Two–Loop QED Corrections in e^+e^- Annihilation and Massive Fermionic Operator Matrix Elements

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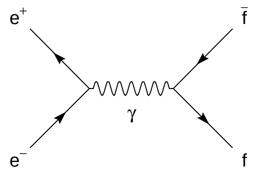
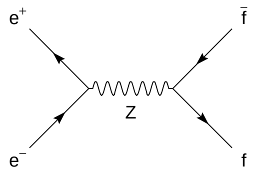
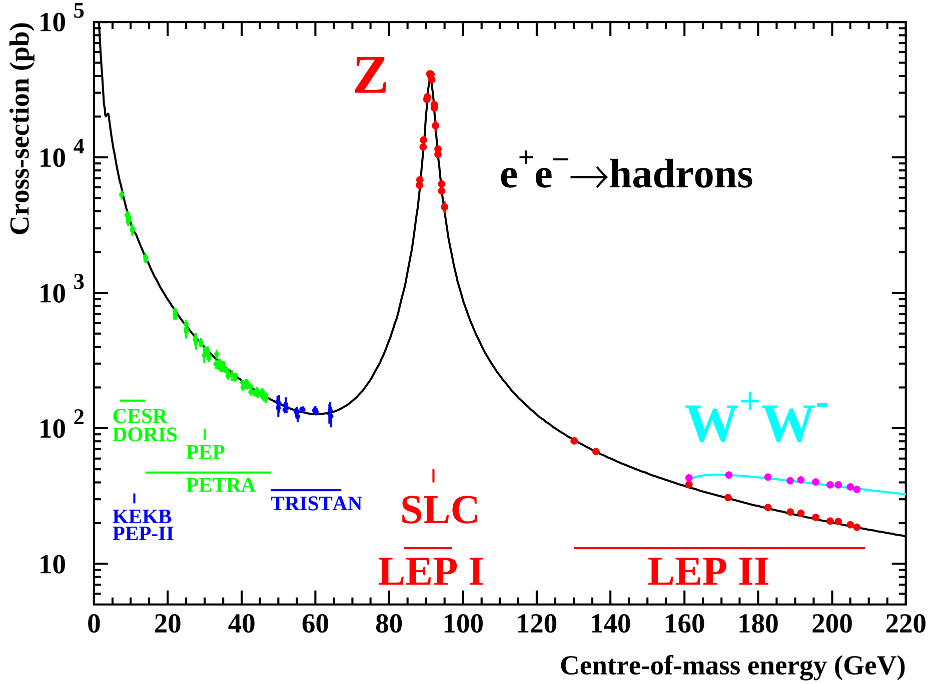
We want to revisit the calculation of the two-loop order initial state radiative corrections to electron-positron annihilation into heavy fermions:







A long time ago, Berends, van Neerven and Burgers (Nucl. Phys. **B297** 1988) 429; E: B304 (1988) 921.) calculated the corrections due to initial state radiation directly (for massive electrons), including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production.

- The process is of central importance at LEP, Giga-Z, and the ILC.
- Recalculate these corrections by a method based on the renormalization group.

1. Introduction

The Born Cross Section : $e^+e^- \to f, \overline{f} \quad f \neq e$

$$\frac{d\sigma^{(0)}(s)}{d\Omega} = \frac{\alpha^2}{4s} N_{C,f} \sqrt{1 - \frac{4m_f}{s}}$$
$$\times \left[\left(1 + \cos^2\theta + \frac{4m_f^2}{s} \sin^2\theta \right) G_1(s) - \frac{8m_f^2}{s} G_2(s) + 2\sqrt{1 - \frac{4m_f^2}{s}} \cos\theta G_3(s) \right]$$
$$\sigma^{(0)}(s) = \frac{4\pi\alpha^2}{3s} N_{C,f} \sqrt{1 - \frac{4m_f}{s}} \left[\left(1 + \frac{2m_f^2}{s} \right) G_1(s) - 6\frac{m_f^2}{s} G_2(s) \right]$$

$$G_1(s) = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \mathrm{Re}[\chi_Z(s)] + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z(s)|^2$$
$$G_2(s) = (v_e^2 + a_e^2) a_f^2 |\chi_Z(s)|^2$$
$$G_3(s) = 2Q_e Q_f a_e a_f \mathrm{Re}[\chi_Z(s)] + 4v_e v_f a_e a_f |\chi_Z(s)|^2.$$

$$\chi_Z(s) = \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$$

2. The Renormalization Group Technique

We represent the observable in Mellin space transforming $z = s'/s \; \epsilon [0,1]$:

The differential scattering cross section $\Sigma(z) = d\sigma_{ij}(z)/ds'$ is considered. This quantity reads in Mellin space

$$\mathbf{M}[\Sigma(z)](N) = \int_0^1 dz z^{N-1} \Sigma(z) \; .$$

In this representation the different Mellin convolutions to be performed in z–space simplify to ordinary products. The following representation is obtained

$$\frac{d\sigma_{ij}(z)}{ds'} = \frac{1}{s} \sigma^{(0}(N) \sum_{l,k} \Gamma_{l,i} \left(N, \frac{\mu^2}{m^2} \right) \tilde{\sigma}_{lk} \left(N, \frac{s'}{\mu^2} \right) \Gamma_{k,j} \left(N, \frac{\mu^2}{m^2} \right) \; .$$

- Here Γ_{li} denote massive operator matrix elements and $\tilde{\sigma}_{lk}$ the massless Wilson coefficients, both being calculated in the $\overline{\rm MS}$ scheme.
- μ is the factorization mass, which cancels in the physical cross section.
- The initial state fermion mass dependence is solely encoded in Γ_{li}.

Γ_{li} and $\tilde{\sigma}_{lk}$ obey the following renormalization group equations:

$$\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\delta_{al}+\gamma_{al}(N,g)\right]\Gamma_{li}\left(N,\frac{\mu^2}{m^2},g(\mu^2)\right) = 0$$

$$\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\delta_{la}\delta_{kb}-\gamma_{la}(N,g)\delta_{kb}-\gamma_{kb}(N,g)\delta_{la}\right]\tilde{\sigma}_{lk}\left(\frac{s'}{m^2},g(\mu^2)\right) = 0$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]\sigma_{ij}\left(\frac{s'}{\mu^2},g(\mu^2)\right) = 0$$

For the process under consideration we obtain to $O(a^2)$:

$$\left[\frac{\partial}{\partial\hat{L}}-\beta_0 a^2\frac{\partial}{\partial a}+\frac{1}{2}\gamma_{ee}(N,a)\right]\Gamma_{ee}\left(N,a,\frac{\mu^2}{m^2}\right)+\frac{1}{2}\gamma_{e\gamma}(N,a)\Gamma_{\gamma e}\left(N,a,\frac{\mu^2}{m^2}\right) = 0$$

$$\left[\frac{\partial}{\partial\hat{L}}-\beta_0 a^2\frac{\partial}{\partial a}-\gamma_{ee}(N,a)\right]\tilde{\sigma}_{ee}\left(N,a,\frac{s'}{\mu^2}\right)-\gamma_{\gamma e}(N,a)\tilde{\sigma}_{e\gamma}\left(N,a,\frac{s'}{\mu^2}\right) = 0$$

where $\partial/\partial\mu$ has been replaced by $2\partial/\partial\hat{L}$, with $\hat{L}=\ln(\mu^2/M^2)$.

