Modular Properties of Superstring Scattering Amplitudes

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Supergravity @ 40

GGI, Florence, October 26 2016

GENERAL SETTING

TO WHAT EXTENT DO DUALITY AND SUPERSYMMETRY CONSTRAIN THEORIES WITH A LARGE AMOUNT OF SUPERSYMMETRY? e.g. Maximal supergravity/Type II string theory

THE LOW ENERGY EXPANSION OF STRING AMPLITUDES

Consider narrowly-focused aspects of the low energy expansion of closed string theory obtained from maximally supersymmetric closed string scattering amplitudes.

- EXPLICIT FEATURES OF LOW ORDER TYPE II STRING PERTURBATION THEORY

  Modular invariants of Riemann surfaces

  Mathematical connections to **MULTIPLE-ZETA VALUES** and their **ELLIPTIC GENERALISATIONS**

With: Eric D'Hoker; Pierre Vanhove; Omer Gurdogan

Recent papers 1502.06698 1509.00363 1512.06779 1603.00839

PART OF A LARGER PROGRAMME INVESTIGATING

- NON-PERTURBATIVE FEATURES OF STRING AMPLITUDES

  Constraints imposed by SUSY, Duality, Unitarity

  Connects perturbative with non-perturbative effects

  MODULAR FORMS; AUTOMORPHIC FORMS FOR HIGHER-RANK GROUPS; ….

  Coefficients of BPS interactions encoding BPS microstate-counting (See also: Bossard+ Pioline)

earlier work: Stephen Miller; Don Zagier; Boris Pioline; Jorge Russo; Rudolfo Russo; Carlos Mafra; Oliver Schlotterer; Anirban Basu; Sav Sethi, Michael Gutperle, ……

Four-Graviton Scattering in Type IIB String Theory

$$A^{(4)}(\epsilon_r, k_r; \Omega) = \mathcal{R}^4\, T^{(4)}(s,t,u;\Omega)$$

$s = -2\, k_1 \cdot k_2$
$t = -2\, k_1 \cdot k_4$
$u = -2\; k_1 \cdot k_3$

$\mathcal{R}$ linearized curvature $\sim\; k_\mu\, k_\nu\, \epsilon_{\rho\sigma}$

One complex modulus $\quad \Omega = \Omega_1 + i\Omega_2 \quad \Omega_2 = \frac{1}{g} = e^{-\phi}$

inverse string coupling constant

Symmetric function of Mandelstam invariants s, t, u (with $s+t+u=0$).

Has an expansion in power series of $\sigma_2 = s^2 + t^2 + u^2$ and $\sigma_3 = s^3 + t^3 + u^3$

(NON-ANALYTIC PIECES ARE ESSENTIAL, BUT WILL BE IGNORED IN THIS TALK)

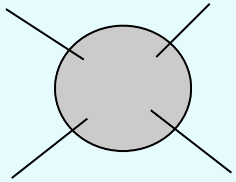
$$T(s,t,u;\Omega) = \sum_{p,q} \mathcal{E}_{(p,q)}(\Omega)\, \sigma_2^p\, \sigma_3^q \quad \to \sim s^{2p+3q} + \ldots$$

Coefficients are $SL(2,\mathbb{Z})$ -invariant functions of scalar fields (moduli, or coupling constants).

To what extent can we determine these coefficients?

Boundary data: String Perturbation Theory $\quad \Omega_2 \to \infty \quad (g \to 0)$

TREE-LEVEL ("VIRASORO" AMPLITUDE)



$$A_0^{(4)}(\epsilon_r, k_r) = g^{-2}\,\mathcal{R}^4\, T_0^{(4)}(s,t,u)$$

$\sigma_n = s^n + t^n + u^n$

$$T_0^{(4)} = \frac{1}{stu}\,\frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' t)\,\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\,\Gamma(1+\alpha' t)\,\Gamma(1+\alpha' u)} \quad = \frac{3}{\sigma_3}\exp\left[\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}{\alpha'}^{2n+1}\sigma_{2n+1}\right]$$

