Holography for $\mathcal{N} = 1^*$ on S^4

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Supergravity was born in 1976

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Progress toward a theory of supergravity*

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(Received 29 March 1976)

As a new approach to supergravity, an action containing only vierbein and Rarita-Schwinger fields ($V_{a\mu}$ and ψ_μ) is presented together with supersymmetry transformations for these fields. The action is explicitly shown to be invariant except for a ψ^5 term in its variation. This term may also vanish, depending on a complicated calculation. (Added note: This term has now been shown to vanish by a computer calculation, so that the action presented here does possess full local supersymmetry.)

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It has inspired many important developments in theoretical physics over the past 40 years!

AdS/CFT

The Large N Limit of Superconformal field theories and supergravity

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Supersymmetric QFT on curved space

Rigid Supersymmetric Theories in Curved Superspace

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GAUGED $N=8$ $d=5$ SUPERGRAVITY

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Received 19 November 1984

5d $\mathcal{N}=8$ gauged $SO(6)$ supergravity

GAUGED $N=8$ SUPERGRAVITY IN FIVE DIMENSIONS ☆

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Received 10 December 1984

COMPACT AND NON-COMPACT GAUGED SUPERGRAVITY THEORIES IN FIVE DIMENSIONS*

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Received 3 December 1985

Motivation

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- Apply gauge/gravity duality to this setup and test holography in a non-conformal setup.

- Study the dynamics of $\mathcal{N} = 1$ theories holographically. Localization on S^4 has not been successful (so far!) for these theories!

Synopsis

- $\mathcal{N} = 1^*$ SYM is a theory of an $\mathcal{N} = 1$ vector multiplet and 3 massive chiral multiplets in the adjoint of the gauge group. It is a massive deformation of $\mathcal{N} = 4$ SYM. There is a unique supersymmetric Lagrangian on S^4. [Pestun], [Festuccia-Seiberg], [NB-Elvang-Freedman-Pufu]

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$$F_{S^4}^{\mathcal{N}=2^*} = -\log \mathcal{Z}_{S^4}^{\mathcal{N}=2^*} = -\frac{N^2}{2}(1 + (mR)^2) \log \frac{\lambda(1 + (mR)^2) e^{2\gamma + \frac{1}{2}}}{16\pi^2} ,$$

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- Precision test of holography! In AdS_5/CFT_4 one typically compares numbers. Here we have a whole function to match.

- Previous results from holography for $\mathcal{N} = 1^*$ and $\mathcal{N} = 2^*$ on $\mathbb{R}^4$. [Freedman-Gubser-Pilch-Warner], [Girardello-Petrini-Porrati-Zaffaroni], [Pilch-Warner], [Buchel-Peet-Polchinski], [Evans-Johnson-Petrini], [Polchinski-Strassler], ... On S^4 the holographic construction is more involved.

Plan

- $\mathcal{N} = 1^*$ SYM theory on S^4
- The supergravity dual
- Holographic calculations
- Outlook

$\mathcal{N} = 1^*$ SYM theory on S^4

$\mathcal{N} = 1^*$ SYM on $\mathbb{R}^4$

The field content of $\mathcal{N} = 4$ SYM is

$$A_\mu \,, \qquad X_{1,2,3,4,5,6} \,, \qquad \lambda_{1,2,3,4} \,.$$

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Organize this into an $\mathcal{N} = 1$ vector multiplet

$$A_\mu\,, \qquad \psi_1 \equiv \lambda_4\,,$$

and 3 chiral multiplets

$$\chi_j = \lambda_j\,, \qquad Z_j = \frac{1}{\sqrt{2}}\big(X_j + iX_{j+3}\big), \qquad j = 1, 2, 3\,.$$

Only $SU(3) \times U(1)_R$ of the $SO(6)$ R-symmetry is manifest.

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The $\mathcal{N} = 1^*$ theory is obtained by turning on (independent) mass terms for the chiral multiplets.

