

Resumming Instantons in $N=2^*$ Theories

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This talk is mainly based on:

- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, “*S-duality and the prepotential in $N=2$ theories (I): the ADE algebras*,” JHEP **1511** (2015) 024, [arXiv:1507.07709](#)
- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, “*S-duality and the prepotential in $N=2$ theories (II): the non-simply laced algebras*,” JHEP **1511** (2015) 026, [arXiv:1507.08027](#)
- M. Billò, M. Frau, F. Fucito, A.L. and J.F. Morales, “*Resumming instantons in $N=2$ theories*,” XIV Marcel Grossmann Meeting, [arXiv:1602.00273](#)

and

- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, A.L. and M. Raman, “*Modular anomaly equations and S-duality in $N=2$ conformal SQCD*,” JHEP **1510** (2015) 091, [arXiv:1507.07476](#)
- S.K. Ashok, E. Dell'Aquila, A.L. and M. Raman, “*S-duality, triangle groups and modular anomalies in $N=2$ SQCD*,” JHEP **1604** (2016) 118, [arXiv:1601.01827](#)
- S.K. Ashok, M. Billò, E. Dell'Aquila, M. Frau, A.L., M. Moskovic, M. Raman, “*Chiral observables and S-duality in $N=2$ $U(N)$ gauge theories*”, [arXiv:1607.08327](#) to be published on JHEP

but it builds on **a very vast literature ...**

Plan of the talk

1. Introduction
2. $N=4$ SYM
3. $N=2^*$ SYM
4. Conclusions

- Non-perturbative effects are important:
 - in **gauge theories**: confinement, chiral symmetry breaking, ...
 - in **string theories**: D-branes, duality, AdS/CFT, ...
- They are essential to complete the perturbative expansion and lead to **results valid at all couplings**
- In supersymmetric theories, tremendous progress has been possible thanks to the development of **localization techniques**

(Nekrasov '02, Nekrasov-Okounkov '03, Pestun '07, ..., Nekrasov-Pestun '13,)
- In superconformal theories these methods allowed us to compute **exactly** several quantities:
 - Sphere partition function and free energy
 - Wilson loops
 - Correlation functions, amplitudes
 - Cusp anomalous dimensions and bremsstrahlung function

- We will focus on **SYM theories in $4d$ with $N=2$ supersymmetry**
 - They are less constrained than the $N=4$ theories
 - They are sufficiently constrained to be analyzed exactly
- We will be interested in studying how **S-duality** on the quantum effective couplings constrains the **prepotential and the observables of $N=2$ theories**

(earlier work by Minahan et al. '96, '97)
- We will make use of these constraints to obtain **exact expressions valid at all couplings**

$N=4$ SYM

$N=4$ SYM

- Consider $N=4$ SYM in $d=4$

- This theory is **maximally supersymmetric** (16 SUSY charges)
- The field content is

A 1 vector

λ^a ($a = 1, \dots, 4$) 4 Weyl spinors

X^i ($i = 1, \dots, 6$) 6 real scalars

- All fields are in the **adjoint** repr. of the gauge group G
- The **β -function vanishes** to all orders in perturbation theory
- If $\langle X^i \rangle = 0$, the theory is **superconformal** (*i.e.* invariant under $SU(2, 2|4)$) also at the quantum level

$N=4$ SYM

- The relevant ingredients of $N=4$ SYM are:
 - The gauge group G (or the gauge algebra \mathfrak{g})
 - The (complexified) coupling constant

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} \in \mathbb{H}_+$$

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- Many exact results have been obtained using:
 - Explicit expressions of scattering amplitudes
 - Integrability
 - AdS/CFT correspondence
 - **Duality**

$N=4$ SYM

- $N=4$ SYM is believed to possess an **exact duality invariance** which contains the electro-magnetic duality S

(Montonen-Olive '77, Vafa-Witten '94, Sen '94, ...)

- If the gauge algebra \mathfrak{g} is simply laced (ADE)
 - S maps the theory to itself but with **electric** and **magnetic** states exchanged
 - It is a **weak/strong** duality, acting on the coupling by

$$S(\tau) = -1/\tau$$

- Together with $T(\tau) = \tau + 1$ ($\theta \rightarrow \theta + 2\pi$), it generates the modular group $\Gamma = \text{SL}(2, \mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad S^2 = -1, \quad (ST)^3 = -1$$

$N=4$ SYM

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- If the algebra \mathfrak{g} of the gauge group G is non-simply laced (BCFG) duality relation still exist, but they are more involved...

