Resumming Instantons in $\mathcal{N}$=2* Theories

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This talk is mainly based on:

- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, *``S-duality and the prepotential in N=2*theories (I): the ADE algebras,''* JHEP **1511** (2015) 024, arXiv:1507.07709
- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, *``S-duality and the prepotential in N=2*theories (II): the non-simply laced algebras,''* JHEP **1511** (2015) 026, arXiv:1507.08027
- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, *``Resumming instantons in N=2* theories,''* XIV Marcel Grossmann Meeting, arXiv:1602.00273

and

- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, A.L. and M. Raman, *``Modular anomaly equations and S-duality in N=2 conformal SQCD,"* JHEP 1510 (2015) 091, arXiv:1507.07476
- S.K. Ashok, E. Dell'Aquila, A.L. and M. Raman, *``S-duality, triangle groups and modular anomalies in N=2 SQCD,''* JHEP 1604 (2016) 118, arXiv:1601.01827
- S.K.Ashok, M.Billò, E.Dell'Aquila, M. Frau, A.L., M.Moskovic, M.Raman, *``Chiral observables and S-duality in N=2* U(N) gauge theories'',* arXiv:1607.08327 to be published on JHEP

but it builds on a very vast literature …

Plan of the talk

1. Introduction
2. *N*=4 SYM
3. *N*=2* SYM
4. Conclusions

- Non-perturbative effects are important:
  - in gauge theories: confinement, chiral symmetry breaking, ...
  - in string theories: D-branes, duality, AdS/CFT, ...
- They are essential to complete the perturbative expansion and lead to results valid at all couplings
- In supersymmetric theories, tremendous progress has been possible thanks to the developement of localization techniques

  (Nekrasov ‘02, Nekrasov-Okounkov ’03, Pestun ‘07, …, Nekrasov-Pestun ‘13, ….)
- In superconformal theories these methods allowed us to compute exactly several quantities:
  - Sphere partition function and free energy
  - Wilson loops
  - Correlation functions, amplitudes
  - Cusp anomalous dimensions and bremsstrahlung function

- We will focus on SYM theories in *4d* with $\mathcal{N}$=2 supersymmetry
  - They are less constrained than the $\mathcal{N}$=4 theories
  - They are sufficiently constrained to be analyzed exactly
- We will be interested in studying how S-duality on the quantum effective couplings constrains the prepotential and the observables of N=2 theories

(earlier work by Minahan et al. '96, '97)

- We will make use of these constraints to obtain exact expressions valid at all couplings

$\mathcal{N}$=4 SYM

N=4 SYM

- Consider *N* =4 SYM in d=4
  - This theory is maximally supersymmetric (16 SUSY charges)
  - The field content is

A		1 vector
λ^a	$(a = 1, \cdots, 4)$	4 Weyl spinors
X^i	$(i = 1, \cdots, 6)$	6 real scalars

  - All fields are in the adjoint repr. of the gauge group G
  - The β−function vanishes to all orders in perturbation theory
  - If $\langle X^i \rangle = 0$, the theory is superconformal (*i.e.* invariant under $\mathrm{SU}(2,2|4)$) also at the quantum level

N=4 SYM

- The relevant ingredients of *N* =4 SYM are:
  - The gauge group G (or the gauge algebra $\mathfrak{g}$)
  - The (complexified) coupling constant

$$\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2} \quad \in \mathbb{H}_+$$

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- Many <u>exact</u> results have been obtained using:
  - Explicit expressions of scattering amplitudes
  - Integrability
  - AdS/CFT correspondence
  - Duality

N=4 SYM

- *N* =4 SYM is believed to possess an exact duality invariance which contains the electro-magnetic duality S

  (Montonen-Olive '77, Vafa-Witten '94, Sen '94, ...)

- If the gauge algebra $\mathfrak{g}$ is simply laced (ADE)
  - S maps the theory to itself but with electric and magnetic states exchanged
  - It is a weak/strong duality, acting on the coupling by

$$S(\tau) = -1/\tau$$

  - Together with $T(\tau) = \tau + 1$ ($\theta \to \theta + 2\pi$), it generates the modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;; \quad S^2 = -1 \;, \quad (ST)^3 = -1$$

$\mathcal{N}$=4 SYM

- $\mathcal{N}$ =4 SYM is believed to possess an exact duality invariance which contains the electro-magnetic duality S
  (Montonen-Olive '77, Vafa-Witten '94, Sen '94, ...)