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$$
\begin{aligned}
\mathcal{L}^{S^4}_{\mathcal{N}=1^*} = \; & \mathcal{L}^{S^4}_{\mathcal{N}=4} \\
& + \frac{2}{R^2}\,\mathrm{tr}\Big(Z_1\tilde{Z}_1 + Z_2\tilde{Z}_2 + Z_3\tilde{Z}_3\Big) \\
& + \mathrm{tr}\,\Big(m_1\tilde{m}_1 Z_1\tilde{Z}_1 + m_2\tilde{m}_2 Z_2\tilde{Z}_2 + m_3\tilde{m}_3 Z_3\tilde{Z}_3\Big) \\
& - \frac{1}{2}\,\mathrm{tr}\,(m_1\chi_1\chi_1 + m_2\chi_2\chi_2 + m_3\chi_3\chi_3 + \tilde{m}_1\tilde{\chi}_1\tilde{\chi}_1 + \tilde{m}_2\tilde{\chi}_2\tilde{\chi}_2 + \tilde{m}_3\tilde{\chi}_3\tilde{\chi}_3) \\
& - \frac{1}{\sqrt{2}}\,\mathrm{tr}\,\Big[m_i\epsilon^{ijk}Z_i\tilde{Z}_j\tilde{Z}_k + \tilde{m}_i\epsilon^{ijk}\tilde{Z}_iZ_jZ_k\Big] \\
& + \frac{i}{2R}\,\mathrm{tr}\,\Big(m_1 Z_1^2 + m_2 Z_2^2 + m_3 Z_3^2 + \tilde{m}_1\tilde{Z}_1{}^2 + \tilde{m}_2\tilde{Z}_2{}^2 + \tilde{m}_3\tilde{Z}_3{}^2\Big) \; .
\end{aligned}
$$

15 (real) relevant operators in the Lagrangian + 1 complex gaugino vev + 1 complexified gauge coupling. Only 18 of these operators are visible as modes in IIB supergravity.

For $m_3 = \tilde{m}_3 = 0$, $m_1 = m_2 \equiv m$ and $\tilde{m}_1 = \tilde{m}_2 \equiv \tilde{m}$ we get the $\mathcal{N} = 2^*$ theory.

Results from localization for $\mathcal{N} = 2^*$

After supersymmetric localization the path integral for the theory on S^4 reduces to a finite dimensional integral over the Coulomb branch moduli. [Pestun]

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The scheme independent quantity is

$$\boxed{\frac{d^3 F_{S^4}}{d(mR)^3} = -2N^2 \frac{mR((mR)^2 + 3)}{((mR)^2 + 1)^2}}$$

This is the unambiguous result one can aim to compute holographically.

The supergravity dual

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- It is a consistent truncation of IIB supergravity on S^5 with fields dual to the lowest dimension operators in $\mathcal{N} = 4$ SYM. [Lee-Strickland-Constable-Waldram], [Baguet-Hohm-Samtleben]
- The gravity dual of $\mathcal{N} = 1^*$ and $\mathcal{N} = 2^*$ on $\mathbb{R}^4$ was constructed first in 5d. [Girardello-Petrini-Porrati-Zaffaroni], [Pilch-Warner]

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The dual of $\mathcal{N} = 1^*$ on S^4 is captured by a 5d $\mathcal{N} = 2$ gauged supergravity with 2 vector and 4 hyper multiplets and a scalar coset

$$O(1,1) \times O(1,1) \times \frac{SO(4,4)}{SO(4) \times SO(4)} \,.$$

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There are special cases which allow for an explicit analysis. We fix $\tilde{m}_j=m_j$

- $m_1=m_2$ and $m_3=0$ - on $\mathbb{R}^4$ this is the Leigh-Strassler flow. [Freedman-Gubser-Pilch-Warner] On S^4 we need 3 scalars.
- $m_1=m_2=m_3$ - on $\mathbb{R}^4$ this is the GPPZ/PS flow. [Girardello-Petrini-Porrati-Zaffaroni], [Polchinski-Strassler]. On S^4 we need 4 scalars.
- $m_2=m_3=0$ - on $\mathbb{R}^4$ this is the $\mathcal{N}=2^*$ PW flow. [Pilch-Warner] On S^4 we need 3 scalars.