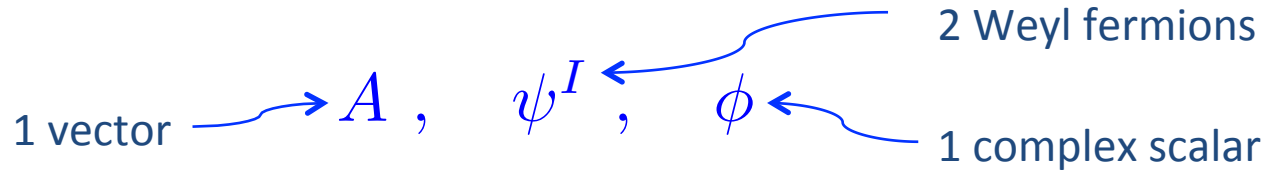
(see Billò et al. '15 and Ashok et al. '16)

- For simplicity I will only describe the case of simply laced algebras \mathfrak{g} , but all the arguments can be generalized to include also the non-simply laced cases

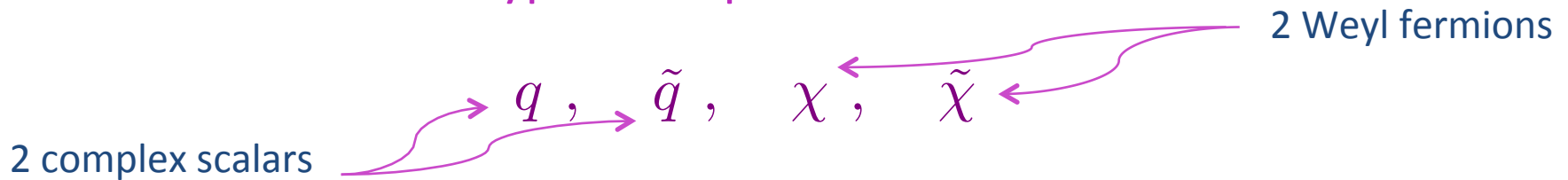
$N=4$ SYM as a $N=2$ theory

Let us decompose the $N=4$ multiplet into

- one $N=2$ vector multiplet



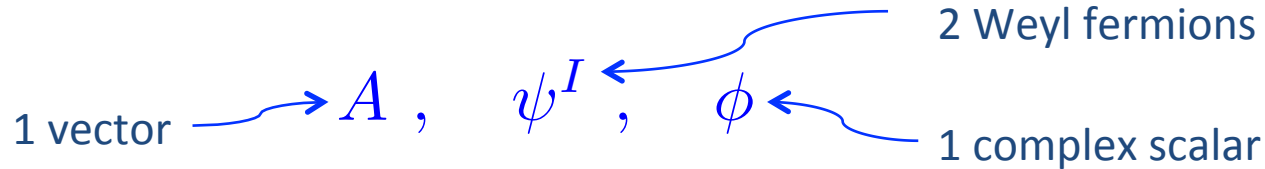
- one $N=2$ hypermultiplet



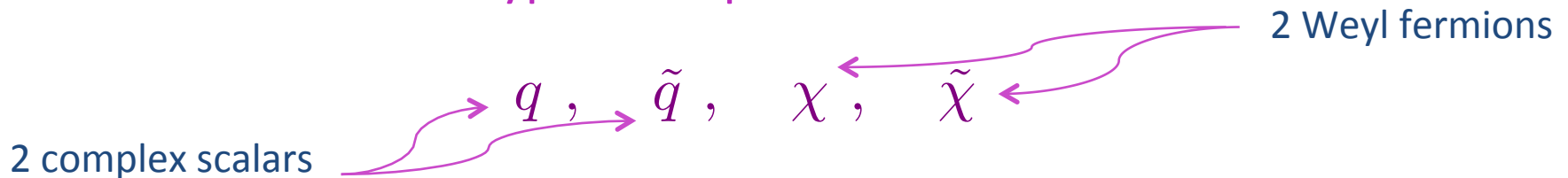
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By introducing the v.e.v.

$$\langle \phi \rangle = a = \text{diag}(a_1, \dots, a_n)$$

- we break the gauge group $G \rightarrow U(1)^n$
- we spontaneously break conformal invariance
- we can describe the dynamics in terms of a **holomorphic prepotential** $F(a)$, as in $N=2$ theories

$N=4$ SYM as a $N=2$ theory

- The prepotential of the $N=4$ theory is simply

$$F = i \pi \tau a^2$$

- The **dual variables** are defined as

$$a_D \equiv \frac{1}{2\pi i} \frac{\partial F}{\partial a} = \tau a$$

- S-duality relates the **electric** variable a to the **magnetic** variable a_D :

$$S \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -a \\ a_D \end{pmatrix}$$

$N=4$ SYM as a $N=2$ theory

- Let's find the **S-dual prepotential**:

$$S(F) = i\pi \left(-\frac{1}{\tau} \right) (a_D)^2 = -i\pi \frac{1}{\tau} a_D^2$$

