 8$ SCFT with a holographic dual, c_T is a universal function of L and G_4,

$$c_T = \frac{32L^2}{\pi G_4}$$

[Chester-Lee-Pufu-Yacoby '14]

- The normalization of the sources can be fixed using the 2-point function

$$c_2 = C_2 \quad \Longrightarrow \quad \mathcal{C} = -\frac{1}{\sqrt{2}} \quad \Longrightarrow \quad c_3 = C_3$$

and the 3-point functions agree!

Conclusions

- Our puzzle has been solved.
- We have a new precision test of AdS_4/CFT_3.
- The 3-point correlators of $\Delta = 1$ scalar operators $\mathcal{O}(\vec{x})$ arise from a finite boundary counterterm in the renormalized supergravity action.
  - This may be generic for 3d SCFTs with holographic duals – the 3-point functions computed from Witten diagrams with the bulk vertex AAA or $A\partial_\mu A\partial_\mu A$ diverge when $d \to 3$ and $\Delta \to 1$.

    [Freedman, Mathur, Matusis, Rastelli '99]
- The relevant counterterm can be obtained by a Bogomolny type argument and/or by requiring supersymmetry of the Legendre transformed renormalized on-shell action.
- The use of Legendre transform and alternate quantization has been clarified in an explicit example.
- The importance of boundary (counter-)terms have been appreciated since the early days of AdS/CFT, see, e.g.,
  [Henningson-Sfetsos '98], [Mueck-Viswanathan '98], [Arutyunov-Frolov '99], [Henneaux '99], . . . ,
  [Bianchi-Freedman-Skenderis '01-'02], . . . ,
  [Belyaev-van Nieuwenhuizen '08], [Grumiller-van Nieuwenhuizen '08], . . . ,
  [Andrianopoli-D'Auria '14], . . .