- If the algebra $\mathfrak{g}$ of the gauge group G is non-simply laced (BCFG) duality relation still exist, but they are more involved… (see Billò et al. '15 and Ashok et al.'16)

- For simplicity I will only describe the case of simply laced algebras $\mathfrak{g}$, but all the arguments can be generalized to include also the non-simply laced cases

N=4 SYM as a *N*=2 theory

Let us decompose the *N*=4 multiplet into

- one *N*=2 vector multiplet

2 Weyl fermions

1 vector $\rightarrow A\ , \quad \psi^I\ , \quad \phi \leftarrow$ 1 complex scalar

- one *N*=2 hypermultiplet

2 Weyl fermions

$$q\ , \quad \tilde{q}\ , \quad \chi\ , \quad \tilde{\chi}$$

2 complex scalars

N=4 SYM as a N=2 theory

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$q\ ,\quad \tilde{q}\ ,\quad \chi\ ,\quad \tilde{\chi}$

2 complex scalars

By introducing the v.e.v.

$$\langle \phi \rangle = a = \mathrm{diag}(a_1, \ldots, a_n)$$

- we break the gauge group $G \to U(1)^n$
- we spontaneously break conformal invariance
- we can describe the dynamics in terms of a holomorphic prepotential $F(a)$, as in N=2 theories

$\mathcal{N}$=4 SYM as a $\mathcal{N}$=2 theory

- The prepotential of the $\mathcal{N}$=4 theory is simply

$$F = i\,\pi\,\tau\,a^2$$

- The dual variables are defined as

$$a_D \equiv \frac{1}{2\pi i}\frac{\partial F}{\partial a} = \tau\,a$$

- S-duality relates the electric variable a to the magnetic variable a_D:

$$S\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -a \\ a_D \end{pmatrix}$$

$\mathcal{N}$=4 SYM as a $\mathcal{N}$=2 theory

- Let's find the S-dual prepotential:

$$S(F) = i\,\pi\Big(-\frac{1}{\tau}\Big)\,(a_D)^2 = -i\,\pi\,\frac{1}{\tau}\,a_D^2$$

- S-duality exchanges the description based on a with its Legendre-transform, based on a_D :

$$\mathcal{L}(F) = F - a\,\frac{\partial F}{\partial a} = i\,\pi\,\tau\,a^2 - 2\pi\,i\,a\,a_D$$