- S-duality exchanges the description based on a with its **Legendre-transform**, based on a_D :

$$\begin{aligned} \mathcal{L}(F) &= F - a \frac{\partial F}{\partial a} = i\pi\tau a^2 - 2\pi i a a_D \\ &= -i\pi \frac{1}{\tau} a_D^2 \end{aligned}$$

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- Thus

$$S(F) = \mathcal{L}(F)$$

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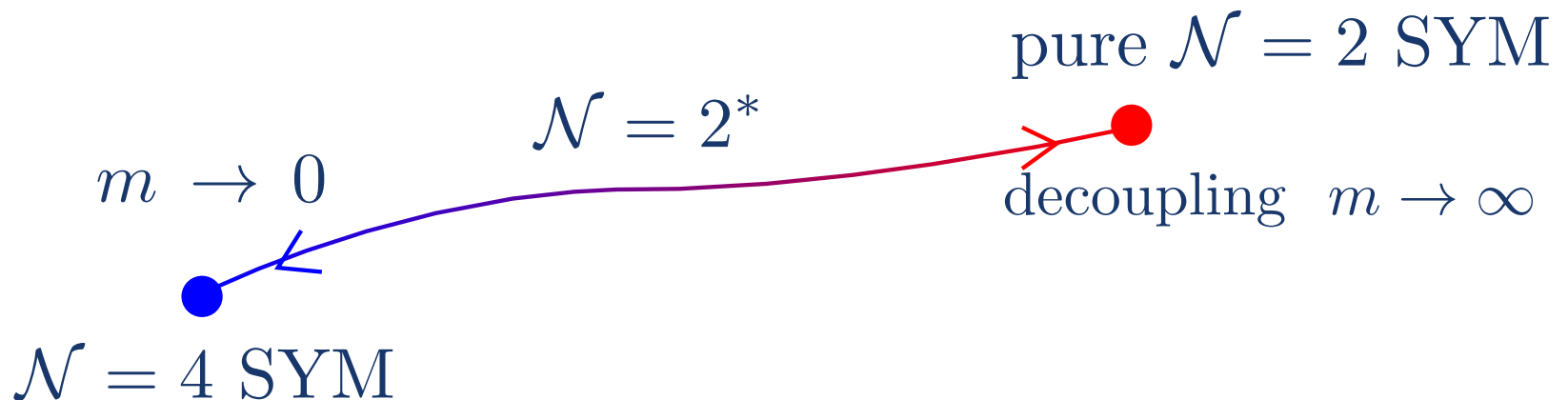
$$S(F) = \mathcal{L}(F)$$

- This structure is present also in **$N=2$ theories** and has important consequences on their **strong coupling dynamics!**

$N=2^*$ SYM

The $N=2^*$ set-up

- The $N=2^*$ theory is a mass deformation of the $N=4$ SYM
- Field content:
 - one $N=2$ vector multiplet for the algebra \mathfrak{g}
 - one $N=2$ hypermultiplet in the adjoint rep. of \mathfrak{g} with mass m
- Half of the supercharges are broken, and we have $N=2$ SUSY
- The β -function still vanishes, but the superconformal invariance is explicitly broken by the mass m

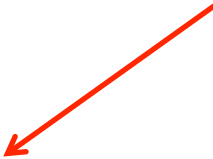


Structure of the $N=2^*$ prepotential

- The $N=2^*$ prepotential contains **classical, 1-loop and non-perturbative terms**

$$F = i \pi \tau a^2 + f \quad \text{with} \quad f = f_{1-loop} + f_{non-pert}$$

- The **1-loop term** reads

$$\frac{1}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \left[-(\alpha \cdot a)^2 \log \left(\frac{\alpha \cdot a}{\Lambda} \right)^2 + (\alpha \cdot a + m)^2 \log \left(\frac{\alpha \cdot a + m}{\Lambda} \right)^2 \right]$$


- $\Psi_{\mathfrak{g}}$ is the set of the roots α of the algebra \mathfrak{g}
 - $\alpha \cdot a$ is the mass of the W-boson associated to the root α
- The **non-perturbative contributions** come from all **instanton sectors** and are proportional to q^k and can be explicitly computed using **localization** for all classical algebras

S-duality and the prepotential

- The **dual variables** are defined as

$$a_D \equiv \frac{1}{2\pi i} \frac{\partial F}{\partial a} = \tau \left(a + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial a} \right)$$

- Applying **S-duality** we get

$$S(F) = i \pi \left(-\frac{1}{\tau} \right) a_D^2 + f \left(-\frac{1}{\tau}, a_D \right)$$

- Computing the **Legendre transform** we get

$$\begin{aligned} \mathcal{L}(F) &= F - 2i\pi a \cdot a_D \\ &= i \pi \left(-\frac{1}{\tau} \right) a_D^2 + f(\tau, a) + \frac{1}{4i\pi\tau} \left(\frac{\partial f}{\partial a} \right)^2 \end{aligned}$$

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S-duality and the prepotential

- Requiring

$$S(F) = \mathcal{L}(F)$$

implies

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \frac{1}{4i\pi\tau} \left(\frac{\partial f}{\partial a}\right)^2$$

Modular anomaly equation!

