FIELD THEORY PARTITION FUNCTIONS FROM SUPERGRAVITY

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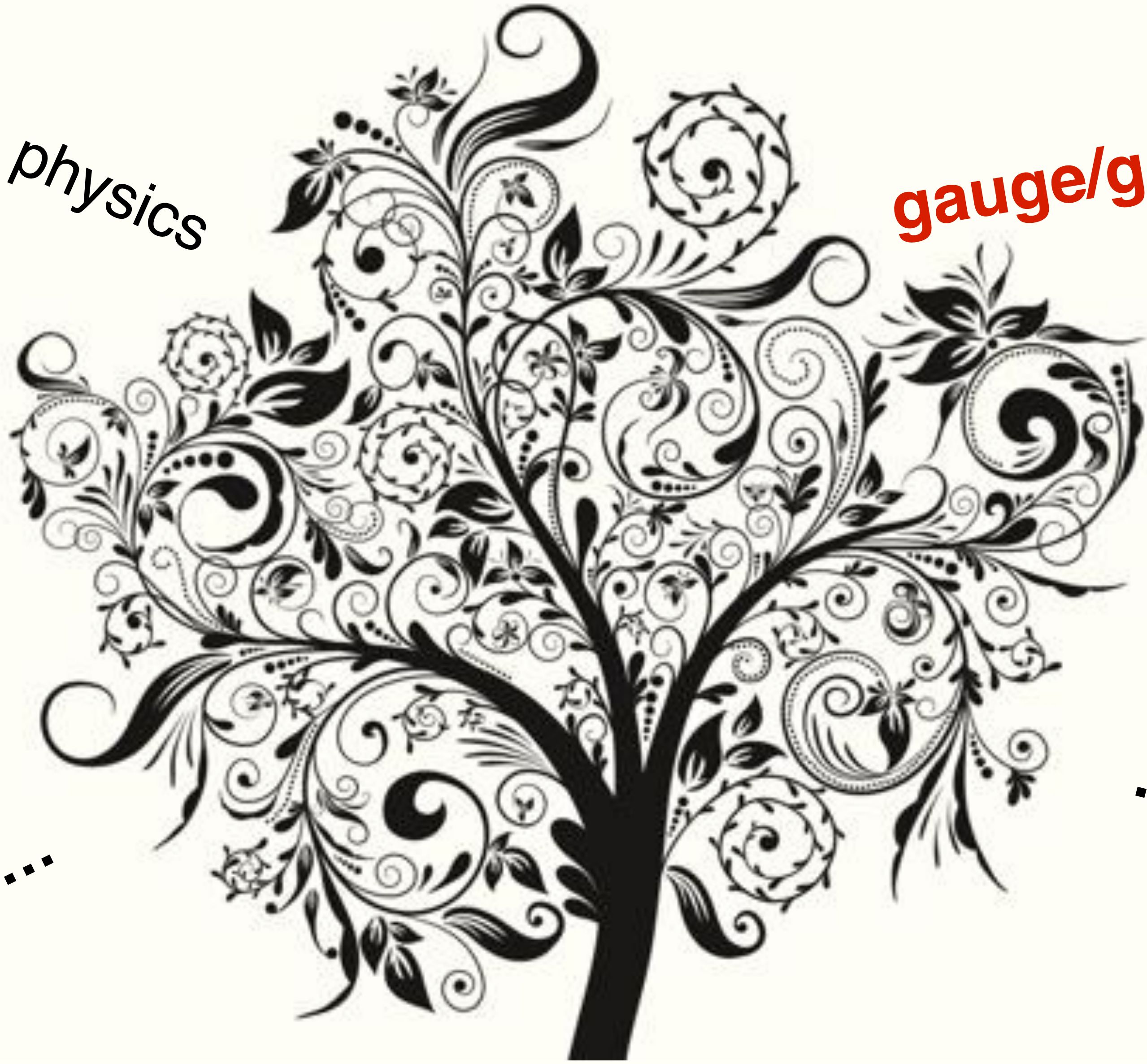
LPTHE — Université Paris 6