The solutions of these equations are

$$\Gamma_{ee}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{ee}^{(0)}(N)L+\Gamma_{ee}^{(0)}(N)\right]+a^2\left[\left\{\frac{1}{8}\gamma_{ee}^{(0)}(N)\left(\gamma_{ee}^{(0)}(N)-2\beta_0\right)+\frac{1}{8}\gamma_{e\gamma}^{(0)}(N)\gamma_{\gamma e}^{(0)}(N)\right\}L^2\right.$$

$$\left.+1+\frac{1}{2}\left\{-\gamma_{ee}^{(1)}(N)+2\beta_0\Gamma_{ee}^{(0)}-\gamma_{ee}^{(0)}(N)\Gamma_{ee}^{(0)}(N)-\gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}(N)\right\}L+\Gamma_{ee}^{(1)}\right]+O(a^3),$$

$$\tilde{\sigma}_{ee}\left(N,a,\frac{s'}{\mu^2}\right) = a\left[-\frac{1}{2}\gamma_{ee}^{(0)}(N)\lambda+\tilde{\sigma}_{ee}^{(0)}(N)\right]+a^2\left[\left\{\frac{1}{2}\gamma_{ee}^{(0)}(N)\left(\gamma_{ee}^{(0)}(N)+\beta_0\right)+\frac{1}{4}\gamma_{e\gamma}^{(0)}(N)\gamma_{\gamma e}^{(0)}(N)\right\}\lambda^2\right.$$

$$\left.+1+\left\{-\gamma_{ee}^{(1)}(N)-\beta_0\tilde{\sigma}_{ee}^{(0)}-\gamma_{ee}^{(0)}(N)\tilde{\sigma}_{ee}^{(0)}(N)-\gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}(N)\right\}\lambda+\tilde{\sigma}_{ee}^{(1)}\right]+O(a^3),$$

$$\Gamma_{\gamma e}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{\gamma e}^{(0)}(N)L+\Gamma_{\gamma e}^{(0)}\right]+O(a^2)$$

$$\tilde{\sigma}_{e\gamma}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{e\gamma}^{(0)}(N)\lambda+\tilde{\sigma}_{e\gamma}^{(0)}\right]+O(a^2)\ ,$$

with the logarithms $L=\ln\left(\frac{\mu^2}{m^2}\right)$ and $\lambda=\ln\left(\frac{s'}{\mu^2}\right)$

2. The Renormalization Group Technique

Introducing the splitting functions in N-space

$$P_{ij}^{(l)}(N) = \int_0^1 dz z^{N-1} P_{ij}^{(l)}(z) = -\gamma_{ij}^{(l)}(N)$$

one obtains

$$\begin{aligned}
\frac{d\sigma_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\Bigg\{1 + a_0\left[P_{ee}^{(0)} \cdot \mathbf{L} + \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right] \\
&+ a_0^2\Bigg\{\left[\frac{1}{2}{P_{ee}^{(0)}}^2 - \frac{\beta_0}{2}P_{ee}^{(0)} + \frac{1}{4}P_{e\gamma}^{(0)} \cdot P_{\gamma e}^{(0)}\right]\mathbf{L}^2 \\
&+ \left[P_{ee}^{(1)} + P_{ee}^{(0)}\left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right) - \beta_0\tilde{\sigma}_{ee}^{(0)} + P_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)}P_{e\gamma}^{(0)}\right]\mathbf{L} \\
&+ \left(2\Gamma_{ee}^{(1)} + \tilde{\sigma}_{ee}^{(1)}\right) + 2\Gamma_{ee}^{0)}\tilde{\sigma}_{ee}^{(0)} + 2\tilde{\sigma}_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)} + {\Gamma_{ee}^{(0)}}^2\Bigg\}\Bigg\}
\end{aligned}$$

with

$$\mathbf{L} = \ln\left(\frac{s'}{m^2}\right) = \ln\left(\frac{s}{m^2}\right) + \ln(z); \quad \hat{\mathbf{L}} \equiv \ln(s/m^2)\ .$$

- Note a series of typos and omissions in Berends et al. (1988)

2. The Renormalization Group Technique

It is convenient to represent the differential scattering cross section in terms of three contributions, the flavor non-singlet terms with a single fermion line (I), those with a closed fermion line (II), and the pure-singlet terms (III). These contributions are :

$$
\begin{aligned}
\frac{d\sigma^{\mathrm{I}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\Biggl\{1 + a_0\left[P^{(0)}_{ee}\cdot\mathbf{L} + \left(\tilde{\sigma}^{(0)}_{ee} + 2\Gamma^{(0)}_{ee}\right)\right] \\
&\qquad + a_0^2\Biggl\{\frac{1}{2}{P^{(0)}_{ee}}^2\mathbf{L}^2 + \left[P^{(1),\mathrm{I}}_{ee} + P^{(0)}_{ee}\left(\tilde{\sigma}^{(0)}_{ee} + 2\Gamma^{(0)}_{ee}\right)\right]\mathbf{L} \\
&\qquad + \left(2\Gamma^{(1),\mathrm{I}}_{ee} + \tilde{\sigma}^{(1),\mathrm{I}}_{ee}\right) + 2\Gamma^{(0)}_{ee}\tilde{\sigma}^{(0)}_{ee} + {\Gamma^{(0)}_{ee}}^2\Biggr\}\Biggr\} \\
\frac{d\sigma^{\mathrm{II}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\Biggl\{-\frac{\beta_0}{2}P^{(0)}_{ee}\mathbf{L}^2 + \left[P^{(1),\mathrm{II}}_{ee} - \beta_0\tilde{\sigma}^{(0)}_{ee}\right]\mathbf{L} + \left(2\Gamma^{(1),\mathrm{II}}_{ee} + \tilde{\sigma}^{(1),\mathrm{II}}_{ee}\right)\Biggr\} \\
\frac{d\sigma^{\mathrm{III}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\Biggl\{\frac{1}{4}P^{(0)}_{e\gamma}\cdot P^{(0)}_{\gamma e}\mathbf{L}^2 + \left[P^{(1),\mathrm{III}}_{ee} + P^{(0)}_{\gamma e}\tilde{\sigma}^{(0)}_{e\gamma} + \Gamma^{(0)}_{\gamma e}P^{(0)}_{e\gamma}\right]\mathbf{L} \\
&\qquad + \left(2\Gamma^{(1),\mathrm{III}}_{ee} + \tilde{\sigma}^{(1),\mathrm{III}}_{ee}\right) + 2\tilde{\sigma}^{(0)}_{e\gamma}\Gamma^{(0)}_{\gamma e}\Biggr\}
\end{aligned}
$$

- $\tilde{\sigma}^{(k)}_{ij}$ denotes the respective contribution of the massless Drell-Yan (DY) cross section.