- This constraint has very deep implications!

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Modular anomaly equation!

- This constraint has very deep implications!
- The **modular anomaly equation** is related to the **holomorphic anomaly equation** of the local CY topological string description of the low-energy effective theory

(BCOV '93, Witten '93, ... Aganagic et al '06, Gunaydin et al '06, Huang et al 09, Huang '13, ...)

Modular anomaly equation

- We organize the quantum prepotential f in a mass expansion

$$f(\tau, a) = \sum_{n=1} f_n(\tau, a) \quad \text{with } f_n \propto m^{2n}$$

- From explicit calculations, one sees that:

- f_1 is only **1-loop** and thus τ -independent

$$f_1(a) = \frac{m^2}{4} \sum_{\alpha \in \Psi_g} \log \left(\frac{\alpha \cdot a}{\Lambda} \right)^2$$

- f_n ($n \geq 2$) are both **1-loop** and **non-perturbative**. They are homogeneous functions

$$f_n(\tau, \lambda a) = \lambda^{2-2n} f_n(\tau, a)$$

(This is because the prepotential has mass dimension 2)

Modular anomaly equation

- The modular anomaly equation

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \frac{\delta}{24} \left(\frac{\partial f}{\partial a}\right)^2, \quad \delta = \frac{6}{i\pi\tau}$$

implies

$$f_n\left(-\frac{1}{\tau}, a_D\right) = f_n(\tau, a) + \dots$$

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implies

$$f_n\left(-\frac{1}{\tau}, a_D\right) = f_n(\tau, a) + \dots$$

- $n = 1$

- Using $f_1(a) = \frac{m^2}{4} \sum_{\alpha \in \Psi_{\mathfrak{g}}} \log\left(\frac{\alpha \cdot a}{\Lambda}\right)^2$ and


requiring that under S-duality $\Lambda \rightarrow \tau \Lambda$, we have

$$f_1(a_D) = f_1(\tau a + \dots) = f_1(a) + \dots \quad \checkmark$$

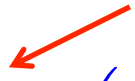
Modular anomaly equation

■ $n = 2$

- Using the definition of the dual variable and the homogeneity property, we have

$$f_2\left(-\frac{1}{\tau}, a_D\right) = f_2\left(-\frac{1}{\tau}, \tau(a + \dots)\right) = \tau^{-2} f_2\left(-\frac{1}{\tau}, a + \dots\right)$$


- In order to solve the equation, we must require that

$$f_2\left(-\frac{1}{\tau}, a + \dots\right) = \tau^2 f_2\left(\tau, a + \dots\right) = \tau^2 f_2\left(\tau, a\right) + \dots$$


- i.e. $f_2(\tau, a)$ should have modular weight 2 under S-duality !
- ## ■ The only quantity with this property is the **second Eisenstein series E_2** (quasi-modular)

Modular anomaly equation

■ Generic n

- The previous analysis can be easily generalized to arbitrary n .
- In order to be able to solve the equation, we must have

$$f_n\left(-\frac{1}{\tau}, a + \dots\right) = \tau^{2n-2} f_n(\tau, a) + \dots$$

- Thus we must require that f_n^g depends on τ through “modular” functions with weight $2n - 2$, *i.e.*

$$f_n(\tau, a) = f_n\left(E_2(\tau), E_4(\tau), E_6(\tau), a\right)$$

where $E_2(\tau), E_4(\tau), E_6(\tau)$ are the Eisenstein series

Eisenstein series

- The Eisenstein series are “modular” forms with a well-known Fourier expansion in $q = e^{2i\pi\tau}$:

$$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \dots$$


$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \dots$$

$$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \dots$$

- E_4 and E_6 are **truly modular forms** of weight 4 and 6

$$E_4\left(-\frac{1}{\tau}\right) = \tau^4 E_4(\tau) \quad , \quad E_6\left(-\frac{1}{\tau}\right) = \tau^6 E_6(\tau)$$

- E_2 is **quasi-modular** of weight 2

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 [E_2(\tau) + \delta] \quad , \quad \delta = \frac{6}{i\pi\tau}$$


- Thus a modular form of weight w is mapped under S into a form of weight w times τ^w , **up to shifts induced by E_2**

