Multiparticle Cuts of Scattering Amplitudes

Pierpaolo Mastrolia

Institute of Theoretical Physics, University of Zürich

RAD COR 2007

Outline

All fundamental processes are reversible

Feynman

- Cutting Loops $\Leftrightarrow$ Sewing Trees
- Unitarity & Cut-Constructibility
- General Algorithm for Multiple-Cuts in D-dim
- Quadruple-Cut
- Double-Cut
- Triple-Cut
- Applications

Spinor Formalism

Xu, Zhang, Chang

Berends, Kleiss, De Causmaeker

Gastmans, Wu

Gunion, Kunzst

- on-shell massless Spinors

$$|i\rangle \equiv |k_i^+\rangle \equiv u_+(k_i) = v_-(k_i) \ , \qquad [i| \equiv \langle k_i^+| \equiv \bar{u}_+(k_i) = \bar{v}_-(k_i) \ ,$$

- $k^2 = 0:$ $\quad k_{a\dot{a}} \equiv k_\mu \sigma^\mu_{a\dot{a}} = \ell^k_a \, \tilde{\ell}^k_{\dot{a}} \qquad \text{or} \qquad \not{k} = |k\rangle[k| + |k]\langle k|$

- Spinor Inner Products

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \varepsilon_{ab}\, \ell_i^a \ell_j^b = \sqrt{|s_{ij}|}\, e^{i\Phi_{ij}} \ , \qquad [i\, j] \equiv \langle i^+|j^-\rangle = \varepsilon_{\dot{a}\dot{b}}\, \tilde{\ell}_i^{\dot{a}}\, \tilde{\ell}_j^{\dot{b}} = -\langle i\, j\rangle^* \ ,$$

with $s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j = \langle i\, j\rangle[j\, i]$.

- Polarization Vector

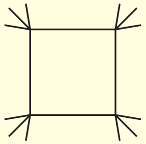
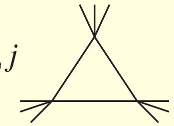
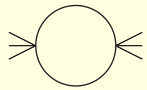
$$\varepsilon^+_\mu(k;q) = \frac{\langle q|\gamma_\mu|k]}{\sqrt{2}\langle q\, k\rangle} \ , \qquad \varepsilon^-_\mu(k;q) = \frac{[q|\gamma_\mu|k\rangle}{\sqrt{2}[k\, q]} \ ,$$

with $\varepsilon^2 = 0 \ , \quad k_\mu \cdot \varepsilon^\pm_\mu(k;q) = 0 \ , \quad \varepsilon^+ \cdot \varepsilon^- = -1$.

Changes in ref. mom. q are equivalent to gauge trasformations.

One Loop Amplitudes

P-V Tensor Reduction

$A = \sum_i c_{4,i}$  $+ \sum_j c_{3,j}$  $+ \sum_k c_{2,k}$  $+ \,\mathrm{rational}$

Since the D-regularised scalar functions associated to **boxes** $(I_4^{(4m)}, I_4^{(3m)}, I_4^{(2m,\mathrm{e})}, I_4^{(2m,\mathrm{h})}, I_4^{(1m)}, I_4^{(0m)})$, **triangles** $(I_3^{(3m)}, I_3^{(2m)}, I_3^{(1m)})$ and **bubbles** (I_2) are analytically known

't Hooft & Veltman (1979)

Bern, Dixon & Kosower (1993)

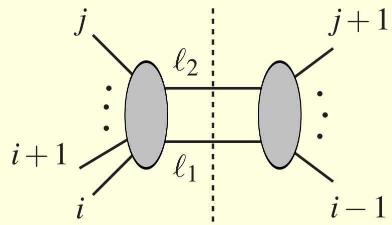
Duplančic & Nižic (2002)

- A is known, once the coefficients c_4, c_3, c_2 and the rational term are known: they all are rational functions of spinor products $\langle i\, j\rangle$, $[i\, j]$

Unitarity & Cut-Constructibility

- Discontinuity accross the Cut

Cut Integral in the P_{ij}^2-channel



$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \int d^4\Phi \, A^{\text{tree}}(\ell_1, i, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)$$

with

$$d^4\Phi = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}) \, \delta^{(+)}(\ell_1^2) \, \delta^{(+)}(\ell_2^2)$$