2. The Renormalization Group Technique

Different ingredients to the calculation :

- Splitting functions to $O(\alpha^2)$

[E. G. Floratos, D. A. Ross and C. T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545]; Nucl. Phys. B **152** (1979) 493; A. Gonzalez-Arroyo, C. Lopez and F. J. Yndurain, Nucl. Phys. B **153** (1979) 161; A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429; E. G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417; G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27; W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437; R. Hamberg and W. L. van Neerven, Nucl. Phys. B **379** (1992) 143; R. K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356; S. Moch and J. A. M. Vermaseren, Nucl. Phys. B **573** (2000) 853 [arXiv:hep-ph/9912355].]

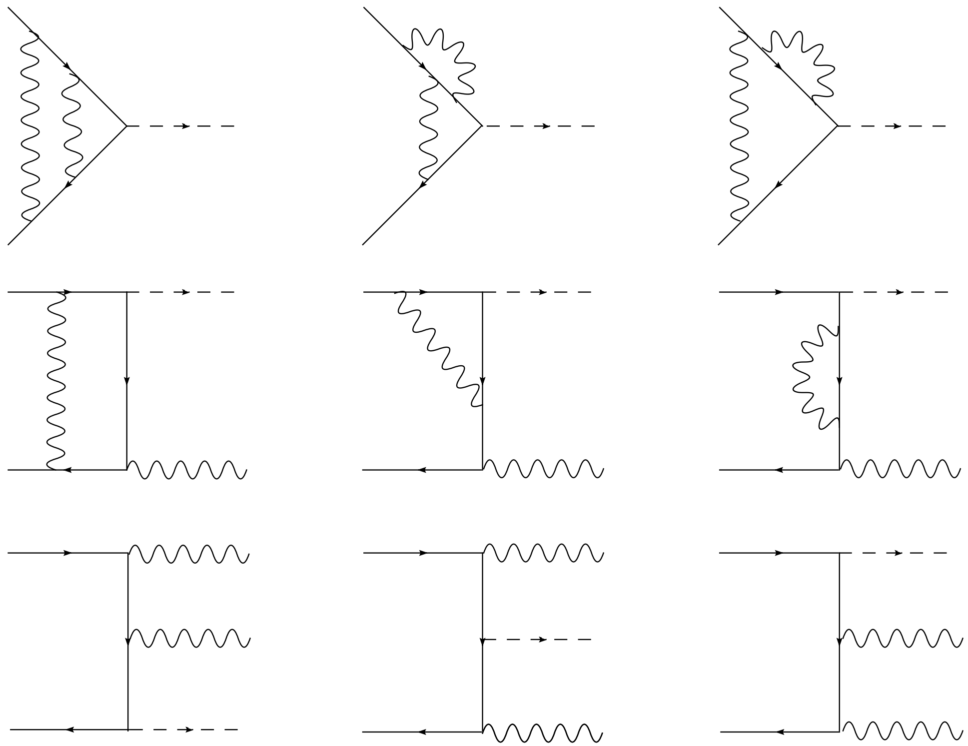
- massless Drell-Yan Cross Section to $O(\alpha^2)$

[R. Hamberg, W. L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [E: B **644** (2002) 403]; R. V. Harlander and W. B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801.]

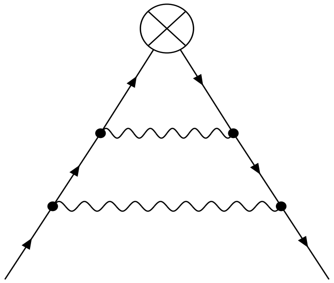
- massive OMEs to $O(\alpha^2)$

$\Longrightarrow$ this calculation; (Some errors at $O(\alpha)$ in earlier work corrected.)

Photonic Diagrams for the massless Drell-Yan Process :



Other diagrams : different pair-production channels.



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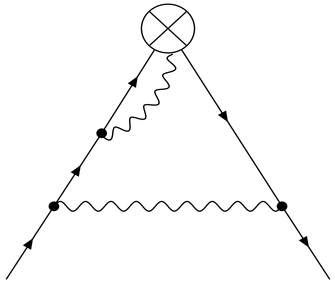

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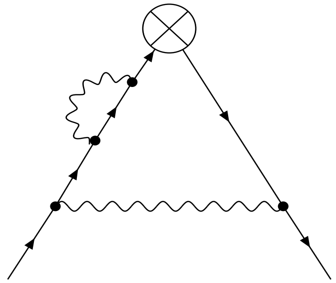

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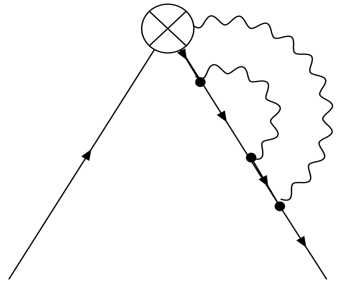

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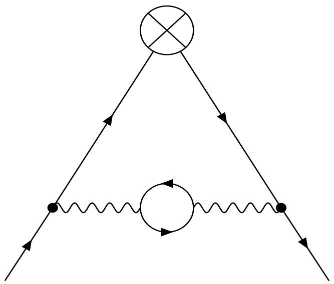

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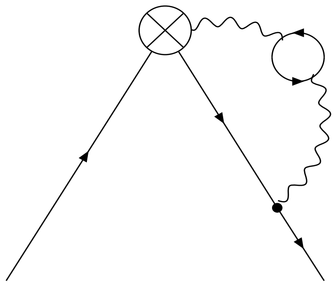

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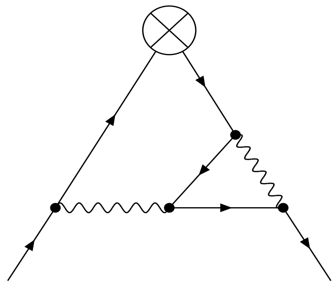

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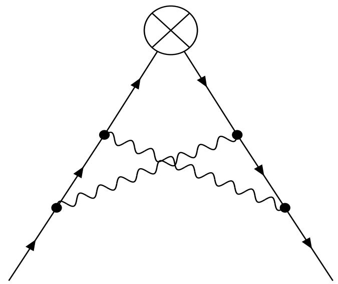