Recursion relation

- S-duality

$$\begin{aligned} f\left(-\frac{1}{\tau}, a_D\right) &= f\left(E_2\left(-\frac{1}{\tau}\right), E_4\left(-\frac{1}{\tau}\right), E_6\left(-\frac{1}{\tau}\right), \tau\left(a + \frac{\delta}{12} \frac{\partial f}{\partial a}\right)\right) \\ &= f\left(E_2 + \delta, E_4, E_6, \left(a + \frac{\delta}{12} \frac{\partial f}{\partial a}\right)\right) \\ &= f(\tau, a) + \delta \left[\frac{\partial f}{\partial E_2} + \frac{1}{12} \left(\frac{\partial f}{\partial a}\right)^2 \right] + \mathcal{O}(\delta^2) \end{aligned}$$

- Modular anomaly equation

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \delta \left[\frac{1}{24} \left(\frac{\partial f}{\partial a}\right)^2 \right]$$

Recursion relation

- S-duality

$$f\left(-\frac{1}{\tau}, a_D\right) = f\left(E_2\left(-\frac{1}{\tau}\right), E_4\left(-\frac{1}{\tau}\right), E_6\left(-\frac{1}{\tau}\right), \tau\left(a + \frac{\delta}{12} \frac{\partial f}{\partial a}\right)\right)$$

$$= f\left(E_2 + \delta, E_4, E_6, \left(a + \frac{\delta}{12} \frac{\partial f}{\partial a}\right)\right)$$

$$= f(\tau, a) + \delta \left[\frac{\partial f}{\partial E_2} + \frac{1}{12} \left(\frac{\partial f}{\partial a}\right)^2 \right] + \mathcal{O}(\delta^2)$$

- Modular anomaly equation

$$f\left(-\frac{1}{\tau}, a_D\right) = f(\tau, a) + \delta \left[\frac{1}{24} \left(\frac{\partial f}{\partial a}\right)^2 \right]$$

Recursion relation

- We thus obtain

$$\frac{\partial f}{\partial E_2} + \frac{1}{24} \left(\frac{\partial f}{\partial a} \right)^2 = 0$$

which implies the following **recursion relation**

(Minahan et al '97)

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24} \sum_{\ell=1}^{n-1} \frac{\partial f_\ell}{\partial a} \frac{\partial f_{n-\ell}}{\partial a}$$

- This allows us to determine f_n from the lower coefficients up to E_2 -independent terms. These are fixed by comparison with **the perturbative expressions** (or the first instanton corrections).
 - The modular anomaly equation is a symmetry requirement; it does not eliminate the need of a dynamical input
- Once this is done, **the result is valid to all instanton orders.**

Exploiting the recursion

- Using this recursive procedure we find

$$f_2 = -\frac{m^4}{24} E_2 C_2^{\mathfrak{g}}$$

$$f_3 = -\frac{m^6}{720} (5E_2^2 + E_4) C_2^{\mathfrak{g}} - \frac{m^6}{576} (E_2^2 - E_4) C_{2;1,1}^{\mathfrak{g}}$$

where $C_2^{\mathfrak{g}}$ and $C_{2;1,1}^{\mathfrak{g}}$ are root lattice sums of \mathfrak{g} defined as

$$C_2^{\mathfrak{g}} = \sum_{\alpha \in \Psi_{\mathfrak{g}}} \frac{1}{(\alpha \cdot a)^2}$$

$$C_{2;1,1}^{\mathfrak{g}} = \sum_{\alpha \in \Psi_{\mathfrak{g}}} \sum_{\beta_1 \neq \beta_2 \in \Psi_{\mathfrak{g}}(\alpha)} \frac{1}{(\alpha \cdot a)^2 (\beta_1 \cdot a) (\beta_2 \cdot a)}$$

with $\Psi_{\mathfrak{g}}(\alpha) = \{\beta \in \Psi_{\mathfrak{g}} : \alpha \cdot \beta = 1\}$

Exploiting the recursion

- For example

$$C_2^{U(2)} = \frac{1}{(a_1 - a_2)^2}$$

$$C_2^{U(3)} = \frac{1}{(a_1 - a_2)^2} + \frac{1}{(a_1 - a_2)^2} + \frac{1}{(a_2 - a_3)^2}$$

and thus

$$f_2^{U(2)} = -\frac{m^4}{24} E_2(\tau) C_2^{U(2)}$$

$$f_2^{U(3)} = -\frac{m^4}{24} E_2(\tau) C_2^{U(3)}$$

- From the Fourier expansion of E_2 we get **the perturbative and all non-perturbative contributions** to the prepotential at order m^4 !
- There are no free parameters !