Supergravity at 40

26/10/2016

The many branches of supergravity



particle physics

gauge/gravity duality

black holes

cosmology

geometry

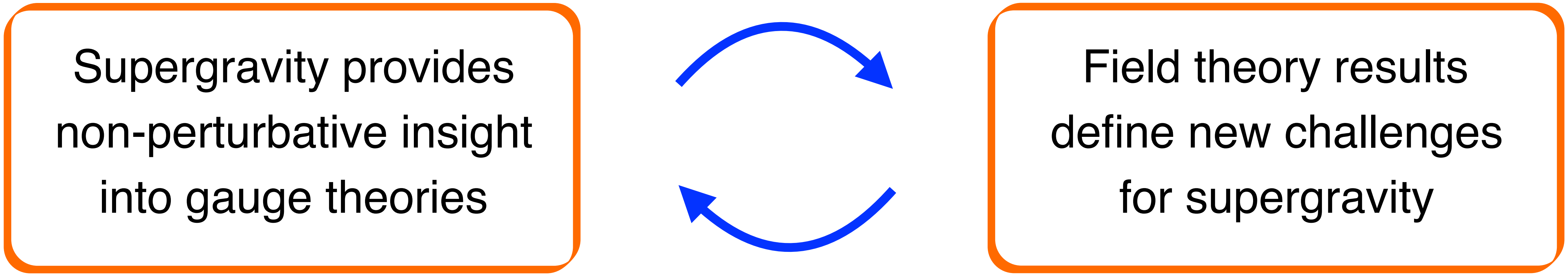
……

……

supergravity

Supergravity is the low-energy limit of string theory
and as such plays a crucial role in the gauge/gravity duality

gauge/gravity duality



Supergravity provides non-perturbative insight into gauge theories

Field theory results define new challenges for supergravity

interplay has led to

- progress in either field
- better understanding of the duality

see also other talks in this conference:
Bobev, Freedman, Martelli, Pilch, ...

In this talk

- Summarize recent results for N=1 SCFT's on a curved four-manifold in part obtained using the technique of **localization**

➜ Opportunity to explore holography on generally curved boundaries

- Focus on the partition function on $M_4 \sim S^1 \times S^3$

- Explain how this poses a puzzle for holography

- Present the solution

Based on: 1606.02724 with Benetti Genolini, Martelli, Sparks + to appear

+ previous work with Assel, Di Pietro, Komargodski, Lorenzen, Martelli

Recent field theory results

- Consider a 4d, N=1 SCFT. Couple it to background fields :

$$S[\Phi; \underset{\substack{\uparrow \\ \text{background} \\ \text{gauge field}}}{A_\mu}, \underset{\substack{\uparrow \\ \text{background} \\ \text{metric on } M_4}}{g_{\mu\nu}}] = S_0[\Phi] + \int (A_\mu \underset{\substack{\uparrow \\ \text{R-current}}}{j^\mu} + g^{\mu\nu} \underset{\substack{\uparrow \\ \text{energy-momentum tensor}}}{T_{\mu\nu}} + \ldots)$$

At least one supercharge is preserved iff M_4 is equipped with a complex structure and a Hermitian metric

Klare, Tomasiello, Zaffaroni;
Dumitrescu, Festuccia, Seiberg

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At least one supercharge is preserved iff M_4 is equipped with a complex structure and a Hermitian metric

Klare, Tomasiello, Zaffaroni;
Dumitrescu, Festuccia, Seiberg

- A priori, partition function depends on all background fields:

$$Z[A_\mu, g_{\mu\nu}] = \int \mathcal{D}\Phi \, e^{-S[\Phi; A_\mu, g_{\mu\nu}]}$$