$$= -i\,\pi\,\frac{1}{\tau}\,a_D^2$$

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- Thus

$$S(F) = \mathcal{L}(F)$$

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- Thus

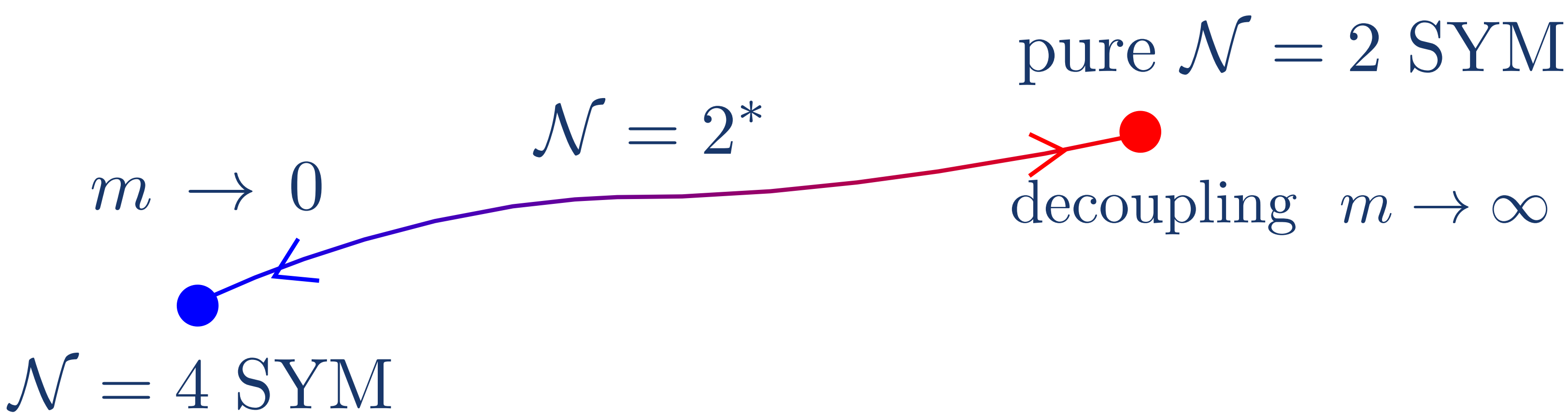
$$S(F) = \mathcal{L}(F)$$

- This structure is present also in $\mathcal{N}$=2 theories and has important consequences on their strong coupling dynamics!

$\mathcal{N}$=2* SYM

The *N*=2* set-up

- The *N*=2* theory is a mass deformation of the *N*=4 SYM
- Field content:
  - one *N*=2 vector multiplet for the algebra $\mathfrak{g}$
  - one *N*=2 hypermultiplet in the adjoint rep. of $\mathfrak{g}$ with **mass *m***
- Half of the supercharges are broken, and we have *N*=2 SUSY
- The β-function still vanishes, but the superconformal invariance is explicitly broken by the mass ***m***



Structure of the $\mathcal{N}$=2* prepotential

- The *N*=2* prepotential contains classical, 1-loop and non-perturbative terms

$$F = i\,\pi\,\tau\,a^2 + f \quad \text{with} \quad f = f_{1-loop} + f_{non-pert}$$

- The 1-loop term reads

$$\frac{1}{4}\sum_{\alpha\in\Psi_{\mathfrak{g}}}\left[-(\alpha\cdot a)^2\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2+(\alpha\cdot a+m)^2\log\left(\frac{\alpha\cdot a+m}{\Lambda}\right)^2\right]$$

  - $\Psi_{\mathfrak{g}}$ is the set of the roots α of the algebra $\mathfrak{g}$
  - $\alpha\cdot a$ is the mass of the W-boson associated to the root α

- The non-perturbative contributions come from all instanton sectors and are proportional to q^k and can be explicitly computed using localization for all classical algebras

(Nekrasov '02, Nekrasov-Okounkov '03, ..., Billò et al 15, ...)

S-duality and the prepotential

- The dual variables are defined as

$$a_D \equiv \frac{1}{2\pi i}\frac{\partial F}{\partial a} = \tau\left(a + \frac{1}{2\pi i\tau}\frac{\partial f}{\partial a}\right)$$

- Applying S-duality we get

$$S(F) = i\,\pi\Big(-\frac{1}{\tau}\Big)a_D^2 + f\Big(-\frac{1}{\tau}, a_D\Big)$$

- Computing the Legendre transform we get

$$\begin{aligned}\mathcal{L}(F) &= F - 2i\pi\, a\cdot a_D \\ &= i\,\pi\Big(-\frac{1}{\tau}\Big)a_D^2 + f(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f}{\partial a}\right)^2\end{aligned}$$

S-duality and the prepotential

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S-duality and the prepotential

- Requiring

$$S(F) = \mathcal{L}(F)$$

implies

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \frac{1}{4i\pi\tau}\left(\frac{\partial f}{\partial a}\right)^2$$

Modular anomaly equation!

- This constraint has very deep implications!

S-duality and the prepotential

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Modular anomaly equation!

- This constraint has very deep implications!
- The modular anomaly equation is related to the holomorphic anomaly equation of the local CY topological string description of the low-energy effective theory

(BCOV '93, Witten '93, … Aganagic et al '06, Gunaydin et al '06, Huang et al 09, Huang '13, …)

Modular anomaly equation

- We organize the quantum prepotential f in a mass expansion

$$f(\tau,a)=\sum_{n=1} f_n(\tau,a) \quad \text{with} \quad f_n \propto m^{2n}$$

- From explicit calculations, one sees that:
  - f_1 is only 1-loop and thus τ-independent

$$f_1(a)=\frac{m^2}{4}\sum_{\alpha\in\Psi_{\mathfrak{g}}}\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2$$

  - $f_n \ (n\geq 2)$ are both 1-loop and non-perturbative. They are homogeneous functions

$$f_n(\tau,\lambda\, a)=\lambda^{2-2n}\, f_n(\tau,a)$$

(This is because the prepotential has mass dimension 2)