Checks on the results

- For the classical algebras A, B, C and D
 - the **ADHM construction** of the k instanton moduli spaces is available
 - the integration of the moduli action over the **instanton moduli spaces** can be performed à la Nekrasov using **localization techniques**
- In principle straightforward; in practice computationally rather intense. **Not many explicit results for the $N=2^*$ theories** in the literature.
- We worked it out:
 - for A_n and D_n with $n < 6$, up to 5 instantons;
 - for C_n with $n < 6$, up to 4 instantons;
 - for B_n with $n < 6$, up to 2 instantons.
- The results **match** the q -expansion of those obtained above
- **For the exceptional algebras our results are predictions!**

Generalizations

- These results can be extended to non-flat space-times by turning-on the so-called Ω background

$$\begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}$$

(Nekrasov '02)

which actually was already present in the localization calculations

- For $\epsilon_1, \epsilon_2 \neq 0$ one finds that the generalized prepotential

$$F = i \pi \tau a^2 + f(a, \epsilon)$$

obeys a **generalized modular anomaly equation**

$$\frac{\partial f}{\partial E_2} + \frac{1}{24} \left(\frac{\partial f}{\partial a} \right)^2 + \frac{\epsilon_1 \epsilon_2}{24} \frac{\partial^2 f}{\partial a^2} = 0$$

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Generalizations

- In the ADE case, this equation can be used to prove that S-duality acts on the prepotential as a **Fourier transform**

$$\exp\left(-\frac{S[F](a_D)}{\epsilon_1\epsilon_2}\right) = \left(\frac{i\tau}{\epsilon_1\epsilon_2}\right)^{n/2} \int d^n x \exp\left(\frac{2\pi i a_D \cdot x - F(x)}{\epsilon_1\epsilon_2}\right)$$

(Billo et al '13)

- This is consistent with viewing
 - a and a_D as **canonically conjugate variables**
 - S-duality as a **canonical transformation** and

$$\mathcal{Z}(a, \epsilon) = \exp\left(-\frac{F(a, \epsilon)}{\epsilon_1\epsilon_2}\right)$$

as a wave-function in this space with $\epsilon_1\epsilon_2$ as **Planck's constant**, in agreement with the topological string

(BCOV '93, Witten '93, Aganagic et al '06, Gunaydin et al '06 ...)

Applications

- Using **Pestun's localization** formula

$$Z_{S^4} = \int d^n x \left| \exp \left(- \frac{F(a, \epsilon)}{\epsilon_1 \epsilon_2} \right) \right|^2 \Big|_{a=i x; \epsilon_1 = \epsilon_2 = \frac{1}{R}}$$

and our **modular anomaly equation**, one can easily prove that the partition function on the sphere Z_{S^4} is **modular invariant** (a result that was expected on general grounds)

- From Z_{S^4} one can compute (by simply doing gaussian integrations) several interesting observables

- Wilson loops
- Zamolodchikov metric
- Correlation functions
- ...

(Pestun '07, ... ,
Baggio, Papadodimas et al '14, '16
Fiol et al '15,
Gerchkovitz, Gomis, Komargodski et al '16)

Applications

- Using **Pestun's localization** formula

$$Z_{S^4} = \int d^n x \left| \exp \left(- \frac{F(a, \epsilon)}{\epsilon_1 \epsilon_2} \right) \right|^2 \Big|_{a=i x; \epsilon_1 = \epsilon_2 = \frac{1}{R}}$$

and our modular anomaly equation, one can easily prove that the partition function on the sphere Z_{S^4} is **modular invariant** (a result that was expected on general grounds)

- From Z_{S^4} one can compute (by simply doing gaussian integrations) several interesting observables
- Our S-duality results could be used to promote these calculations to the **fully non-perturbative regime**

Chiral correlators

- Other observables of the theory are the chiral correlators

$$\langle \text{Tr} \phi^n \rangle = \sum_{i=1}^N a_i^n + \dots$$

- They can be computed using equivariant localization

(Bruzzo et al. 03, Losev et al. 03, Flume et al. 04, Billò et al. '12)

- The results can be expressed in terms of modular functions and lattice sums

(Ashok et al. '16)

Chiral correlators

- Using the explicit results for $\langle \text{Tr } \phi^n \rangle$, it is possible to change basis and find the quantum symmetric polynomials in the a 's

$$A_n(\tau, a) = \sum_{i_1 < i_2 < \dots < i_n} a_{i_1} a_{i_2} \dots a_{i_n} + \dots$$

that transform as modular form of weight n $S(A_n) = \tau^n A_n$

$$A_1 = \sum_{i_1} a_{i_1}$$

$$A_2 = \sum_{i_1 < i_2} a_{i_1} a_{i_2} + \binom{N}{2} \frac{m^2}{12} E_2 + \frac{m^4}{288} (E_2^2 - E_4) C_2 + \dots$$

Chiral correlators

- These expressions

$$A_1 = \sum_{i_i} a_{i_1}$$

$$A_2 = \sum_{i_i < i_2} a_{i_1} a_{i_2} + \binom{N}{2} \frac{m^2}{12} E_2 + \frac{m^4}{288} (E_2^2 - E_4) C_2 + \dots$$

coincide with the solution of the modular anomaly equation satisfied by the A_n 's

$$\frac{\partial A_n}{\partial E_2} + \frac{1}{24} \frac{\partial A_n}{\partial a} \frac{\partial f}{\partial a} = 0$$

that can be obtained directly from their S-duality properties!