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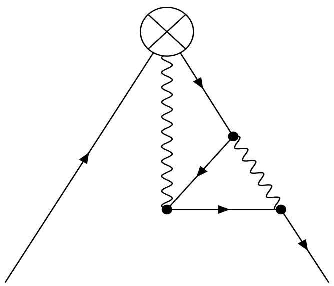

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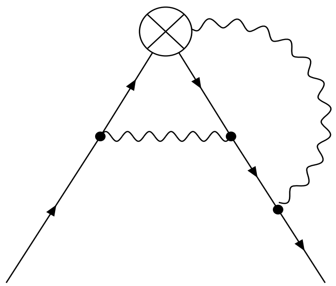

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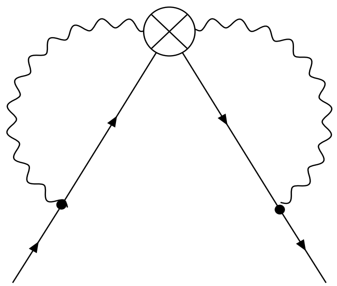

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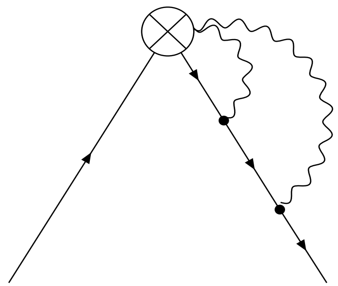
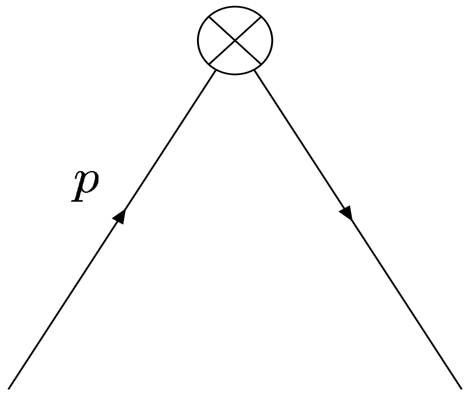


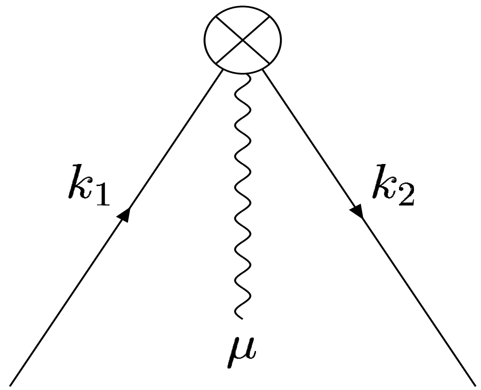
(20)

Two-loop diagrams contributing to the massive operator matrix element $A_{ee}(N, \alpha)$.
The antisymmetric diagrams count twice.

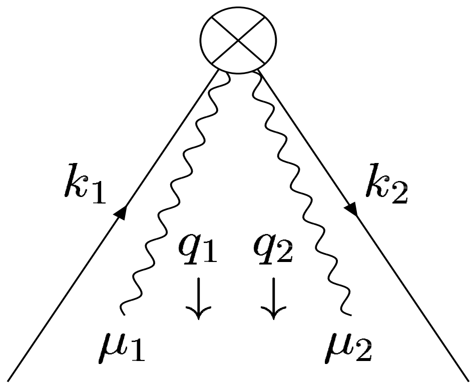
3. The Calculation



$$\not\Delta(\Delta.p)^{N-1}$$



$$-e\Delta_\mu \sum_{j=1}^{N-1}(\Delta.k_1)^{j-1}(\Delta.k_2)^{N-1-j}$$



$$e^2 \not\Delta \Delta_{\mu_1}\Delta_{\mu_2}\sum_{j=1}^{N-2}\sum_{i=1}^{j}(\Delta.k_1)^{i-1}(\Delta.k_2)^{N-2-j}\left\{[\Delta.\,(k_1-q_2)]^{j-i}+[\Delta.\,(k_2+q_2)]^{j-i}\right\}$$

3. The Calculation

Renormalized Operator-Matrix Elements :

$$A_{ij}^{(k)} = \delta_{ij} + a(\mu^2)\left[\hat{A}_{ij}^{(1)} + Z_{O,ij}^{-1,(1)} + \hat{\Gamma}_{ij}^{-1,(1)}\right]$$
$$+ a^2(\mu^2)\left[\hat{A}_{ij}^{(2)} + \delta m \frac{d}{dm}\hat{A}_{ij}^{(1)} + Z_{O,ik}^{-1,(1)}\hat{A}_{kj}^{(1)} + Z_{ij}^{-1,(2)}\right.$$
$$\left. + \left\{\hat{A}_{ik}^{(1)} + Z_{O,ik}^{-1,(1)}\right\}\hat{\Gamma}_{kj}^{-1,(1)} + \hat{\Gamma}_{ij}^{-1,(2)}\right].$$

UV Un-renormalized Operator-Matrix Elements : after mass renormalization

$$\hat{A}_{ee}^{(1)} = aS_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2}\left\{-\frac{1}{\varepsilon}P_{ee}^{(0)} + \Gamma_{ee}^{(0)} + \varepsilon\overline{\Gamma}_{ee}^{(0)}\right\}$$
$$\hat{A}_{ee}^{(2),\mathrm{I}} = a^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon}\left\{\frac{1}{2\varepsilon^2}P_{ee}^{(0)}\otimes P_{ee}^{(0)} - \frac{1}{2\varepsilon}\left[P_{ee}^{(1),\mathrm{I}} + 2\Gamma_{ee}^{(0)}\otimes P_{ee}^{(0)}\right] + \Gamma_{ee}^{(1),\mathrm{I}}\right\}$$
$$\hat{A}_{ee}^{(2),\mathrm{II}} = a^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon}\left\{\frac{1}{\varepsilon^2}\beta_0 P_{ee}^{(0)} - \frac{1}{\varepsilon}\left[\frac{1}{2}P_{ee}^{(1),\mathrm{II}} + 2\beta_0\Gamma_{ee}^{(0)}\right] + \Gamma_{ee}^{(1),\mathrm{II}}\right\}$$
$$\hat{A}_{ee}^{(2),\mathrm{III}} = a^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon}\left\{\frac{1}{2\varepsilon^2}P_{e\gamma}^{(0)}\otimes P_{\gamma e}^{(0)} - \frac{1}{\varepsilon}\left\{\frac{1}{2}P_{ee}^{(1),\mathrm{III}} + \Gamma_{\gamma e}^{(0)}P_{e\gamma}^{(0)}\right\} + \Gamma_{ee}^{(1),\mathrm{III}}\right\}$$

- cf. also : [M. Buza et al., Nucl. Phys. B **472** (1996) 611; I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. B **780** (2007) 40.]