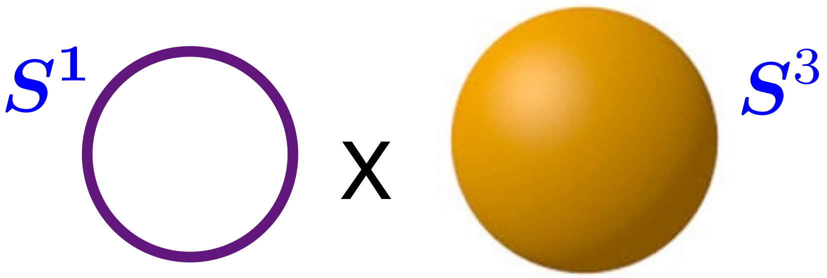
- Supersymmetry implies that Z does not depend on Hermitian metric and is a holomorphic function of the complex structure parameters

Closset, Dumitrescu, Festuccia, Komargodski

susy Ward identities : $\delta_{\text{Hermitian}}(\log Z) = 0$

Recent field theory results

- $M_4 \sim S^1 \times S^3$ ⟷ complex Hopf surfaces $\mathcal{H}_{p,q}$ with complex structure parameters p, q



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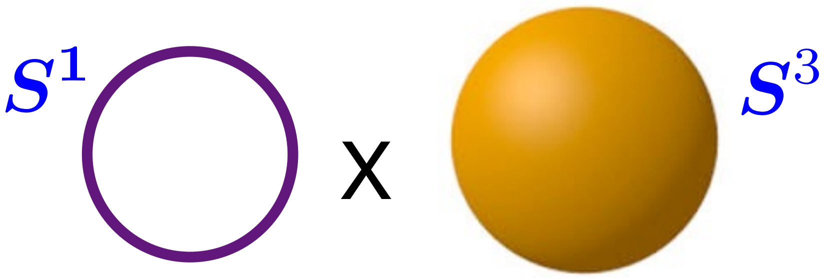


- Localization yields: $Z[\mathcal{H}_{p,q}] = \mathrm{e}^{-\mathcal{F}(p,q)}\mathcal{I}(p\,,q)$ Assel, D.C., Martelli

prefactor

superconformal index

$$\mathcal{I}(p,q) = \mathrm{tr}\left[(-1)^F p^{J+J'-\frac{R}{2}} q^{J-J'-\frac{R}{2}}\right]$$

fugacities

only depends on p, q

✔ susy Ward identities

supersymmetric Casimir energy

$$Z[\mathcal{H}_{p,q}] = e^{-\mathcal{F}(p,q)} \mathcal{I}(p\,,q)$$

superconformal index

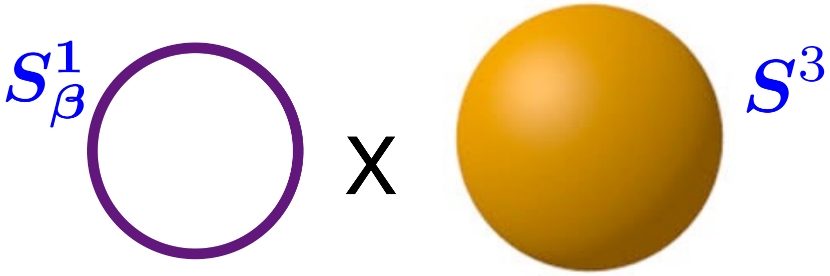
$$\mathcal{F}(p,q) = \beta E_{\mathrm{susy}}(b_1, b_2) \qquad p = e^{-\beta b_1}\,, \quad q = e^{-\beta b_2}$$

$$E_{\mathrm{susy}} = \frac{2}{3}(b_1+b_2)(a-c) + \frac{2}{27}\frac{(b_1+b_2)^3}{b_1 b_2}(3\,c - 2\,a)$$

supersymmetric Casimir energy

a, c central charges

- $$Z = \int \mathcal{D}\phi\, e^{-S[\phi]} = \mathrm{Tr}\, e^{-\beta H}$$



$Z \to e^{-\beta E_{\mathrm{susy}}}$ as $\beta \to \infty$ (projects to ground state)