Conclusions

Conclusions

- The requirement that the **duality group** acts simply as in the $N=4$ theories also in the mass-deformed cases leads to a **modular anomaly equation**
- This allows one to efficiently reconstruct the mass-expansion of the prepotential and the chiral correlators **resumming all instanton corrections** into (quasi-)modular forms of the duality group
- The existence of such modular anomaly equations seems to be **a general feature** for many observables!

Conclusions

- The requirement that the **duality group** acts simply as in the $N=4$ theories also in the mass-deformed cases leads to a **modular anomaly equation**
- This allows one to efficiently reconstruct the mass-expansion of the prepotential and the chiral correlators **resumming all instanton corrections** into (quasi-)modular forms of the duality group
- A similar pattern (although a bit more intricate) arises in **$N=2$ SQCD theories** with $N_f=2N_c$ fundamental flavours, where it has been possible to describe the structure of the low energy effective theory at the special vacuum

(Ashok et al. '15 and '16)

Conclusions

- This approach can be profitably used in other contexts to study the consequences of **S-duality** on:
 - theories formulated in **curved spaces** (e.g. S^4)
 - correlation functions of chiral and anti-chiral operators
 - other observables (e.g. Wilson loops, cusp anomaly, ...)
 - more general extended observables (surface operators, ...)
 - ...

with the goal of studying the **strong-coupling regime**

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with the goal of studying the **strong-coupling regime**

Thank you for your attention



In May 2016 we celebrated the 10th anniversary of the GGI:



Workshop at the Galileo Galilei Institute in Spring 2008
May 2nd - June 20th
New Directions Beyond the Standard Model in Field and String Theory

The beginning of the LHC experimental program in 2007 marks a urgent to undertake a detailed study of possible extensions of the Standard Model that offer an explanation for the origin of the electroweak scale and its connection with other scales in particle physics. In recent years new ideas on the hierarchy problem have been proposed with a great impact in particle phenomenology and cosmology. On the formal side, the gauge/gravity duality conjecture has led to new computational methods for studying unquenched gauge theories, and D-brane engineering has provided new realizations of gauge symmetry and supersymmetry breaking. Moreover recent considerations in string theory have also showed the possibility of stabilizing all moduli fields, opening the way to a thorough phenomenological analysis. The purpose of the workshop, which includes a short conference, is to bring together leading experts in field theory experts to discuss ideas and debate the interaction between these considerations in preparation for the exciting LHC experimental results.

Chairman: Carlo Rovelli (University of Torino), Eraldo D'Ada (CNR, Trieste), Ivo Z. Stepanović (ICTP), Tony Cheng (University of Minnesota), Alex Fodor (Université d'Artois in Lensing)

Topics:

1. Electroweak symmetry breaking
2. Supersymmetry models and supersymmetry breaking
3. String vacua and moduli building
4. Warped compactifications and holography
5. Modifications of gravity and cosmological implications

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

Non-Perturbative Methods in Strongly Coupled Gauge Theories
April 14, 2008 - June 27, 2008

The recent study of QCD has revealed very subtle features in the behavior of quarks and gluons at intermediate and low energies that are not captured by perturbative methods. Theoretical studies of strongly coupled gauge theories have also become an important component of the study of the quark-hadron transition from the quark-gluon plasma to the hadron gas.

Topics:

- Dual string
- Dual matter
- Lattices
- Chiralty

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

New Horizons for Modern Cosmology
19 January - 13 March 2009

The success of the standard cosmological model has many exciting consequences and also several open questions that are still being pursued. The identification of dark energy, dark matter and modified theories of gravity and gravity as emergent phenomena.

Topics:

- Dark energy
- Dark matter
- Lattices
- Chiralty

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

New Perspectives in String Theory
6 April - 19 June 2009

The last few years have seen a renaissance of the Standard Model as a simplified framework. This change represents a paradigm shift in the way we think about the Standard Model and its possible extensions.

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

Indirect Searches for New Physics at the time of LHC
February 13, 2010 - March 28, 2010

The main topics of the workshop include:

1. Electroweak symmetry breaking
2. Supersymmetry models and supersymmetry breaking
3. String vacua and moduli building
4. Warped compactifications and holography
5. Modifications of gravity and cosmological implications

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30 extended workshops with more than 3500 participants

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

Statistical mechanics, integrability and combinatorics
May 11, 2015 - July 8, 2015

The last few years have seen a renaissance of the Standard Model as a simplified framework. This change represents a paradigm shift in the way we think about the Standard Model and its possible extensions.