$\Longrightarrow$ Project onto the $e^- \to e^-, e^+ \to e^+$ transition elements, respectively.

All the diagrams can be written in terms of integrals of these type:

$$A^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta\cdot k_1)^a (\Delta\cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$$B^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{k_2\cdot p(\Delta\cdot k_1)^a (\Delta\cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

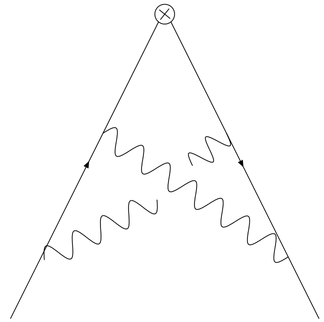
$$F^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta\cdot k_1)^a (\Delta\cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta\cdot p)^j (\Delta\cdot k_1)^{n-1-j}$$

$$G^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta\cdot k_1)^a (\Delta\cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta\cdot k_1)^j (\Delta\cdot k_2)^{n-1-j}$$

where,

$$D_1 = k_1^2 - m^2 \ ; \quad D_2 = k_2^2 - m^2 \ ; \quad D_3 = (k_1 - p)^2 \ ;$$
$$D_4 = (k_1 - k_2)^2 \ ; \quad D_5 = (k_2 - k_1 + p)^2 - m^2 \ ; \quad D_6 = (k_2 - p)^2$$

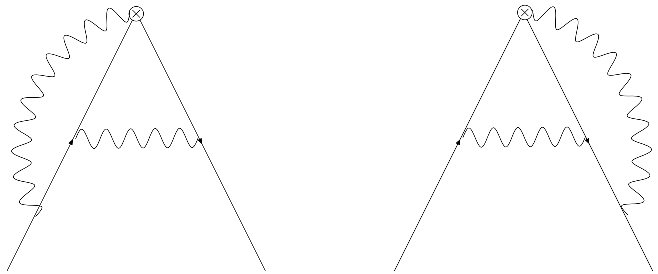
For example, in terms of these integrals, the crossed box gives



$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma^\mu(\not p - \not k_1 + \not k_2 + m)\gamma^\nu(\not k_2 + m)\not\Delta(\not k_2 + m)\gamma_\mu(\not k_1 + m)\gamma_\nu(\not p + m)]}{D_1 D_2^2 D_3 D_4 D_5} (\Delta\cdot k_2)^n =$$

$$= (D-4)(D-2)\left[2A^{0,n+1}_{12001} - 2A^{0,n+1}_{02110} - A^{1,n}_{01111} + A^{1,n}_{11011} - A^{1,n}_{11101} + A^{1,n}_{11110} + (\Delta\cdot p)A^{0,n}_{01111}\right]$$
$$+(D-8)(D-2)\left[A^{0,n+1}_{01111} - A^{0,n+1}_{11011} - (\Delta\cdot p)A^{0,n}_{10111} + (\Delta\cdot p)A^{0,n}_{11101}\right] + 16(D-3)m^2\left[A^{0,n+1}_{12011} + A^{0,n+1}_{12101}\right]$$
$$+4(D-4)m^2\left[(\Delta\cdot p)A^{0,n}_{11111} - A^{0,n+1}_{11111}\right] + 8m^2 A^{0,n+1}_{02111} + 8m^2 A^{0,n+1}_{12110} + 32m^4 A^{0,n+1}_{12111}$$
$$4(D-2)\left[A^{0,n+1}_{11101} - A^{0,n+1}_{11110} - 2B^{0,n+1}_{12011} - 2B^{0,n+1}_{12101} + (\Delta\cdot p)A^{0,n}_{11011}\right] \; ,$$

Another example:



$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma^\mu (\not{k}_1 + m) \not{\Delta} (\not{k}_2 + m)(\not{p} - \not{k}_1 + \not{k}_2 + m) \not{\Delta} (\not{p} + m)]}{D_1 D_2 D_3 D_4 D_5} \sum_{j=0}^{n-1} (\Delta \cdot k_1)^j (\Delta \cdot k_2)^{n-1-j} =$$

$$= 2(D-4) \left[G^{2,0}_{01111} - G^{2,0}_{11011} - G^{2,0}_{11101} + G^{2,0}_{11110} + (\Delta \cdot p) G^{0,0}_{10111} \right] - 4G^{0,2}_{01111} + 4G^{0,2}_{11011} + 4(\Delta \cdot p) G^{0,1}_{10111}$$
$$+2(D-2) \left[G^{0,1}_{11011} - G^{1,0}_{11011} - G^{1,0}_{11110} - (\Delta \cdot p) G^{0,0}_{01111} + (\Delta \cdot p) G^{0,0}_{10111} - (\Delta \cdot p) G^{0,0}_{11101} \right]$$
$$+2(D-6) m^2 \left[G^{1,1}_{11011} - G^{1,1}_{01111} \right] + 4(D-3)(\Delta \cdot p) \left[G^{1,0}_{11101} + G^{1,0}_{01111} \right] + 4G^{1,1}_{11101} + 4G^{1,1}_{11110}$$
$$+8m^2 G^{1,1}_{11111} - 8m^2 G^{2,0}_{11111} - 4(\Delta \cdot p) G^{0,1}_{11101} + 8m^2 (\Delta \cdot p) G^{0,1}_{11111} - 2D(\Delta \cdot p) G^{1,0}_{10111} + 8m^2 (\Delta \cdot p) G^{1,0}_{11111}$$

3. The Calculation

Type A integrals

ν_1	ν_2	ν_3	ν_4	ν_5	$(a\ ,\ b)$
1	1	1	1	0	
1	1	1	0	1	
1	1	0	1	1	
1	0	1	1	1	0,n
0	1	1	1	1	1,n
2	1	1	1	0	n,0
1	2	1	1	0	n,1
1	2	1	0	1	
2	1	0	1	1	
0	2	1	1	1	
2	0	1	1	1	n,0
1	1	1	0	2	n,1
1	1	0	1	2	
1	1	1	2	0	0,n
0	1	1	2	1	1,n
1	0	1	1	2	
1	0	1	2	1	
2	1	1	0	2	0,n
1	2	1	0	2	n,0
3	1	1	1	0	
0	3	1	1	1	n,0
1	3	1	0	1	
2	2	1	1	0	
2	1	2	1	0	
1	1	3	1	0	
1	2	2	1	0	
0	1	3	1	1	
0	2	2	1	1	0,n
3	1	1	0	1	
1	1	2	0	2	
2	1	2	0	1	
1	0	3	1	1	
3	0	1	1	1	
2	0	2	1	1	