- superalgebra implies the BPS condition $E_{\mathrm{susy}} = \frac{1}{r}\langle R \rangle$ on round $S^1_\beta \times S^3_r$

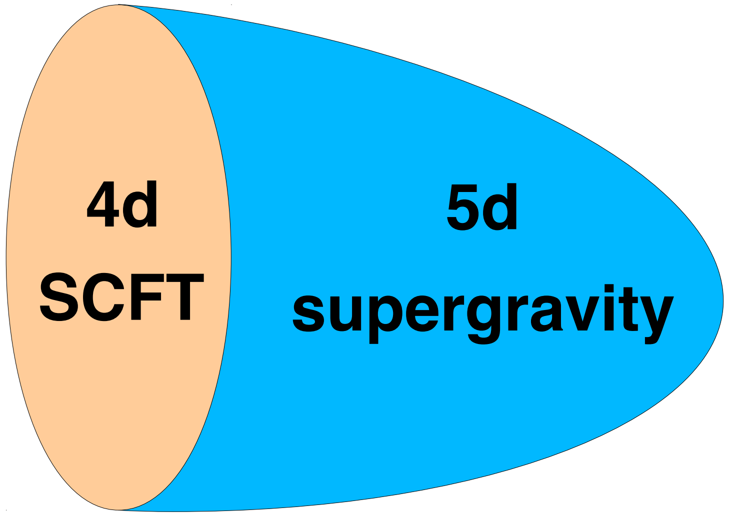
➔ **vacuum charged!**

gauge/gravity duality

QFT

supergravity
string theory

Holographic dictionary



M_4	$=$	∂M_5
QFT background fields		boundary conditions for gravity multiplet
$g^{\text{backgnd}},\ A^{\text{backgnd}}$		boundary values of $g^{\text{bulk}},\ A^{\text{bulk}}$
at large N_c $\quad -\log Z[M_4]$	$=$	$S_{\text{on-shell}}[M_5]$
central charges $\quad c = a$	$=$	$\pi^2 \ell^3 / \kappa_5^2$ 5d grav. coupling
Casimir energy		grav. energy of a solution dual to CFT vacuum
R-charge		charge of the solution under graviphoton

match??

Holographic Casimir energy

$$Z[\mathcal{H}_{p,q}] \;=\; \mathrm{e}^{-\mathcal{F}(p,q)}\mathcal{I}(p\,,q)$$

superconformal index

$$\mathcal{F}(p,q) = \beta E_{\mathrm{susy}}(b_1\,,b_2)$$

$$E_{\mathrm{susy}} = \frac{2}{3}\,(b_1+b_2)\,(a-c) + \frac{2}{27}\,\frac{(b_1+b_2)^3}{b_1b_2}(3\,c-2\,a)$$

dominates Z at large Nc ➜ **prediction for dual supergravity solutions**

- There should be a family of susy solutions with $\partial M_5 = \mathcal{H}_{p,q}$ such that :

$$S_{\mathrm{5d\;sugra}}[M_5] \;=\; \frac{2}{27}\beta\frac{(b_1+b_2)^3}{b_1b_2}\frac{\pi^2\ell^3}{\kappa_5^2}$$

$$c = a = \pi^2\ell^3/\kappa_5^2$$

Holographic Casimir energy

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$$c = a = \pi^2\ell^3/\kappa_5^2$$

$b_1 = b_2 = \dfrac{1}{r}$ round $S^1_\beta \times S^3_r$

Holographic renormalization

On-shell action a priori divergent

$$\boldsymbol{S = \frac{1}{2\kappa_5^2}\int_{M_5} d^5x\sqrt{g}\left(R[g] - F^2 + \frac{12}{\ell^2}\right) - A\wedge F\wedge F}$$