Tree-level SUPERGRAVITY

R^4 $\quad d^4R^4$ $\quad d^6R^4$ $\quad d^8R^4$

$s^k R^4 \sim d^{2k} R^4$

$$= \frac{3}{\sigma_3} + 2\zeta(3)\,{\alpha'}^3 + \zeta(5)\,{\alpha'}^5\sigma_2 + \frac{2\,\zeta(3)^2}{3}\,{\alpha'}^6\,\sigma_3 + \frac{\zeta(7)}{2}\,{\alpha'}^7\,\sigma_2^2$$
$$+ \frac{2\,\zeta(3)\,\zeta(5)}{3}\,{\alpha'}^8\,\sigma_2\,\sigma_3 + \frac{\zeta(9)}{4}\,{\alpha'}^8\,\sigma_2^3 + \frac{2}{27}(2\,\zeta(3)^2 + \zeta(9))\,{\alpha'}^9\,\sigma_3^2 + \ldots$$

$d^{10}R^4$ $\quad d^{12}R^4$

$\sigma_2 = s^2 + t^2 + u^2$

$\sigma_3 = s^3 + t^3 + u^3$

INFINITE SERIES of $d^{2k}R^4$ terms. COEFFICIENTS ARE POWERS OF **ODD RIEMANN** ζ **VALUES** WITH RATIONAL COEFFICIENTS

Generalisation to N-particle scattering involves **Multiple Zeta Values**.

ZETA VALUES AND MULTIPLE-ZETA VALUES

VERY BRIEF REVIEW

ZETA VALUES:

- Special values of POLYLOGARITHMS $$Li_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a} \qquad \zeta(a) = Li_a(1)$$

  Even zeta values $\zeta(2n) = c_n \, \pi^{2n}$ Odd zeta values $\zeta(2n+1)$ transcendental?

MULTI-ZETA VALUES (MZV's)

- Special values of MULTIPLE POLYLOGARITHMS $$Li_{a_1,\ldots,a_r}(z_1,\ldots,z_r) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} \left(\frac{z_\ell}{k_\ell}\right)^{a_\ell}$$

$$\zeta(a_1,\ldots,a_r) = Li_{a_1,\ldots,a_r}(1,\ldots,1)) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} k_\ell^{-a_\ell}$$

"weight" $w = \sum_{\ell=1}^{r} a_\ell$ "depth" r

- MZV are numbers with algebraic properties inherited from the algebraic properties of multiple polylogarithms – "STUFFLE" and "SHUFFLE" relations.

  e.g. first non-trivial (irreducible) case is weight $w = 8$

$$350\,\zeta(3,5) = 875\,\zeta(6,2) + 240\,\zeta(2)^4 - 1400\,\zeta(3)\,\zeta(5)$$

- THE DIMENSION d_w OF THE SUBSPACE OF MZV'S OF WEIGHT w OVER $\mathbb{Q}$

$$\sum_{w=0}^{\infty} d_w\, x^w = \frac{1}{1-x^2-x^3}$$

N-particle tree amplitudes

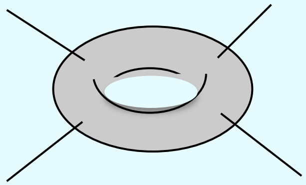
OPEN-STRING TREES: For $N>4$ coefficients of higher derivative interactions of order ${\alpha'}^n$
(Yang-Mills) are multiple zeta values with weight n (Stieberger, Broedel, Mafra, Schlotterer)

CLOSED-STRING TREES: For $N>4$ coefficients are *single-valued* MZV's (svMZV's) (Brown)
(gravity) (Schlotterer, Stieberger)

- Special values of **single-valued** multiple polylogarithms – NO MONODROMIES (generalisations of BLOCH-WIGNER dilogarithm $\mathrm{Im}\,(\mathrm{Li}_2(z) + \log(1-z)\,\log|z|)$)
- Kills even zeta values $\zeta_{sv}(2n) = 0$ Also $\zeta_{sv}(2n+1) = 2\zeta(2n+1)$- ODD ZETA'S ONLY
- First non-trivial case is $\zeta_{sv}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$
  weight $w = 11$
- Role of the KLT construction?

HOW DOES THIS GENERALIZE TO HIGHER GENUS ??