Modular anomaly equation

- The modular anomaly equation

$$f\Big(-\frac{1}{\tau},a_D\Big)=f(\tau,a)+\frac{\delta}{24}\left(\frac{\partial f}{\partial a}\right)^2 \quad , \qquad \delta=\frac{6}{i\pi\tau}$$

implies

$$f_n\Big(-\frac{1}{\tau},a_D\Big)=f_n(\tau,a)+\cdots$$

Modular anomaly equation

- The modular anomaly equation

$$f\Big(-\frac{1}{\tau}, a_D\Big) = f(\tau, a) + \frac{\delta}{24}\left(\frac{\partial f}{\partial a}\right)^2 \quad , \qquad \delta = \frac{6}{i\pi\tau}$$

implies

$$f_n\Big(-\frac{1}{\tau}, a_D\Big) = f_n(\tau, a) + \cdots$$

- n = 1
  - Using $f_1(a) = \dfrac{m^2}{4}\displaystyle\sum_{\alpha\in\Psi_{\mathfrak{g}}} \log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2$ and

    requiring that under S-duality $\Lambda \;\rightarrow\; \tau\,\Lambda$, we have

$$f_1\left(a_D\right) = f_1\left(\tau a + \cdots\right) = f_1\left(a\right) + \cdots \quad ✓$$

Modular anomaly equation

- n = 2
  - Using the definition of the dual variable and the homogeneity property, we have

$$f_2\Big(-\frac{1}{\tau}, a_D\Big) = f_2\Big(-\frac{1}{\tau}, \tau(a+\cdots)\Big) = \tau^{-2} f_2\Big(-\frac{1}{\tau}, a+\cdots\Big)$$

  - In order to solve the equation, we must require that

$$f_2\Big(-\frac{1}{\tau}, a+\cdots\Big) = \tau^2 f_2\Big(\tau, a+\cdots\Big) = \tau^2 f_2\Big(\tau, a\Big) + \cdots$$

  - i.e. $f_2(\tau, a)$ should have modular weight 2 under S-duality !

- The only quantity with this property is the second Eisenstein series E_2 (quasi-modular)

Modular anomaly equation

- Generic n

  - The previous analysis can be easily generalized to arbitrary *n.*
  - In order to be able to solve the equation, we must have

$$f_n\Big(-\frac{1}{\tau}, a+\cdots\Big) = \tau^{2n-2}\, f_n\Big(\tau, a\Big) + \cdots$$

  - Thus we must require that $f_n^{\mathfrak{g}}$ depends on τ through "modular" functions with weight $2n-2$, *i.e.*

$$f_n(\tau,a) = f_n\Big(E_2(\tau), E_4(\tau), E_6(\tau), a\Big)$$

where $E_2(\tau), E_4(\tau), E_6(\tau)$ are the Eisenstein series

Eisenstein series

- The Eisenstein series are “modular” forms with a well-known Fourier expansion in $q = e^{2i\pi\tau}$:

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \cdots$$

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \cdots$$

- E_4 and E_6 are truly modular forms of weight 4 and 6

$$E_4\Big(-\frac{1}{\tau}\Big) = \tau^4\, E_4(\tau) \quad , \quad E_6\Big(-\frac{1}{\tau}\Big) = \tau^6\, E_6(\tau)$$

- E_2 is quasi-modular of weight 2

$$E_2\Big(-\frac{1}{\tau}\Big) = \tau^2\,\big[E_2(\tau) + \delta\big] \quad , \quad \delta = \frac{6}{i\pi\tau}$$

- Thus a modular form of weight w is mapped under S into a form of weight w times τ^w, up to shifts induced by E_2

Recursion relation

- S-duality

$$f\left(-\frac{1}{\tau},a_D\right)=f\left(E_2(-\frac{1}{\tau}),E_4(-\frac{1}{\tau}),E_6(-\frac{1}{\tau}),\tau\left(a+\frac{\delta}{12}\frac{\partial f}{\partial a}\right)\right)$$

$$=f\left(E_2+\delta,E_4,E_6,\left(a+\frac{\delta}{12}\frac{\partial f}{\partial a}\right)\right)$$

$$=f\left(\tau,a\right)+\delta\left[\frac{\partial f}{\partial E_2}+\frac{1}{12}\left(\frac{\partial f}{\partial a}\right)^2\right]+\mathcal{O}(\delta^2)$$