5d supergravity action $\boldsymbol{S_{\mathbf{bulk}}}$

Holographic renormalization

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5d supergravity action $S_{\mathbf{bulk}}$

$$+\frac{1}{\kappa_5^2}\int_{\partial M_5} d^4x\sqrt{h}\left(K - \frac{3}{\ell} - \frac{\ell}{4}R[h]\right)$$

divergent counterterms

Holographic renormalization

On-shell action a priori divergent

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divergent counterterms

$$+ \frac{1}{\kappa_5^2}\int_{\partial M_5} d^4x\sqrt{h}\left(\lambda_1 R_{ijkl}[h]^2 + \lambda_2 R_{ij}[h]^2 + \lambda_3 R[h]^2 + \lambda_4 F_{ij}^2\right)$$

finite boundary terms ➜ parameterize different schemes

Holographic renormalization

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finite boundary terms ➜ parameterize different schemes

- For round $S^1_\beta \times S^3_r$ the most natural candidate is pure Anti de Sitter space

Choosing $\lambda_1 = \lambda_2 = \lambda_3 = 0$

$$S = \frac{3}{4}\frac{\beta}{r}\frac{\pi^2\ell^3}{\kappa_5^2} \neq \frac{16}{27}\frac{\beta}{r}\frac{\pi^2\ell^3}{\kappa_5^2}$$

does **NOT match** the susy QFT result!

…could think about adjusting the λ’s … but …

Holographic renormalization

- $\frac{1}{\sqrt{g^{\text{bdy}}}} \frac{\delta S}{\delta A_i^{\text{bdy}}} = j^i$ holographic R-current ➜ holographic R-charge $\langle R \rangle$

however unavoidably for $\mathbf{AdS_5}$: $\langle R \rangle = 0$

➜ $\langle E \rangle \neq \frac{1}{r} \langle R \rangle$ BPS relation violated

unless one tunes λ's such that

$\langle E \rangle = \frac{1}{r} \langle R \rangle = 0$ misses Casimir energy

…something is not working here

New supergravity solutions

Benetti Genolini, D.C., Martelli, Sparks

We constructed a very general AlAdS solution perturbatively near the boundary

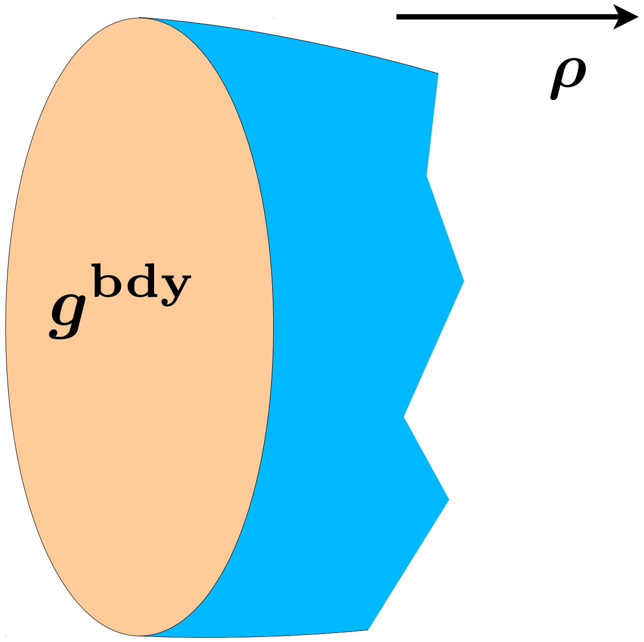
$$g^{\mathrm{bulk}} = \frac{\mathrm{d}\rho^2}{\rho^2} + \frac{1}{\rho^2}\left[g^{\mathrm{bdy}} + g^{(2)}\rho^2 + \left(g^{(4)} + \tilde{g}^{(4)}\log\rho^2\right)\rho^4 + \ldots\right]$$