The gravity dual of $\mathcal{N} = 2^*$ on S^4

The Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[-\mathcal{R} + 12\frac{\partial_\mu \eta \partial^\mu \eta}{\eta^2} + 4\frac{\partial_\mu z \partial^\mu \tilde{z}}{\left(1 - z\tilde{z}\right)^2} + \mathcal{V}\right],$$

$$\mathcal{V} \equiv -\frac{4}{L^2}\left(\frac{1}{\eta^4} + 2\eta^2 \frac{1 + z\tilde{z}}{1 - z\tilde{z}} + \frac{\eta^8}{4}\frac{(z - \tilde{z})^2}{(1 - z\tilde{z})^2}\right).$$

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To preserve the isometries of S^4 take the "domain-wall" Ansatz

$$ds^2 = L^2 e^{2A(r)} ds^2_{S^4} + dr^2 \,, \quad \eta = \eta(r) \,, \quad z = z(r) \,, \quad \tilde{z} = \tilde{z}(r) \,.$$

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$$\eta = e^{\phi/\sqrt{6}}\,, \qquad z = \frac{1}{\sqrt{2}}\big(\chi + i\psi\big)\,, \qquad \tilde{z} = \frac{1}{\sqrt{2}}\big(\chi - i\psi\big)\,.$$

The masses of the scalars around the AdS_5 vacuum are

$$m_\phi^2 L^2 = m_\chi^2 L^2 = -4\,, \qquad\qquad m_\psi^2 L^2 = -3\,.$$

The 3 scalars, $\{\phi, \psi, \chi\}$, are dual to 3 relevant operators

$$\mathcal{O}_\phi\,, \quad \Delta_{\mathcal{O}_\phi} = 2\,; \qquad \mathcal{O}_\psi\,, \quad \Delta_{\mathcal{O}_\psi} = 3\,; \qquad \mathcal{O}_\chi\,, \quad \Delta_{\mathcal{O}_\chi} = 2\,.$$

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For $\chi = 0$ recover the truncation for $\mathcal{N} = 2^*$ on $\mathbb{R}^4$. [Pilch-Warner]

The BPS equations

Plug the Ansatz in the supersymmetry variations of the 5d $\mathcal{N} = 8$ theory and use the "conformal Killing spinors" on S^4

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to derive the BPS equations (arising from $\delta\lambda_\alpha = \delta\psi_{\mu\alpha} = 0$)

$$z' = \frac{3\eta'(z\tilde{z}-1)\left[2(z+\tilde{z})+\eta^6(z-\tilde{z})\right]}{2\eta\left[\eta^6\left(\tilde{z}^2-1\right)+\tilde{z}^2+1\right]} \, ,$$

$$\tilde{z}' = \frac{3\eta'(z\tilde{z}-1)\left[2(z+\tilde{z})-\eta^6(z-\tilde{z})\right]}{2\eta\left[\eta^6\left(z^2-1\right)+z^2+1\right]} \, ,$$

$$(\eta')^2 = \frac{\left[\eta^6\left(z^2-1\right)+z^2+1\right]\left[\eta^6\left(\tilde{z}^2-1\right)+\tilde{z}^2+1\right]}{9L^2\eta^2(z\tilde{z}-1)^2} \, ,$$

$$e^{2A} = \frac{(z\tilde{z}-1)^2\left[\eta^6\left(z^2-1\right)+z^2+1\right]\left[\eta^6\left(\tilde{z}^2-1\right)+\tilde{z}^2+1\right]}{\eta^8\left(z^2-\tilde{z}^2\right)^2} \, .$$