Type B integrals

ν_1	ν_2	ν_3	ν_4	ν_5	$(a\ ,\ b)$
2	1	1	1	0	
2	1	1	0	1	n,0
2	1	0	1	1	
1	2	1	1	0	
1	2	1	0	1	0,n
1	2	0	1	1	

Type F and G integrals

1	1	1	1	0	
1	1	1	0	1	1,0
1	1	0	1	1	0,1
0	1	1	1	1	1,1
1	2	1	0	1	2,0
1	1	1	0	2	
0	2	1	1	1	
					0,0
1	0	1	1	1	1,0
					0,1
0	1	1	1	1	0,0
1	1	0	1	1	0,2
2	1	1	1	0	1,1

In total there are 155+ integrals

3. The Calculation

We do the 5-propagator integrals using the following IBP identities.

$$A^{n,0}_{11111} = \frac{1}{\epsilon}\left(A^{n,0}_{12101} - A^{n,0}_{02111} + A^{n,0}_{11102}\right)$$

$$A^{0,n}_{11111} = \frac{1}{\epsilon}\left(A^{0,n}_{21011} + A^{0,n}_{11120} + A^{0,n}_{10121} - A^{0,n}_{10121} + A^{0,n}_{11012} - A^{0,n}_{10112}\right)$$

$$A^{n,0}_{21111} = \frac{1}{\epsilon}\left(A^{n,0}_{22101} + A^{n,0}_{21102}\right) + \frac{1}{\epsilon(1+\epsilon)}\left(2A^{n,0}_{03111} - 2A^{n,0}_{13101} - A^{n,0}_{12102}\right)$$

$$\begin{aligned} A^{0,n}_{12111} = \frac{1}{\epsilon}\Big[& - A^{0,n}_{12210} - A^{0,n}_{02211} - A^{0,n}_{12102} - A^{0,n}_{22101} \\ & + \frac{1}{1+\epsilon}\Big(-2A^{0,n}_{31101} + 2A^{0,n}_{30111} - A^{0,n}_{21210} + 2A^{0,n}_{20211} - A^{0,n}_{21102} \\ & \quad - A^{0,n}_{21201} - 2A^{0,n}_{11310} - 2A^{0,n}_{10311} - A^{0,n}_{01311} - A^{0,n}_{11202}\Big)\Big] \end{aligned}$$

$$
\begin{aligned}
A^{0,n}_{12111} \quad = \quad & \int_0^1 dx\ x^n \left\{ \frac{2}{3}\mathrm{Li}_2(1-x) + \frac{1}{3}\ln^2(x) + \frac{2}{3(1-x)}\ln(x) + \frac{2}{3}\zeta_2 \right. \\
& - \frac{1}{3}(1-x)^{-3-2\epsilon}\ln^2(x) - \frac{2}{3}(1-x)^{-2-2\epsilon}\ln(x) \\
& + \frac{2}{3}(1-x)^{-1-2\epsilon} + \epsilon(1-x)^{-1-2\epsilon}(2\zeta_2 - 1) \\
& + (-1)^n \left[\frac{4}{3}\mathrm{Li}_2(-x) - \frac{2}{3}\left(1 - \frac{2}{(1+x)^3}\right)\mathrm{Li}_2(1-x) \right. \\
& \qquad - \left(1 - \frac{1}{(1+x)^3}\right)\ln^2(x) + \frac{4}{3}\ln(x)\ln(1+x) \\
& \qquad + \frac{2}{3(1+x)}\left(\frac{2}{(1+x)^2} - \frac{1}{(1+x)} - 1\right)\ln(x) \\
& \qquad \left.\left. - \frac{2}{3(1+x)} + \frac{4}{3(1+x)^2} + \frac{2}{3}\zeta_2 \right] \right\}
\end{aligned}
$$

There are also integration by parts identities for the G integrals with 5 propagators:

$$G^{0,1}_{11111} = \frac{1}{1-\epsilon}\left(A^{n,0}_{11111} + (1+n)A^{0,n}_{11111} + G^{0,1}_{12101} + G^{0,1}_{11102} + G^{0,1}_{02111}\right)$$

$$G^{1,0}_{11111} = \frac{1}{1-\epsilon}\left(nA^{1,n-1}_{11111} + +G^{1,0}_{12101} + G^{1,0}_{11102} + G^{1,0}_{02111}\right)$$

$$G^{1,1}_{11111} = \frac{1}{1-\epsilon}\left(A^{n+1,0}_{11111} + (1+n)A^{1,n}_{11111} + +G^{1,1}_{12101} + G^{1,1}_{11102} + G^{1,1}_{02111}\right)$$

$$G^{2,0}_{11111} = \frac{1}{1-\epsilon}\left(nA^{2,n-1}_{11111} + +G^{1,1}_{12101} + G^{1,1}_{11102} + G^{1,1}_{02111}\right)$$

There are similar relations for the F integrals. The factors of n multiplying the A integrals in these relations can be reabsorbed into x^n by integration by parts.

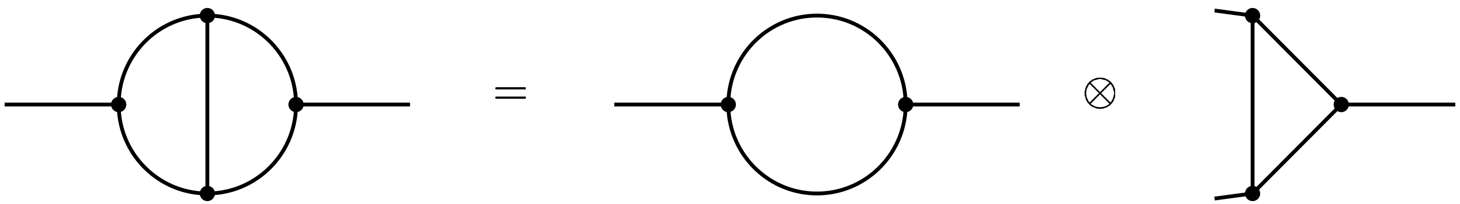
3. The Calculation

We checked the integrals by several means:

- Use of the Mathematica package Tarcer (R. Mertig and R. Scharf, hep-ph/9801383), which allows to check the integrals for a few low moments.
- Use of Mellin-Barnes integrals (V.A. Smirnov, hep-ph/9905323), and then use of the MB package (M. Czakon, hep-ph/0511200) to integrate numerically.
  - Advantage: can check very high moments at some numerical accuracy.
  - Disadvantage: couldn't find contour for some of our integrals.
- Additional: integration by parts identities.