- loop-Reconstruction

Bern, Dixon, Dunbar & Kosower
Bern & Morgan; Anastasiou & Melnikov
Bedford, Brandhuber, Mc Namara, Spence & Travaglini

- channel-by-channel reconstruction of the loop-interal: $\delta^{(+)}(p^2) \leftrightarrow 1/(p^2 - i0)$
- loop-tools integrations: PV-tensor reduction & integration-by-parts identitities

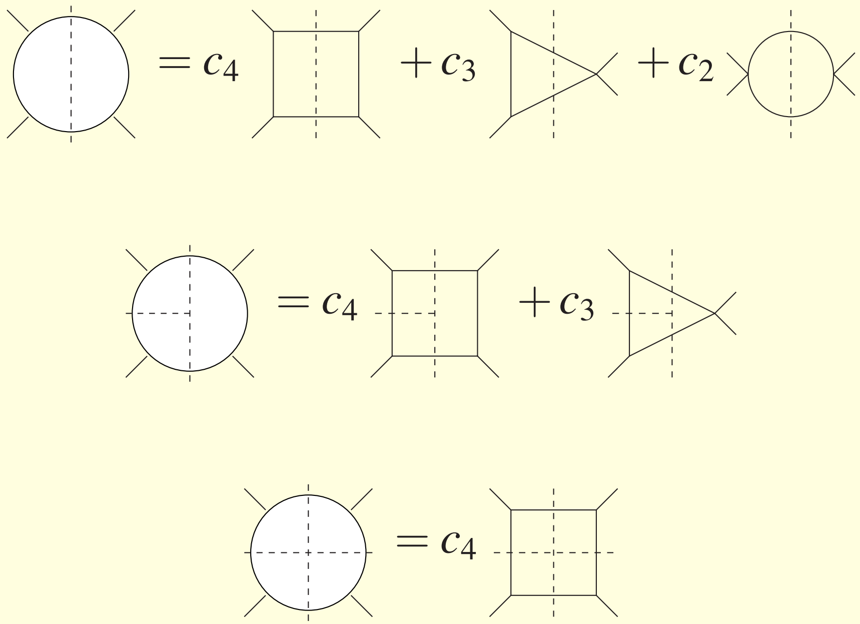
- Unitarity-motivated loop-mometum decomposition Ossola, Papadopoulos & Pittau; Forde; Ellis, Giele & Kunszt

$\longrightarrow$ talks by Forde, Kunszt, Papadopoulos

Generalised Unitarity

- coefficients show up entangled in a given cut: how do we disentangle them?

The polylogarithmic structure of boxes, triangles, and bubbles is different. Therefore their multiple cuts have specific signature which enable us to distinguish unequivocally among them.

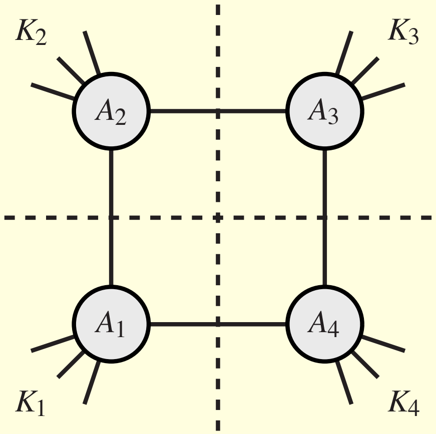


- Cuts in 4-dim carry informations about the *coefficients*
- Cuts in 4-dim do not carry any informations about the *rational term*
- Cuts in D-dim detect also *rational term*

Quadruple Cuts

Boxes

- Multiple Cuts   Bern, Dixon, Dunbar, Kosower (1994)



The discontinuity across the leading singularity, *via* quadruple cuts, is **unique**, and corresponds to the **coefficient** of the master **box**   Britto, Cachazo, Feng (2004)

$$c_{4,i} \quad \propto \quad A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}$$

with a frozen loop momentum: $\ell^\mu = \alpha K_1^\mu + \beta K_2^\mu + \gamma K_3^\mu + \delta \varepsilon^\mu_{\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$

Double-Cut Phase Space Measure

- 4-dim LIPS Cacahazo, Svrček & Witten

$$\ell_0^2 = 0 \, , \quad \ell_0 = |\ell_0\rangle[\ell_0| \equiv t|\ell\rangle[\ell|$$

$$\Rightarrow \int d^4\Phi = \int d^4\ell_0 \, \delta^{(+)}(\ell_0^2) \, \delta^{(+)}((\ell_0 - K)^2) = \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | K | \ell]} \int t \, dt \, \delta^{(+)}\left(t - \frac{K^2}{\langle \ell | K | \ell]} \right)$$