- Modular anomaly equation

$$f\left(-\frac{1}{\tau},a_D\right)=f(\tau,a)+\delta\left[\frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2\right]$$

Recursion relation

- S-duality

$$f\left(-\frac{1}{\tau}, a_D\right) = f\left(E_2(-\frac{1}{\tau}), E_4(-\frac{1}{\tau}), E_6(-\frac{1}{\tau}), \tau\left(a + \frac{\delta}{12}\frac{\partial f}{\partial a}\right)\right)$$

$$= f\left(E_2 + \delta, E_4, E_6, \left(a + \frac{\delta}{12}\frac{\partial f}{\partial a}\right)\right)$$

$$= f\left(\tau, a\right) + \delta\left[\frac{\partial f}{\partial E_2} + \frac{1}{12}\left(\frac{\partial f}{\partial a}\right)^2\right] + \mathcal{O}(\delta^2)$$

- Modular anomaly equation

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \delta\left[\frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2\right]$$

Recursion relation

- We thus obtain

$$\frac{\partial f}{\partial E_2} + \frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2 = 0$$

which implies the following recursion relation

(Minahan et al '97)

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24}\sum_{\ell=1}^{n-1}\frac{\partial f_\ell}{\partial a}\,\frac{\partial f_{n-\ell}}{\partial a}$$

  - This allows us to determine f_n from the lower coefficients up to E_2-independent terms. These are fixed by comparison with the perturbative expressions (or the first instanton corrections).
  - The modular anomaly equation is a symmetry requirement; it does not eliminate the need of a dynamical input

- Once this is done, the result is valid to all instanton orders.

Exploiting the recursion

- Using this recursive procedure we find

$$f_2 = -\frac{m^4}{24} E_2 \, C_2^{\mathfrak{g}}$$

$$f_3 = -\frac{m^6}{720} (5E_2^2 + E_4) C_2^{\mathfrak{g}} - \frac{m^6}{576} (E_2^2 - E_4) C_{2;1,1}^{\mathfrak{g}}$$

where $C_2^{\mathfrak{g}}$ and $C_{2;1,1}^{\mathfrak{g}}$ are root lattice sums of $\mathfrak{g}$ defined as

$$C_2^{\mathfrak{g}} = \sum_{\alpha \in \Psi_{\mathfrak{g}}} \frac{1}{(\alpha \cdot a)^2}$$

$$C_{2;1,1}^{\mathfrak{g}} = \sum_{\alpha \in \Psi_{\mathfrak{g}}} \sum_{\beta_1 \neq \beta_2 \in \Psi_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha \cdot a)^2 (\beta_1 \cdot a)(\beta_2 \cdot a)}$$

with $\Psi_{\mathfrak{g}}(\alpha) = \{\beta \in \Psi_{\mathfrak{g}} \, : \, \alpha \cdot \beta = 1\}$

Exploiting the recursion

- For example

$$C_2^{\mathrm{U}(2)} = \frac{1}{(a_1 - a_2)^2}$$

$$C_2^{\mathrm{U}(3)} = \frac{1}{(a_1 - a_2)^2} + \frac{1}{(a_1 - a_2)^2} + \frac{1}{(a_2 - a_3)^2}$$

and thus

$$f_2^{\mathrm{U}(2)} = -\frac{m^4}{24} E_2(\tau)\, C_2^{\mathrm{U}(2)}$$

$$f_2^{\mathrm{U}(3)} = -\frac{m^4}{24} E_2(\tau)\, C_2^{\mathrm{U}(3)}$$

- From the Fourier expansion of E_2 we get the perturbative and all non-perturbative contributions to the prepotential at order m^4 !
- There are no free parameters !

Checks on the results

- For the classical algebras A, B, C and D
  - the ADHM construction of the *k* instanton moduli spaces is avaliable
  - the integration of the moduli action over the instanton moduli spaces can be performed à la Nekrasov using localization techniques
- In principle straightforward; in practice computationally rather intense. Not many explicit results for the N=2* theories in the literature.
- We worked it out:
  - for A_n and D_n with n<6, up to 5 instantons;
  - for C_n with n<6, up to 4 instantons;
  - for B_n with n<6, up to 2 instantons.
- The results match the q-expansion of those obtained above
- For the exceptional algebras our results are predictions!