3. The Calculation

Using the decomposition of propagator-type integrals as a product (convolution) of two one-loop integrals (I. Bierenbaum and S. Weinzierl, hep-ph/0308311):



we can write our integrals as Mellin-Barnes ones in just two variables:

$$A^{a,b}_{\nu_1\nu_2\nu_3\nu_4\nu_5} = \frac{(m^2)^{\nu_{12345}+D}(\Delta\cdot p)^{a+b}}{\prod_{k=1}^{5}\Gamma(\nu_k)}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}d\sigma\int_{-i\infty}^{+i\infty}d\tau\sum_{j=0}^{b}\sum_{i=0}^{j}(-1)^j\frac{b!}{(b-j)!(j-i)!i!}\Gamma(-\sigma)\Gamma(-\tau)$$
$$\Gamma(\sigma+\tau+\nu_{345}-D/2)\frac{\Gamma(-\sigma-\tau+\nu_{12}-D/2)}{\Gamma(\nu_1-\sigma)\Gamma(\nu_2-\tau)}\frac{\Gamma(\sigma+\tau+j-i+\nu_3)\Gamma(\sigma+i+\nu_4)}{\Gamma(2\sigma+\tau+j+\nu_{34})}$$
$$\frac{\Gamma(-\tau-\nu_{345}+j+D)\Gamma(\tau+\nu_5)}{\Gamma(j-\nu_{34}+D)}\frac{\Gamma(-\sigma+a-i+\nu_1)\Gamma(2\sigma+\tau-2\nu_1+\nu_2+D)}{\Gamma(\sigma+\tau-\nu_{12}+a-i+D)}$$

Additional IBP identities:

$$A^{n,0}_{31110} = \frac{1}{1-\epsilon}\left(A^{n,0}_{32100} + \frac{2}{\epsilon}\left(A^{n,0}_{23100} + \frac{3}{1+\epsilon}\left(A^{n,0}_{14100} - A^{n,0}_{04110}\right)\right)\right)$$

$$0 = n(\Delta \cdot p)A^{0,n-1}_{11011} - A^{0,n}_{11020} + A^{0,n}_{11002} - 2B^{0,n}_{12011} - 2m^2 A^{0,n}_{11012}$$

$$0 = -\epsilon A^{0,n}_{11101} + n(\Delta \cdot p)A^{0,n-1}_{11101} - 2B^{0,n}_{12101} - 2m^2 A^{0,n}_{21101} - A^{0,n}_{01201} + A^{0,n}_{10102} - A^{0,n}_{11002}$$

$$0 = nA^{n-1,1}_{11101} - A^{n,0}_{21001} + A^{n,0}_{21100} - A^{n,0}_{20101} - 2B^{n,0}_{21101} + A^{n,0}_{11200} - A^{n,0}_{10201} - A^{n,0}_{11002} + A^{n,0}_{10102} + 2m^2 A^{n,0}_{11102}$$

$$0 = n(\Delta \cdot p)A^{0,n-1}_{11110} - 2m^2 A^{0,n}_{21110} - 2B^{0,n}_{12110} - A^{0,n}_{01210} + A^{0,n}_{21010}$$

4. First Results

The 2–loop corrections to the process $e^+e^- \to Z^0$ can be organized in the following form :

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s)\left\{1 + a_0\left[T_{11}\hat{\mathbf{L}} + T_{10}\right] + a_0^2\left[T_{22}\hat{\mathbf{L}}^2 + T_{21}\hat{\mathbf{L}} + T_{20}\right]\right\}$$

$$a_0 = \frac{\alpha(m_e^2)}{4\pi}$$

- Universal Corrections : $T_{ii}(z)$ $\Longrightarrow$ depend on LO splitting functions and β_0

$$T_{11} = 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) = 4\left[\frac{1+z^2}{1-z}\right]_+ ,$$

$$T_{22} = \Big\{64\mathcal{D}_1(z) + 48D_0(z) + (18 - 32\zeta_2)\delta(1-z)$$

$$-32\frac{\ln(z)}{1-z} - 32(1+z)\ln(1-z) + 24(1+z)\ln(z) - 8(5+z)\Big\}$$

$$+\frac{2}{3}\Big\{8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z)\Big\} + 16\Big\{\frac{1}{2}(1-z)\ln(z) + \frac{1}{4}(1-z) + \frac{1}{3}\frac{1}{3z}(1-z^3)\Big\} .$$

4. First Results

- $O(\alpha)$ Term : $T_{10}(z)$ $\Longrightarrow$ depend on LO OME + LO DY

$$T_{10} = -4\left[\frac{1+z^2}{1-z}\right]_+ + 2(4\zeta_2 - 1)\delta(1-z)$$

$$T_{11}\hat{\mathbf{L}} + T_{10} = P_{ee}^{(0)}(z)\left[\hat{\mathbf{L}} - 1\right] + 2(4\zeta_2 - 1)\delta(1-z)\ .$$

Complete 1-Loop Result.

- $O(\alpha^2\hat{\mathbf{L}})$ Terms : $T_{21}(z)$ $\Longrightarrow$ depend on LO,NLO splitting fcts., LO OME + LO DY

Contributions to the three main processes I-III :

$$\begin{aligned} T_{21}^{\mathrm{I}} = 16\Bigg\{ & -8\mathcal{D}_1(z) - (7 - 4\zeta_2)\mathcal{D}_0(z) + \left(-\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3\right)\delta(1-z) \\ & + \left(\frac{1+z^2}{1-z}\right)\left[\ln(z)\ln(1-z) - \ln^2(z) + \frac{11}{4}\ln(z)\right] \\ & + (1+z)\left[4\ln(1-z) + \frac{1}{4}\ln^2(z) - \frac{7}{4}\ln(z) - 2\zeta_2\right] - \ln(z) + 3 + 4z\Bigg\} \end{aligned}$$

$$
\begin{aligned}
T_{21}^{\mathrm{II}} &= 16\Bigg\{\frac{4}{3}\mathcal{D}_1(z) - \frac{10}{9}\mathcal{D}_0(z) - \frac{17}{12}\delta(1-z) \\
&\quad -\frac{2}{3}\frac{\ln(z)}{1-z} - \frac{1}{3}(1+z)\left[2\ln(1-z) - \ln(z)\right] - \frac{1}{9} + \frac{11}{9}z\Bigg\} \\
T_{21}^{\mathrm{III}} &= 16\Bigg\{(1+z)\left[2\mathrm{Li}_2(1-z) - \ln^2(z) + 2\ln(z)\ln(1-z)\right] \\
&\quad + \left(\frac{4}{3}\frac{1}{z} + 1 - z - \frac{4}{3}z^2\right)\ln(1-z) - \left(\frac{2}{3}\frac{1}{z} + 1 - \frac{1}{2}z - \frac{4}{3}z^2\right)\ln(z) \\
&\quad -\frac{8}{9}\frac{1}{z} - \frac{8}{3} + \frac{8}{3}z + \frac{8}{9}z^2\Bigg\}
\end{aligned}
$$

So far agreement with Berends et al. (1988).