- D-dim LIPS Anastasiou, Britto, Feng, Kunszt, PM

$$\int d^{4-2\epsilon}\Phi \quad = \quad \chi(\epsilon) \int d\mu^{-2\epsilon} \int d^4\Phi_\mu \, ,$$

$$L = \ell_0 + zK \, , \qquad \text{with } \ell_0^2 = 0 \, , \qquad \ell_0 \equiv t|\ell\rangle[\ell| \qquad z_0 = \frac{1 - \sqrt{1 - \frac{4\mu^2}{K^2}}}{2} \, ,$$

$$\Rightarrow \int d^4\Phi_\mu \quad = \quad \int d^4L \, \delta^{(+)}(L^2 - \mu^2) \, \delta^{(+)}((L-K)^2 - \mu^2)$$

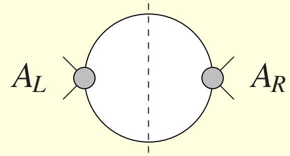
$$= \quad \int dz \, \delta(z - z_0) \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | K | \ell]} \int t \, dt \, \delta^{(+)}\left(t - \frac{(1-2z)K^2}{\langle \ell | K | \ell]} \right)$$

Double-Cut ⊕ Spinor-Integration

Britto, Buchbinder, Cachazo & Feng (2005); Britto, Feng & PM (2006)

Anastasiou, Britto, Feng, Kunszt & PM (2006)

Britto & Feng (2006)



$$M = \chi(\varepsilon) \int d\mu^{-2\varepsilon}\, \Delta\,, \qquad \Delta = \int d^4\Phi_\mu\, A_L^{\rm tree} \otimes A_R^{\rm tree}$$

- t-integration ⊕ Schouten identity

$$\int t\, dt\, \delta\left(t - \frac{(1-2z)K^2}{\langle\ell|K|\ell]}\right) \frac{A_L^{\rm tree}(\ell,z,t)\; A_R^{\rm tree}(\ell,z,t)}{\langle\ell|K|\ell]} = \sum_i G_i\left(|\ell\rangle, z\right) \frac{[\eta\,\ell]^n}{\langle\ell|P_1|\ell]^{n+1}\langle\ell|P_2|\ell]} \equiv \sum_i T_i$$

the 4D-discontinuity reads,

$$\Delta = \sum_i \int dz\, \delta(z - z_0) \int \langle\ell\, d\ell\rangle[\ell\, d\ell]\; T_i$$

1. $P_1 = P_2 = K$ (momentum across the cut) $\Rightarrow$ 2-point function (cut-free term)
2. $P_1 = K$, $P_2 \neq K$, or $P_1 \neq P_2 \neq K \Rightarrow$ n-point functions with $n \geq 3$ (Log-term)

Log-term of 4D-Double Cut

- Feynman Parametrization: $P_1 = K$, $P_2 \neq K$, or $P_1 \neq P_2 \neq K$

$$T_i = (n+1)\int dx\,(1-x)^n\, G_i(|\ell\rangle, z)\,\frac{[\eta\,\ell]^n}{\langle\ell|R|\ell]^{n+2}}\,, \qquad \not{R} = x\not{P}_1 + (1-x)\not{P}_2$$

- Integration-by-Parts in $|\ell]$

$$[\ell\, d\ell]\frac{[\eta\,\ell]^n}{\langle\ell|P|\ell]^{n+2}} = \frac{[d\ell\ \partial_{|\ell]}]}{(n+1)}\frac{[\eta\,\ell]^{n+1}}{\langle\ell|P|\ell]^{n+1}\langle\ell|P|\eta]}\,.$$

- Integration in $|\ell\rangle$: Holomorphic δ-function (Cauchy-Pompeiu's Formula) Cachazo, Svrcek, Witten; Cachazo; Britto, Cachazo, Feng

$$F_i \;=\; \int \langle\ell\, d\ell\rangle[\ell\, d\ell]\; T_i = \int dx\,(1-x)^n \int \langle\ell\, d\ell\rangle[d\ell\ \partial_{|\ell]}]\frac{G_i(|\ell\rangle, z)\ [\eta\,\ell]^{n+1}}{\langle\ell|R|\ell]^{n+1}\langle\ell|R|\eta]}$$