Generalizations

- These results can be extended to non-flat space-times by turning-on the so-called Ω background

$$\begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}$$

(Nekrasov '02)

which actually was already present in the localization calculations

- For $\epsilon_1, \epsilon_2 \neq 0$ one finds that the generalized prepotential

$$F = i\,\pi\,\tau\,a^2 + f(a,\epsilon)$$

obeys a generalized modular anomaly equation

$$\frac{\partial f}{\partial E_2} + \frac{1}{24}\left(\frac{\partial f}{\partial a}\right)^2 + \frac{\epsilon_1\epsilon_2}{24}\,\frac{\partial^2 f}{\partial a^2} = 0$$

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Generalizations

- In the ADE case, this equation can be used to prove that S-duality acts on the prepotential as a Fourier transform

$$\exp\left(-\frac{S[F](a_D)}{\epsilon_1\epsilon_2}\right) = \left(\frac{i\,\tau}{\epsilon_1\epsilon_2}\right)^{n/2}\int d^n x\,\exp\left(\frac{2\pi\,i\,a_D\cdot x - F(x)}{\epsilon_1\epsilon_2}\right)$$

(Billo et al '13)

- This is consistent with viewing
  - a and a_D as canonically conjugate variables
  - S-duality as a canonical transformation and

$$\mathcal{Z}(a,\epsilon) = \exp\left(-\frac{F(a,\epsilon)}{\epsilon_1\epsilon_2}\right)$$

as a wave-function in this space with $\epsilon_1\epsilon_2$ as Planck's constant, in agreement with the topological string

(BCOV '93, Witten '93,Aganagic et al '06, Gunaydin et al '06 …)

Applications

- Using Pestun's localization formula

$$Z_{S^4} = \int d^n x \, \left| \exp\left(- \frac{F(a,\epsilon)}{\epsilon_1 \epsilon_2} \right) \right|^2 \Bigg|_{a = i\,x ; \epsilon_1 = \epsilon_2 = \frac{1}{R}}$$

  and our modular anomaly equation, one can easily prove that the partition function on the sphere Z_{S^4} is modular invariant (a result that was expected on general grounds)

- From Z_{S^4} one can compute (by simply doing gaussian integrations) several interesting observables
  - Wilson loops
  - Zamolodchikov metric
  - Correlation functions
  - …

(Pestun '07, … ,
Baggio, Papadodimas et al '14, '16
Fiol et al '15,
Gerchkovitz, Gomis, Komargodski et al '16)

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- From Z_{S^4} one can compute (by simply doing gaussian integrations) several interesting observables

- Our S-duality results could be used to promote these calculations to the fully non-perturbative regime

Chiral correlators

- Other observables of the theory are the chiral correlators

$$< Tr\phi^n >= \sum_{i=1}^{N} a_i^n + \cdots$$

- They can be computed using equivariant localization

(Bruzzo et al. 03, Losev et al. 03, Flume et al. 04, Billò et al. '12)

- The results can be expressed in terms of modular functions and lattice sums

(Ashok et al. '16)

Chiral correlators

- Using the explicit results for $< Tr\,\phi^n >$, it is possible to change basis and find the quantum symmetric polynomials in the a's

$$A_n(\tau, a) = \sum_{i_1<i_2<\cdots<i_n} a_{i_1} a_{i_2} \cdots a_{i_n} + \cdots$$

that transform as modular form of weight n $S(A_n) = \tau^n A_n$

$$A_1 = \sum_{i_i} a_{i_1}$$

$$A_2 = \sum_{i_i<i_2} a_{i_1} a_{i_2} + \binom{N}{2} \frac{m^2}{12} E_2 + \frac{m^4}{288} (E_2^2 - E_4) C_2 + \cdots$$

Chiral correlators

- These expressions

$$A_1 = \sum_{i_i} a_{i_1}$$

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coincide with the solution of the modular anomaly equation satisfied by the A_n's

$$\frac{\partial A_n}{\partial E_2} + \frac{1}{24} \frac{\partial A_n}{\partial a} \frac{\partial f}{\partial a} = 0$$

that can be obtained directly from their S-duality properties!