Genus one



$$\mathcal{A}_1^{(4)}(\epsilon_r, k_r) = \frac{\pi}{16}\,\mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d\tau^2}{y^2}\,\mathcal{B}_1(s,t,u;\tau)$$

Integral over complex structure $\tau = x + iy$

$$\mathcal{B}_1(s,t,u;\tau) = \frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4} d^2z\,\exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right)$$

Vertex operator Corr. function

Green function

Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

(MBG, D'Hoker, Russo, Vanhove)

Expanding in a power series in momenta gives (with $\alpha' = 4$)

$$\frac{1}{w!}\,\frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4} d^2z_i \left(\sum_{0<i<j\le 4} s_{ij}\,G(z_i - z_j)\right)^{w} = \sum_i \sigma_2^{p_i}\,\sigma_3^{q_i}\, j^{(p_i,q_i)}(\tau)$$

$$\sum_i (2p_i + 3q_i) = w$$

Coefficients of higher derivative interactions

Modular invariants for surface

Feynman diagrams on toroidal world-sheet

Coefficients of higher derivative interactions:

(genus-one generalisation of the tree-level ζ values)

$$\Xi^{(p,q)} = \int_{\mathcal{M}_1} \frac{d^2\tau}{y^2}\, j^{(p,q)}(\tau)$$

“Modular Graph Functions”

(D’Hoker, MBG, Vanhove)

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$j^{(p,q)}(\tau)$ is sum of world-sheet Feynman diagrams.

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

Each of these is a modular function - invariant under $SL(2,\mathbb{Z})$

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

The Green function on a torus of complex structure $\tau = x + iy$:

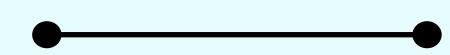
$$G(z) = -\ln\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{\pi}{2y}(z - \bar{z})^2 \qquad z = u + \tau v$$

doubly periodic function

$$= \sum_{(m,n)\neq(0,0)} \hat{G}(m,n) e^{2\pi i(mu - nv)} + 2\ln\left(2\pi\,|\eta(\tau)|^2\right)$$

Momentum-space Propagator: integer world-sheet momenta $m, n \in \mathbb{Z}$

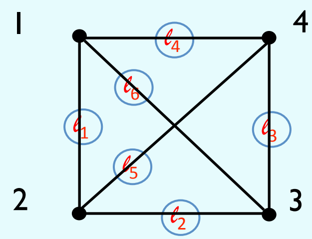
$$\hat{G}(m,n) = \frac{y}{|m\tau + n|^2}$$



General contribution to 4-particle amplitude: $i\,,\, j = 1, 2, 3, 4$

Modular function

$$D_{\ell_1,\ell_2,\ell_3,\ell_4;\ell_5,\ell_6} =$$



ℓ_S labels number of propagators on line S

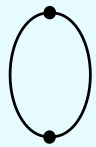
“Weight” $w = \ell_1 + \ell_2 + \cdots + \ell_6$ contributes to $D^{2w}\mathcal{R}^4$

World-sheet Feynman diagrams

Multiple sums:

Non-holomorphic SL(2) Eisenstein series

e.g. $D^4\,\mathcal{R}^4$



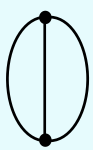
D_2

$$= \sum_{(m,n)\neq(0,0)} \frac{y^2}{|m\tau+n|^4} \equiv E_2(\tau)$$

$$E_s(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|m\tau+n|^{2s}}$$

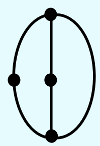
e.g. $C_{a,b,c}$ sequence $\quad w=a+b+c \quad (w-1)$ vertices $\quad D^{2w}\mathcal{R}^4$

(two-loop diagrams)



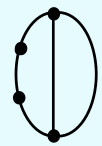
$C_{1,1,1}\equiv D_3$

$D^6\mathcal{R}^4$



$C_{2,2,1}\equiv D_{1,1,1,1;1}$

$D^{10}\,\mathcal{R}^4$



$C_{3,1,1}\equiv D_{2,1,1,1}$

$D^{10}\,\mathcal{R}^4$

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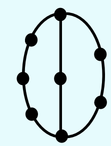


$C_{4,3,2}$

$D^{18}\,\mathcal{R}^4$

.....

$$C_{a,b,c}(\tau) = \sum_{\substack{(m_r,n_r)\neq(0,0)\\ \sum_i m_i=0=\sum_j n_j}} \frac{y^{a+b+c}}{|m_1\tau+n_1|^{2a}|m_2\tau+n_2|^{2b}|m_3\tau+n_3|^{2c}}$$

Direct analysis looks forbidding. But these functions satisfy simple Laplace equations with Laplacian $\Delta = y^2\left(\partial_x^2 + \partial_y^2\right)$

Simple examples of LAPLACE EQUATIONS :

Eisenstein series

$w = 3$ $$\Delta\left(C_{1,1,1} - E_3\right) = 0$$

SOLUTION: $$C_{1,1,1} = E_3 + \zeta(3)$$ (also Zagier)

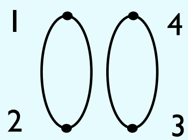
$w = 4$ $$(\Delta - 2)\, C_{2,1,1} = 9E_4 - E_2^2$$

INHOMOGENEOUS LAPLACE EIGENVALUE EQUATIONS

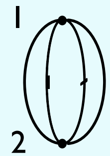
$w = 5$ $$(\Delta - 6)\, C_{3,1,1} = \frac{6}{5}E_5 + \frac{\zeta(5)}{10} + 16E_5 - 4E_2\, E_3\,.$$

$w > 5$ Degeneracy – simultaneous inhomogeneous Laplace eigenvalue equations.