- boundary metric:

$$g^{\mathrm{bdy}} = \mathrm{d}\tau^2 + (\mathrm{d}\psi + a)^2 + 4\mathrm{e}^{w}\mathrm{d}z\mathrm{d}\bar{z}$$

with $\mathrm{d}a = \mathrm{i}\,u\,\mathrm{e}^{w}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ $\quad u(z,\bar{z})$, $w(z,\bar{z})$ arbitrary

describes $S^1 \times S^3$ topology (and more)



- graviphoton field A^{bulk} also determined

$$A^{\mathrm{bdy}} = -\frac{1}{\sqrt{3}}\left[\frac{\mathrm{i}}{8}u\,\mathrm{d}\tau + \frac{1}{4}u(\mathrm{d}\psi + a) + \frac{\mathrm{i}}{4}(\partial_{\bar{z}}w\,\mathrm{d}\bar{z} - \partial_z w\,\mathrm{d}z) + \boxed{\gamma\mathrm{d}\psi}\right]$$

- 4 free subleading functions $k_1(z,\bar{z})$, $k_2(z,\bar{z})$, $k_3(z,\bar{z})$, $k_4(z,\bar{z})$

New supergravity solutions

- non-trivial:

susy involves solving 6th-order equation for auxiliary Kähler metric

$$\nabla^2\left(\frac{1}{2}\nabla^2 R+\frac{2}{3}R_{pq}R^{pq}-\frac{1}{3}R^2\right)+\nabla^m(R_{mn}\partial^n R) \;=\; 0$$

D.C., Lorenzen, Martelli

⚠ bulk metric has complex components (but real in Lorentzian signature)

⚠ on-shell action gauge-dependent due to Chern-Simons term $\int A\wedge F\wedge F$

➜ crucial to fix the gauge properly

- Killing spinors independent of time
- gauge field globally well-defined

Supersymmetric holographic renormalization

Vary the boundary data keeping complex structure on ∂M fixed

⟷ vary the functions $u(z,\bar{z})$, $w(z,\bar{z})$ with globally def. variations

$$\delta S = \int_{M_4} \mathrm{d}^4 x \sqrt{g^{\mathrm{bdy}}} \left(\tfrac{1}{2} T^{ij} \delta g^{\mathrm{bdy}}_{ij} + j^i \delta A^{\mathrm{bdy}}_i \right)$$

⚠ There is no choice of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $\delta_u S = 0 = \delta_w S$

➜ **Holographic renormalization violates field theory susy Ward identities!**

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➜ Holographic renormalization violates field theory susy Ward identities!

new boundary term ΔS_{new} such that $S_{\mathrm{susy}} = S + \Delta S_{\mathrm{new}}$ satisfies $\delta S_{\mathrm{susy}} = 0$

$$\Delta S_{\mathrm{new}}[M_4] = \frac{1}{\kappa_5^2} \int_{M_4} (\mathrm{i} A^{\mathrm{bdy}} \wedge \Phi + \Psi)$$

S is taken with

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

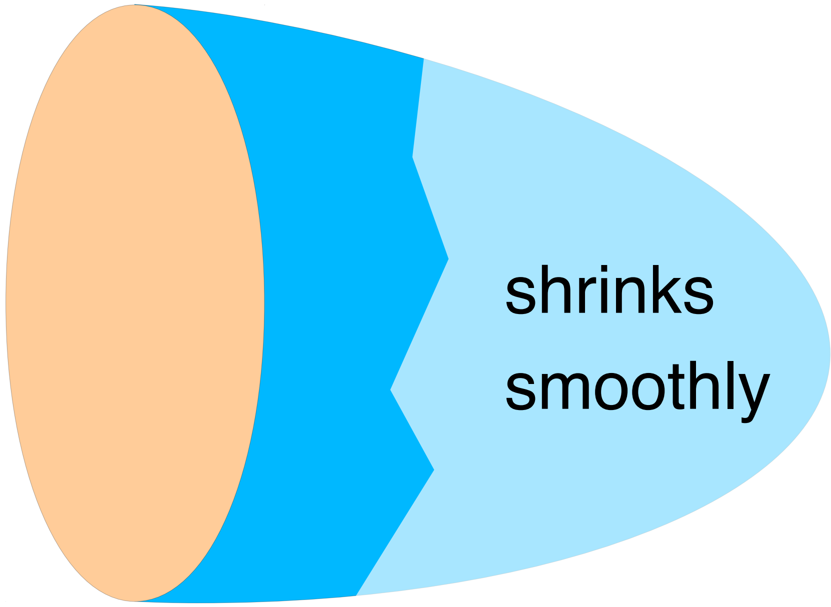
$$\Phi = \tfrac{1}{72\sqrt{3}} \left(u^3 + 4u\Box w \right) \mathrm{i}\,\mathrm{e}^w \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge (2\,\mathrm{d}\psi + \mathrm{i}\,\mathrm{d}\tau)$$