Conclusions

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- The requirement that the duality group acts simply as in the *N*=4 theories also in the mass-deformed cases leads to a modular anomaly equation
- This allows one to efficiently reconstruct the mass-expansion of the prepotential and the chiral correlators resumming all instanton corrections into (quasi-)modular forms of the duality group
- The existence of such modular anomaly equations seems to be a general feature for many observables!

Conclusions

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- This allows one to efficiently reconstruct the mass-expansion of the prepotential and the chiral correlators resumming all instanton corrections into (quasi-)modular forms of the duality group
- A similar pattern (although a bit more intricate) arises in N=2 SQCD theories with $N_f=2N_c$ fundamental flavours, where it has been possible to describe the structure of the low energy effective theory at the special vacuum

(Ashok et al. ’15 and ‘16)

Conclusions

- This approach can be profitably used in other contexts to study the consequences of S-duality on:

  - theories formulated in curved spaces (*e.g.* S^4)
  - correlation functions of chiral and anti-chiral operators
  - other observables (e.g. Wilson loops, cusp anomaly, …)
  - more general extended observables (surface operators, ...)
  - …

  with the goal of studying the strong-coupling regime

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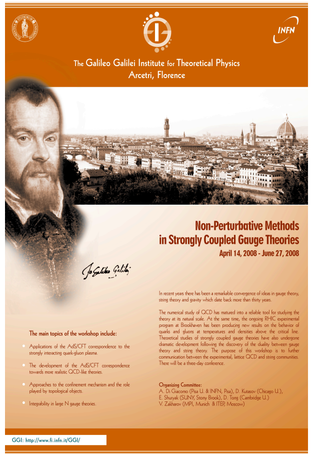
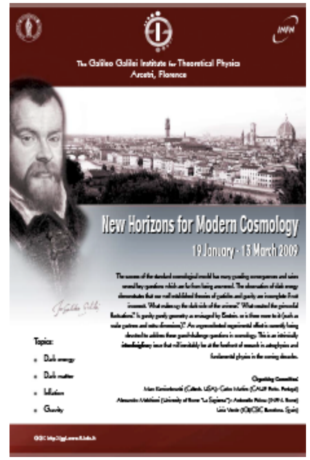
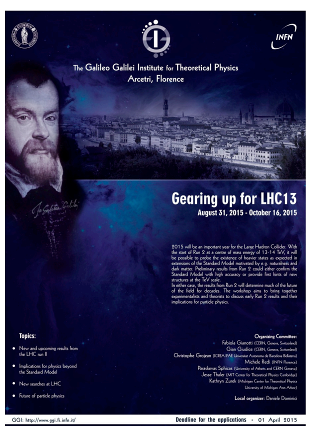
Thank you for your attention

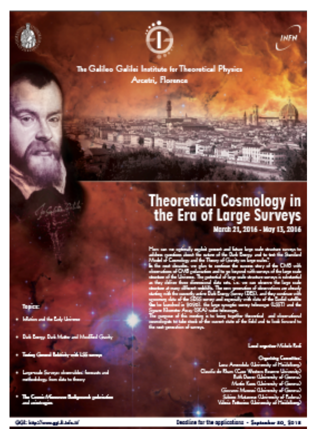
In May 2016 we celebrated the 10th anniversary of the GGI:



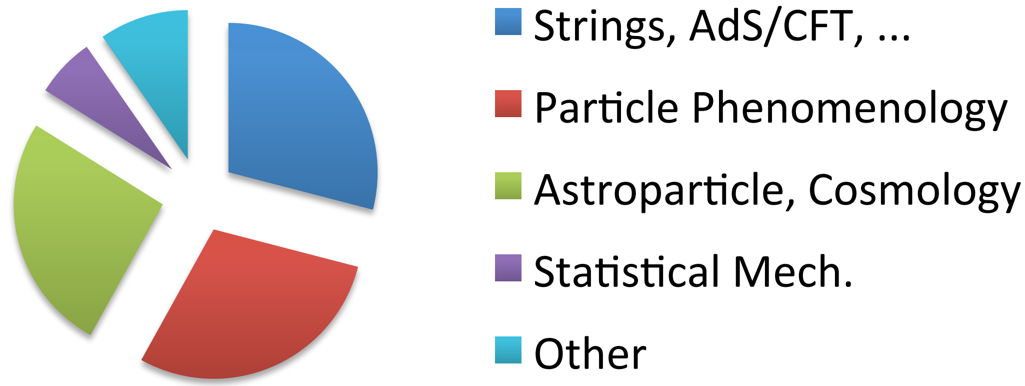


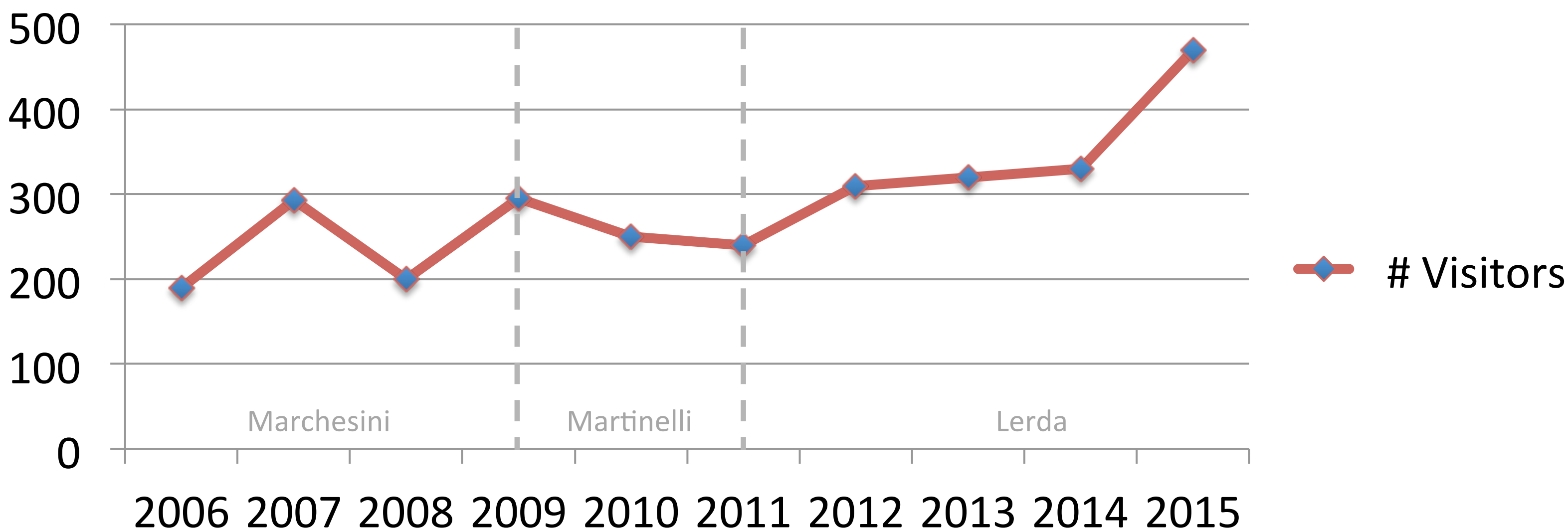
30 extended workshops with more than 3500 participants





30 Workshops





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Lezioni Avanzate di Campi E Stringhe

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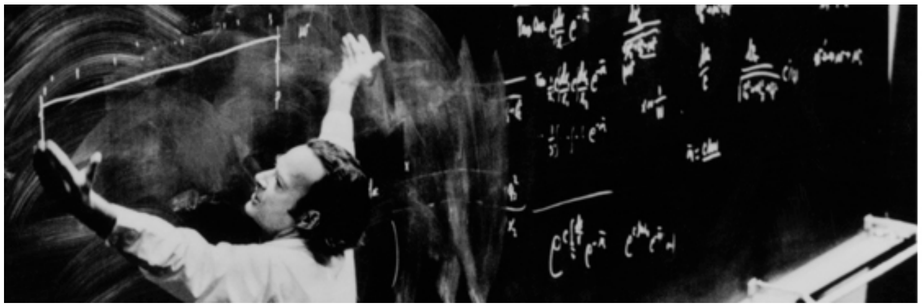

GGI LECTURES ON THE THEORY OF FUNDAMENTAL INTERACTIONS

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Happy birthday to Supergravity !!

Happy birthday to GGI !!