UV and IR expansion

The (constant curvature) metric on $\mathbb{H}^5$ is

$$ds_5^2 = dr^2 + L^2 \sinh^2\left(\frac{r}{L}\right) ds_{S^4}^2 \ .$$

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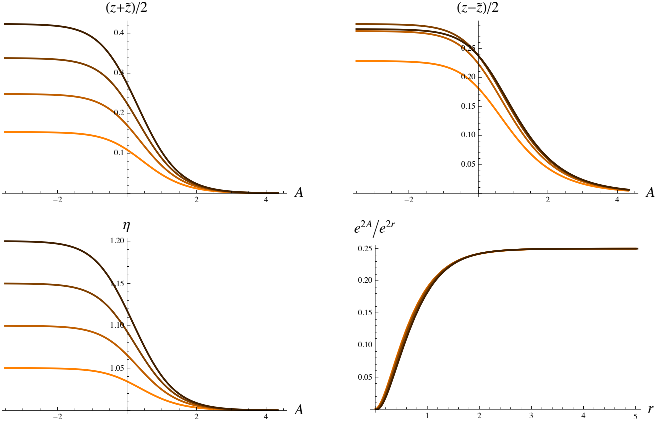
Impose that at $r = r_*$ (IR) the S^4 shrinks to zero size. Solve the BPS equations close to $r = r_*$, and require that the solution is smooth. There is only one free parameter, η_0, controlling this expansion.

Numerical solutions

One can find numerical solutions by "shooting" from the IR to the UV. There is a one (complex) parameter family parametrized by η_0, so

$$v = v(\eta_0) \,, \qquad \text{and} \qquad \mu = \mu(\eta_0) \,.$$



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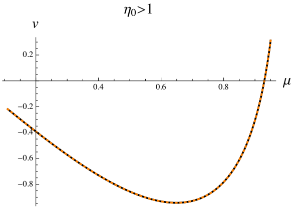
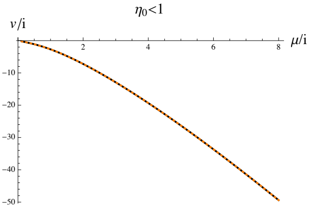
For real η_0 one finds the following results for $v(\mu)$





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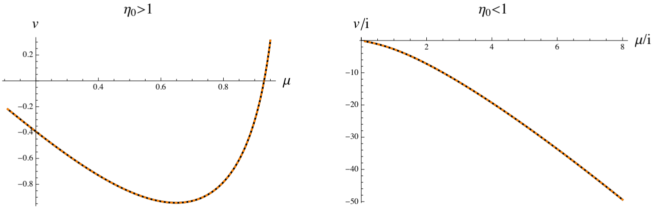

From the numerical results one can extract the following dependence

$$\boxed{v(\mu) = -2\mu - \mu\,\log(1-\mu^2)}$$

Holographic calculations

Calculating $F_{S^4}^{\mathcal{N}=1^*}$ from supergravity

- By the holographic dictionary the partition function of the field theory is mapped to the on-shell action of the supergravity dual. [Maldacena], [GKP], [Witten]
- The on-shell action diverges and one has to regularize it using holographic renormalization. [Skenderis], ...

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- The on-shell action diverges and one has to regularize it using holographic renormalization. [Skenderis], ...
- There is a subtlety here. If we insist on using a supersymmetric regularization scheme there is a particular finite counterterm that has to be added. Only with it one can successfully compare $\frac{d^3F}{d\mu^3}$ with the field theory result.
- Without knowing this finite counterterm we can only hope to match $\frac{d^5F}{d\mu^5}$ with field theory.

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The full renormalized 5d action is

$$S_{\mathsf{ren}} = S_{\mathsf{5D}} + S_{\mathsf{ct}} + S_{\mathsf{finite}} \; .$$

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$$\frac{dF^{\text{SUGRA}}}{d\mu} = \frac{N^2}{2\pi^2}\,\text{vol}(S^4)\Big(4\mu - 12v(\mu)\Big) = N^2\Big(\,\frac{1}{3}\mu - v(\mu)\Big)\,.$$

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Set $\mu = imR$ and compare this to field theory

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Lo and behold!

$$\boxed{\frac{d^3F_{S^4}^{\mathcal{N}=2^*}}{d(mR)^3} = \frac{d^3F_{S^4}^{\text{SUGRA}}}{d\mu^3}}$$