$$\Psi = \tfrac{1}{2^{11}3^2} \left(-19u^4 - 48u^2 \Box w \right) \mathrm{d}^4 x \sqrt{g^{\mathrm{bdy}}}$$

Supersymmetric holographic renormalization

We propose that $S_{\text{susy}}[M_5] = S[M_5] + \Delta S_{\text{new}}[M_4]$

should be identified with the SCFT $-\log Z_{\text{susy}}[M_4]$



Although we don't know the solution in the interior, under a topological assumption we can actually evaluate the on-shell action

- S_{bulk} reduces to a boundary term
- 4 subleading functions $k_1(z,\bar{z}),\ k_2(z,\bar{z}),\ k_3(z,\bar{z}),\ k_4(z,\bar{z})$ drop out !

Supersymmetric holographic renormalization

field theory prediction: $S = \beta E_{\rm susy} = \frac{2}{27}\beta\frac{(b_1+b_2)^3}{b_1 b_2}\frac{\pi^2\ell^3}{\kappa_5^2}$

$$S_{\rm susy} = S + \Delta S_{\rm new} = -\frac{\gamma^2}{27}\int_{M_4}{\rm d}^4x\,\sqrt{g^{\rm bdy}}\,\Box w\,\frac{\ell^3}{\kappa_5^2}$$

Supersymmetric holographic renormalization

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$$8\pi^2\beta\,\frac{b_1+b_2}{b_1 b_2}$$

$$S_{\text{susy}} = S + \Delta S_{\text{new}} = -\frac{\gamma^2}{27}\int_{M_4}\mathrm{d}^4x\,\sqrt{g^{\text{bdy}}}\,\Box w\,\frac{\ell^3}{\kappa_5^2}$$

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from $A^{\text{bdy}} = \ldots + \gamma\, \mathrm{d}\psi$

$$\gamma = \tfrac{1}{2}(b_1+b_2)$$

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$$= \frac{2}{27}\beta\frac{(b_1+b_2)^3}{b_1 b_2}\frac{\pi^2\ell^3}{\kappa_5^2}$$

✓

also computed $\langle R\rangle$ via $\frac{1}{\sqrt{g^{\mathrm{bdy}}}}\frac{\delta S_{\mathrm{susy}}}{\delta A_i^{\mathrm{bdy}}} = j^i$

✓ BPS relation

What did we learn?

- Standard holographic renormalization in 5d violates susy
- Identified boundary terms $\Delta S_{\rm new}$ that restore susy Ward identities
- Constructed asymptotic solutions such that

$$S_{\rm susy} = S + \Delta S_{\rm new} = \frac{2}{27}\beta\frac{(b_1+b_2)^3}{b_1 b_2}\frac{\pi^2\ell^3}{\kappa_5^2}$$

✔ field theory

? $\Delta S_{\rm new}$ in covariant form?

? which boundary fields involved?

First principle derivation such that bulk + boundary action is supersymmetric?

Where are we heading to?