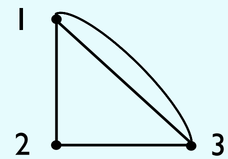
COEFFICIENTS OF $D^8\,\mathcal{R}^4$ (WEIGHT-4)



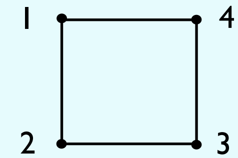
$D_2^2 = E_2^2$



D_4

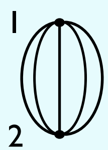


$D_{2,1,1} \equiv C_{2,1,1}$

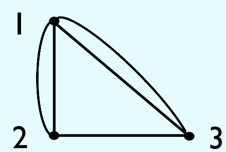


$D_{1,1,1,1}$

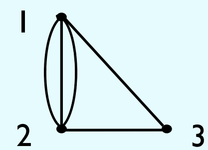
COEFFICIENTS OF $D^{10}\,\mathcal{R}^4$ (WEIGHT-5)



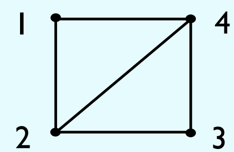
D_5



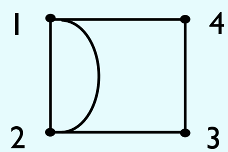
$D_{2,2,1}$



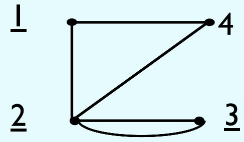
$D_{3,1,1}$



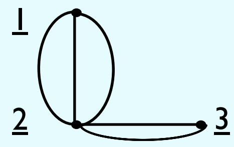
$D_{1,1,1,1;1} \equiv C_{2,2,1}$



$D_{2,1,1,1} \equiv C_{3,1,1}$



$D_{1,1,1}\, D_2$



$D_3\, D_2$

Relation to single-valued Elliptic Multiple Polylogarithms

(D'Hoker, MBG, Gurdogan, Vanhove)

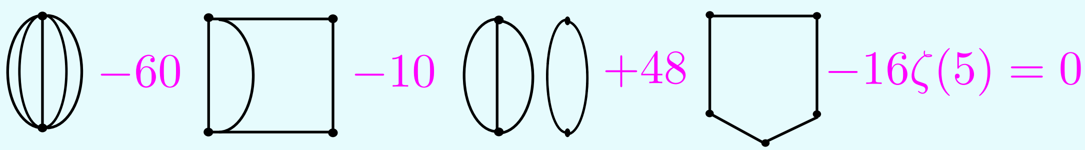
A Modular Graph Function is a Single-Valued Elliptic Multiple Polylogarithm evaluated at a special value of its argument

As with MZV's, these elliptic functions satisfy a fascinating SET OF POLYNOMIAL RELATIONSHIPS – we have found a few of these (with great difficulty!) See also Basu

Examples of polynomial relationships:

e.g. weight 5

$$D_5 - 60\, C_{3,1,1} - 10\, E_2\, C_{1,1,1} + 48\, E_5 - 16\, \zeta(5) = 0$$



polynomial of weight 5 in functions of different depth (different no. of loops).

e.g. weight 6

$$-3\, D_{411} + 109\, C_{222} + 408\, C_{321} + 36\, C_{411} + 18\, C_{211}\, E_2 + 12\, E_3^2 - 211\, E_6 + 12\, E_3 \zeta_3 = 0$$

polynomial of weight 6 in functions of different depth.

General Conjecture

Modular graph functions of a given weight satisfy polynomial relations with rational coefficients

Elliptic generalisation of the rational polynomial relations between single-valued MZV's

Question:

What is the basis of modular graph functions?