Holography for $\mathcal{N} = 1^*$

For the $\mathcal{N} = 1^*$ theory (and more general massive $\mathcal{N} = 1$ theories) we have performed a counterterm analysis (in "old minimal" rigid 4d supergravity) and showed that there is an ambiguity in F_{S^4}

$$F_{S^4} \to F_{S^4} + f_1(\tau, \bar{\tau}) + f_2(\tau, \bar{\tau}) \sum_{j=1}^{3} m_j \tilde{m}_j R^2 \ .$$

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At order m_j^4 the dependence of F_{S^4} for small m_i must be

$$F_{S^4} \approx N^2 \left[A \sum_{j=1}^{3} m_j^4 + B \left(\sum_{j=1}^{3} m_j^2 \right)^2 \right] \ .$$

We can aim at computing the constants A and B holographically.

No results from localization to guide us!

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From the numerical analysis for the three special limits of $\mathcal{N} = 1^*$ we find

$$F_{\mathcal{N}=2^*} \approx -\frac{m^4 N^2}{4} \ , \quad F_{\text{1-mass}} \approx -0.235 m^4 N^2 \ , \quad F_{m_1=m_2=m_3} \approx -0.043 m^4 N^2 \ .$$

These 3 results are compatible with $A \approx -0.346$ and $B \approx 0.111$ and we have $A + 2B = -\frac{1}{8}$.

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For the equal mass model the supergravity results predict that the gaugino condensate is

$$\langle \mathrm{Tr}(\lambda\lambda + \tilde{\lambda}\tilde{\lambda}) \rangle = \frac{4}{\pi^2} m^3 N^2 \qquad \rightarrow \qquad \langle \mathrm{Tr}(\lambda\lambda + \tilde{\lambda}\tilde{\lambda}) \rangle = \frac{4}{\pi^2} m_1 m_2 m_3 N^2 \,.$$

This is just a glimpse of the detailed holographic results we have extracted from our supergravity solutions!

Summary

- We found a 5d supegravity dual of $\mathcal{N} = 2^*$ SYM on S^4.
- After careful holographic renormalization we computed the universal part of the free energy of this theory.
- The result is in exact agreement with the supersymmetric localization calculation in field theory.
- This is a precision test of holography in a non-conformal Euclidean setting.
- Extension of these results to $\mathcal{N} = 1^*$ SYM where our results can be viewed as supergravity "lessons" for the dynamics of the gauge theory.
- The results generalize readily to $\mathcal{N} = 2^*$ mass deformations of quiver gauge theories obtained by $\mathbb{Z}_k$ orbifolds of $\mathcal{N} = 4$ SYM.
  [Azeyanagi-Hanada-Honda-Matsuo-Shiba]

Outlook

- Uplift of the $\mathcal{N} = 1^*$ solutions to IIB supergravity. Relation to Polchinski-Strassler? Holographic calculation of Wilson or 't Hooft line vevs. Study probe D3- and D7-branes. [in progress]
- Holography for $\mathcal{N} = 1^*$ on other 4-manifolds. [Cassani-Martelli], ...
- Extensions to other $\mathcal{N} = 2$ theories in 4d with holographic duals, e.g. pure $\mathcal{N} = 2$ SYM? [Gauntlett-Kim-Martelli-Waldram], [in progress]
- Extensions to other dimensions.

Outlook

- Can we see some of the large N phase transitions argued to exist by Russo-Zarembo in IIB string theory?
- Revisit supersymmetric localization for $\mathcal{N} = 1$ theories on S^4. Can one find the exact partition function (modulo ambiguities)?
- For 4d $\mathcal{N} = 2$ conformal theories Z_{S^4} leads to the Zamolodchikov metric. What is the "meaning" of Z_{S^4} for "gapped" theories?
- Understand the role of $SL(2, \mathbb{Z})$ in $\mathcal{N} = 1^*$? [Vafa-Witten], [Donagi-Witten], [Dorey], [Aharony-Dorey-Kumar]
- Systematic understanding of supersymmetric finite counterterms in holographic renormalization? [Assel-Cassani-Martelli], ...
- Broader lessons for holography from localization?

Happy Birthday Supergravity!