All Two–Loop massive OMEs calculated.

Example :

$$
\begin{aligned}
T_{17} =\;& \\
& -136(1-x) + 2\,\frac{\left(-12+9\,x+6\,x^2+5\,x^3\right)\ln^2(x)}{(-1+x)^2} - 8\,\frac{(2\,x-1)\,\mathrm{S}_{12}(1-x)}{-1+x} + \frac{2}{3}\,\frac{\left(x^2+8-8\,x\right)(\ln(x))^3}{-1+x} \\
& -4\,\frac{\left(-3\,x^2+x^3+25\,x-11\right)\mathrm{Li}_2(x)}{-1+x} + 4\,(1-x)(x-3)\,\mathrm{Li}_2(1-x) + 8\,\frac{\left(6-5\,x+3\,x^2\right)\ln(x)}{-1+x} \\
& -4\,\frac{\left(-7\,x^2+6\,x-6+3\,x^3\right)\mathrm{Li}_3(1-x)}{-1+x} + \left(16\,x^2+20\,x-16\right)\ln(1-x)\,\mathrm{Li}_2(1-x) - 96\,\frac{x\mathrm{Li}_2(-x)}{-1+x} \\
& + \left(-8\,x^2-24\,x+8\right)\ln(x)\,\mathrm{Li}_2(1-x) + \left(-14\,x-8\,x^2+2\right)(\ln(x))^2\ln(1-x) + (16\,x+16)\,\mathrm{Li}_2(-x)\ln(x) \\
& + \left(4\,x^2-28\,x+16\right)\zeta(2) + \Bigg(-32+32\,x+(20\,x-20)(\ln(1-x))^2 - 8\,\frac{\left(2\,x+x^3+1\right)\ln(1-x)\ln(x)}{-1+x} \\
& +8\,x^2\zeta(2) + (-16\,x+40)\ln(x) + (64-56\,x)\ln(1-x) + \left(16-8\,x^2-16\,x\right)\mathrm{Li}_2(x) \\
& -8\,\frac{x\left(2-x+x^2\right)\mathrm{Li}_2(1-x)}{-1+x} + 4\,\frac{\left(-4+4\,x+x^2\right)(\ln(x))^2}{-1+x}\Bigg)\frac{1}{\varepsilon} + \left(-24\,x+24+12\,x^2\right)\zeta(3) \\
& + (88-112\,x)\ln(1-x) + (-20\,x-20)\,\mathrm{Li}_3(x) + \left(-12\,x^2-20\,x+12\right)\mathrm{S}_{12}(x) + (-32-32\,x)\,\mathrm{Li}_3(-x) \\
& + \left(-48+48\,x+16\,\frac{\ln(x)}{-1+x} + (16-16\,x)\ln(1-x)\right)\frac{1}{\varepsilon^2} \\
& +(-1)^n\Bigg[-136 - 4\,\frac{\left(-1-9\,x^2-15\,x+x^3\right)\zeta(2)}{x+1} + 8\,\frac{\left(7\,x^2+14\,x+3\right)\mathrm{Li}_2(-x)}{x+1} + 4\,(-1+x)(x+1)\,\mathrm{Li}_2(x) \\
& -8\,\frac{\left(3\,x^2+3+8\,x\right)\mathrm{Li}_3(-x)}{x+1} + 136\,x + 8\,\frac{\left(x^2+4\,x^3-3\right)(\ln(x))^2}{(x+1)^2} - 4/3\,\frac{\left(4\,x^2+11\,x+4\right)(\ln(x))^3}{x+1} \\
& +4\,\frac{\left(8\,x+1+x^2\right)\zeta(3)}{x+1} + (-48-88\,x)\ln(x) - 16\,\frac{\left(4\,x+x^2+1\right)\mathrm{Li}_3(x)}{x+1} - 16\,\frac{\left(3\,x^2+3+8\,x\right)\mathrm{S}_{12}(-x)}{x+1} \\
& +8\,\frac{\left(1+x^2\right)\ln(x)\,\mathrm{Li}_2(-x)}{x+1} - 4\,\frac{\left(8\,x+1+x^2\right)\ln(x+1)(\ln(x))^2}{x+1} + 8\,\frac{\left(4\,x+x^2+1\right)}{x+1}\left(\ln(x)\,\mathrm{Li}_2(x) + 2\mathrm{Li}_2(-x)\right) \\
& +4\,(-1+x)(x+1)\ln(x)\ln(1-x) - 8\,\frac{\left(3\,x^2+3+8\,x\right)\ln(x+1)\,\zeta(2)}{x+1} + 4\,\frac{\left(x^2+4\,x-1\right)(-1+x)^2\,\mathrm{Li}_2(1-x)}{(x+1)^2} \\
& +8\,\frac{\left(7\,x^2+14\,x+3\right)\ln(x)\ln(x+1)}{x+1} - 16\,\frac{\left(3\,x^2+3+8\,x\right)\ln(x+1)\,\mathrm{Li}_2(-x)}{x+1} - 4\,\frac{\left(4\,x+x^2+1\right)\zeta(2)\ln(x)}{x+1} \\
& -8\,\frac{\left(3\,x^2+3+8\,x\right)\ln(x)(\ln(x+1))^2}{x+1} - \left(16\,\frac{\left(4\,x+x^2+1\right)\ln(x)}{x+1} + 48(1-x)\right)\frac{1}{\varepsilon^2} \\
& + \Bigg(32(1-x) + 8\,\frac{\left(4\,x+x^2+1\right)\zeta(2)}{x+1} + (-72\,x+40)\ln(x) + 16\,\frac{\left(4\,x+x^2+1\right)\ln(x+1)\ln(x)}{x+1} \\
& +8\,\frac{\left(7\,x+2\,x^2+2\right)(\ln(x))^2}{x+1}\Bigg)\frac{1}{\varepsilon}\Bigg]
\end{aligned}
$$

Assemble the $O(\alpha^2)$ result combining the $O(\alpha^2)$ contributions to the OME's and the DY cross section.

5. Conclusions

- The $O(\alpha^2)$ corrections to $e^+e^- \to Z^0$ in the on-mass-shell case can be calculated using the Renormalization Group Method. All terms but the power suppressed contributions $O(m_e^2/s)$ can be derived.

- The Mass Effects are contained in the massive operator matrix elements, which are process independent quantities.

- The complete process thus is assembled out of the massive 2-loop operator matrix elements and the corresponding massless contributions to the Drell-Yan process.

- We finished the calculation of the massive OMEs to $O(\alpha^2)$. The cross section is currently being assembled.
We agree with Berends et al. up to the $O(\alpha^2 \ln(s/m_e^2))$ terms.

- Final results will be given soon.