Elliptic generalisation of the known basis of single-valued MZV's

Integration over fundamental domain

Genus-one expansion coefficients :

Integrating over τ - using the earlier relations - gives the one-loop expansion:

$$A_1^{(4)} = \frac{\pi}{3}\left(1 + 0\,\sigma_2 + \frac{\zeta(3)}{3}\,\sigma_3 + 0\,\sigma_2^2 + \frac{116\,\zeta(5)}{5}\,\sigma_2\,\sigma_3 \dots\right)\mathcal{R}^4$$

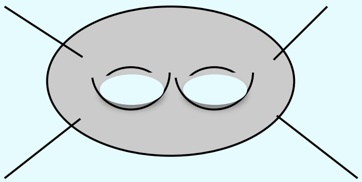
$\mathcal{R}^4$ $\quad d^4\,\mathcal{R}^4$ $\quad d^6\,\mathcal{R}^4$ $\quad d^8\,\mathcal{R}^4$ $\quad d^{10}\,\mathcal{R}^4$

+ non-analytic threshold piece

These coefficients are analogous to the tree-level coefficients:

What is the connection between them ?

Genus Two



Amplitude is explicit but difficult to study. (D'Hoker, Gutperle, Phong)

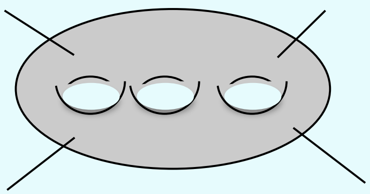
Low energy expansion: (D'Hoker, MBG, Pioline, R. Russo)

Result:

$$A_2^{(4)} = g_s^2 \left(\frac{4}{3}\zeta(4)\, \sigma_2 R^4 \; + 4\zeta(4)\sigma_3 R^4 + \dots \right)$$

$d^4\, R^4$ $\qquad$ $d^6\, R^4$

Genus Three



Technical difficulties analysing 3-loops. Gomez and Mafra evaluated the leading low energy behaviour using PURE SPINOR FORMALISM, giving

$$A_3^{(4)} = g_s^4 \left(\frac{4}{27}\, \zeta(6)\, \sigma_3 + \dots \right) \mathcal{R}^4$$

$d^6\, R^4$

Higher Orders

New problems - No explicit expression

Non-perturbative extension

$$T(s,t,u;;\{\mu_d\}) = \sum_{p,q} \mathcal{E}_{(p,q)}(\{\mu_d\})\, \sigma_2^p\, \sigma_3^q \rightarrow \sim s^{2p+3q} + \ldots$$

$E_{d+1}(\mathbb{Z})$ - invariant functions compactification on d-torus

moduli $\{\mu_d\}$

e.g. $SL(2,\mathbb{Z})$ duality - modulus Ω

- Nonlinear supersymmetry + $SL(2,\mathbb{Z})$ duality lead to Laplace equations with solutions:

$\mathcal{R}^4$ ($\frac{1}{2} - BPS$) $\quad \mathcal{E}_{(0,0)}(\Omega) = E_{\frac{3}{2}}(\Omega)$ NON-RENORMALISATION BEYOND 1 LOOP

$d^4\,\mathcal{R}^4$ ($\frac{1}{4} - BPS$) $\quad \mathcal{E}_{(1,0)}(\Omega) = E_{\frac{5}{2}}(\Omega)$ NON-RENORMALISATION BEYOND 2 LOOPS

Eisenstein series $E_s(\Omega)$ has two power-behaved terms Ω_2^s, Ω_2^{1-s} with ζ - valued coefficients.

perturbative tree and (s-1/2)-loop

$d^6\,\mathcal{R}^4$ ($\frac{1}{8} - BPS$) $\quad \mathcal{E}_{(0,1)}(\Omega)$ not an Eisenstein series NON-RENORMALISATION BEYOND 3 LOOPS

- Coefficients of all power-behaved terms agree precisely with explicit perturbative string calculations.
- Generalisations to HIGHER-RANK GROUPS involve MAXIMAL PARABOLIC LANGLANDS EISENSTEIN SERIES.

  Toroidal compactifications

- Correct ½-BPS and ¼-BPS instanton orbits – correspond to all the expected wrapped branes.

(See also Bossard + Pioline)

The coefficients of the UV divergences in maximal supergravity up to 3 loops in dimensions > 4 are precisely reproduced by log terms in modular coefficients.

To what extent do string theory dualities constrain the structure of perturbative supergravity? – ultraviolet divergences??

Fantasy:

Superstring perturbation theory is free of UV divergences. Can we understand the UV properties of supergravity by careful consideration of the low energy limit of string theory?