Directions

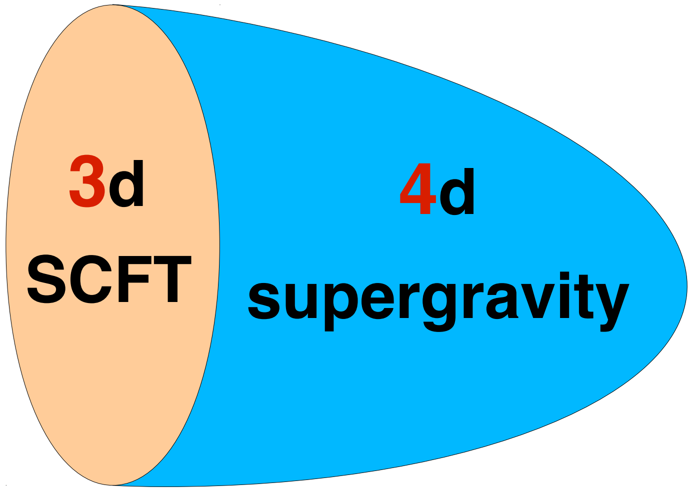
- Refined tests of gauge/gravity duality
- Clarify role of boundary terms in supergravity
- Revisit the path integral of supergravity using localization methods

... thank you !

Extra slides

A similar story in one dimension less

- We considered
  pure gauged Euclidean N = 2 supergravity in 4d
  and studied
  on-shell action as a function of boundary data



- We found that:
  - supersymmetric Ward identities **are** satisfied
    with **standard** holographic counterterms ➜ **no new terms required**
  - on-shell action only depends on a “transverse holomorphic foliation”
    of the boundary geometry ✔ field theory
  - for certain solutions with self-dual graviphoton, the on-shell action can be evaluated explicitly and matches the field theory partition function ✔

Supersymmetric regularization

- need to renormalize UV divergences

  ➜ crucial to use a regularization scheme that preserves supersymmetry

  - One way to assess this : check susy Ward identities at the end

- Even if regularization is supersymmetric, there may be ambiguities,

  ➜ important to classify them in order to extract physically meaningful result

Example

Gerchkovitz, Gomis, Komargodski

Partition function of N=2 SCFT's on S^4 as a function of marginal couplings τ, $\bar{\tau}$ computes the Kähler potential $K(\tau, \bar{\tau})$ on the conformal manifold.
Ambiguous by $F(\tau) + \overline{F}(\bar{\tau})$: correspond to Kähler transformations.

The importance of being supersymmetric

- $Z_{\mathrm{non-susy}}$ is ambiguous due to local counterterm :

$$-\log Z_{\mathrm{non-susy}} + \underbrace{b \int d^4x \sqrt{g}\, R^2}_{\neq 0 \text{ on } S^1 \times S^3}$$

➜ Casimir energy on $S^1 \times S^3$ is scheme-dependent

- **In susy case :**

ambiguities are gauge-invariant (new minimal) supergravity actions of dim = 4

◆ they are all F-terms

◆ F-terms vanish on susy backgrounds with 2 supercharges of opposite R-charge

$$-\log Z_{\mathrm{susy}} + \underbrace{b \int d^4x \sqrt{g}\, \left[(R + 6V_\mu V^\mu)^2 - 8F_{\mu\nu}F^{\mu\nu}\right] + \text{fermions}}_{= 0 \text{ on } S^1 \times S^3\,!}$$

The importance of being supersymmetric

- ✤ If an N=1 SCFT is defined on a supersymmetric background preserving two Euclidean supercharges, and using new minimal sugra

- ✤ then the dependence of the supersymmetric partition function on the background is free of ambiguities

➜ $$E_{\text{susy}} = \frac{2}{3}\,(b_1 + b_2)\,(a - c) + \frac{2}{27}\,\frac{(b_1 + b_2)^3}{b_1 b_2}(3\,c - 2\,a)$$

is an intrinsic observable of the SCFT