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
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Gearing up for LHC13
August 31, 2015 - October 14, 2015

The last few years have seen a renaissance of the Standard Model as a simplified framework. This change represents a paradigm shift in the way we think about the Standard Model and its possible extensions.

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

Theoretical Cosmology in the Era of Large Surveys
March 23, 2016 - May 13, 2016

The last few years have seen a renaissance of the Standard Model as a simplified framework. This change represents a paradigm shift in the way we think about the Standard Model and its possible extensions.

OSQ: <http://www.ggi.it>

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Arezzo, Florence

Conformal Field Theories and Renormalization Group Flows in Dimensions ≥ 2
May 21, 2016 - July 8, 2016

The last few years have seen a renaissance of the Standard Model as a simplified framework. This change represents a paradigm shift in the way we think about the Standard Model and its possible extensions.

OSQ: <http://www.ggi.it>

The Galileo Galilei Institute for Theoretical Physics
Arezzo, Florence

Supergravity: what next?
September 7, 2016 - October 28, 2016

The last few years have seen a renaissance of the Standard Model as a simplified framework. This change represents a paradigm shift in the way we think about the Standard Model and its possible extensions.

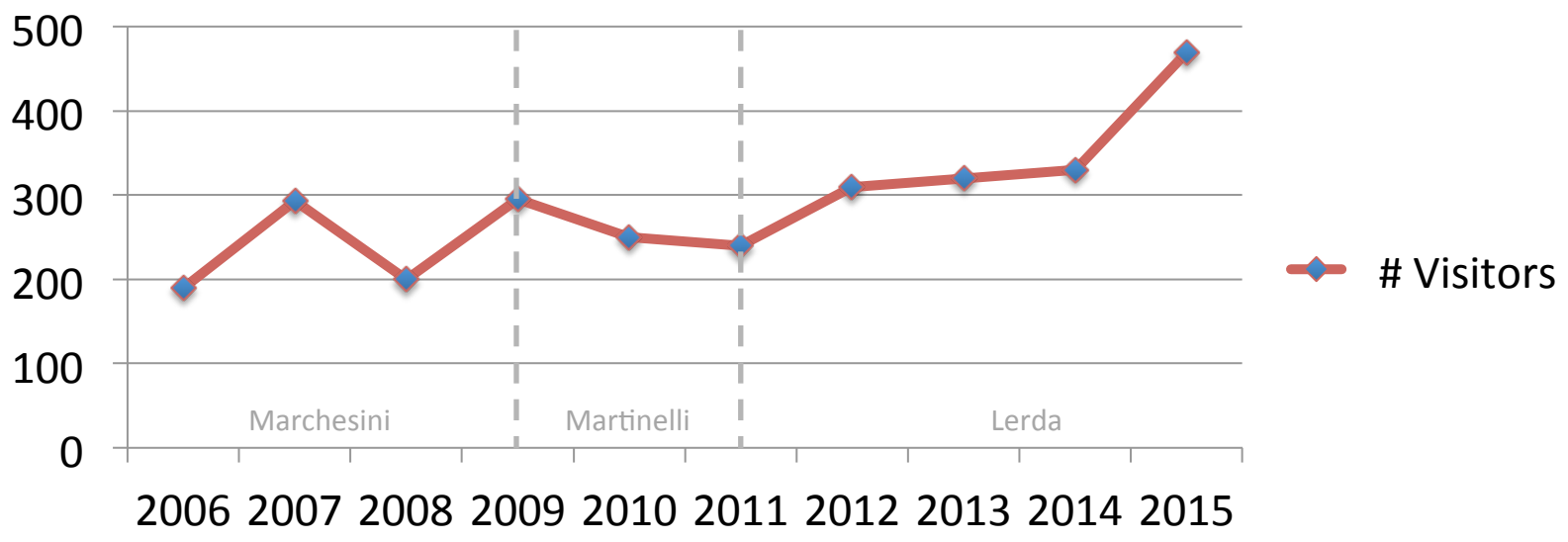
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30 Workshops



- Strings, AdS/CFT, ...
- Particle Phenomenology
- Astroparticle, Cosmology
- Statistical Mech.
- Other





LACES

Lezioni Avanzate di Campi E Stringhe

Galileo Galilei Institute for Theoretical Physics, Arcetri - Italy



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Galileo Galilei Institute for Theoretical Physics
Firenze

WINTER PHD SCHOOL ON *STATISTICAL FIELD THEORIES*

The Galileo Galilei Institute for Theoretical Physics
Arcetri Firenze



The Galileo Galilei Institute for Theoretical Physics
Arcetri, Florence



**Frontiers in Nuclear and
Hadronic Physics**
School

Happy birthday to Supergravity !!



Happy birthday to GGI !!