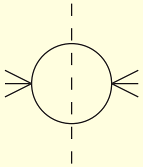
$$=\; \int dx\,(1-x)^n\left\{\frac{G_i(\not{R}|\eta], z)}{(R^2)^{n+1}} + \sum_j \lim_{\ell\to\ell_{ij}}\langle\ell\,\ell_{ij}\rangle\frac{G_i(|\ell\rangle, z)\ [\eta\,\ell]^{n+1}}{\langle\ell|R|\ell]^{n+1}\langle\ell|R|\eta]}\right\} = F_i^{(1)} + F_i^{(2)}$$

where $|\ell_{ij}\rangle$ are the simple poles of G_i, and $R^2 = a(x-x_1)(x-x_2)$

- Double-Cut

$$M = \chi(\epsilon)\int d\mu^{-2\epsilon}\int dz\,\delta(z-z_0)\sum_i\left(F_i^{(1)} + F_i^{(2)}\right)$$

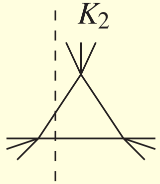
- I_2



$$= \int d^4\ell \, \delta^{(+)}(\ell^2) \, \delta^{(+)}((\ell - K)^2) = K^2 \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | K | \ell]^2} = 1 \, ;$$

The discontinuity of a bubble is **rational** !!!

- I_3^{3m}



$$= \int d^4\ell \, \delta^{(+)}(\ell^2) \, \frac{\delta^{(+)}((\ell - K_1)^2)}{(\ell + K_3)^2} = \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | K_1 | \ell] \langle \ell | Q | \ell]} = \int_0^1 dx \int \frac{\langle \ell \, d\ell \rangle [\ell \, d\ell]}{\langle \ell | R | \ell]^2} = \int_0^1 dx \frac{1}{R^2}$$

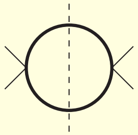
$$\not{Q} = (K_3^2 / K_1^2) \not{K}_1 + \not{K}_3 \, , \quad \not{R} = (1 - x) \not{K}_1 + x \not{Q} \; \Rightarrow R^2 \text{ quadratic in } x$$

The discontinuity of a 3m-Triangle is a $\ln$**(irrational argument)** !!!

- I_4

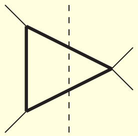
The double cut detect box-coefficient as well. One can show that the discontinuity of a 1m-,2m-,3m-box is a $\ln$**(rational argument)** – but boxes are known from 4-ple cuts.

- I_2



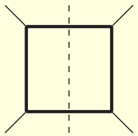
$$= \sqrt{1 - \frac{4\mu^2}{K^2}}$$

- I_3^{1m}



$$= \frac{1}{K^2} \ln \left(\frac{1 - \sqrt{1 - \frac{4\mu^2}{K^2}}}{1 + \sqrt{1 - \frac{4\mu^2}{K^2}}} \right)$$

- I_4^{0m}



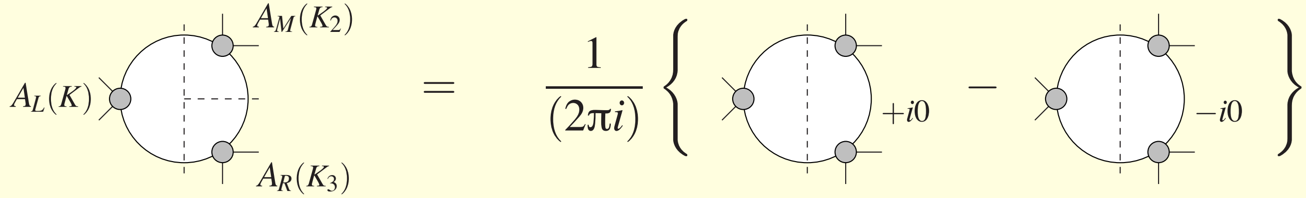
$$= \frac{2}{st\sqrt{1 - \frac{4\mu^2(s+t)}{s}}} \ln \left(\frac{1 - \sqrt{1 - \frac{4\mu^2(s+t)}{s}}}{1 + \sqrt{1 - \frac{4\mu^2(s+t)}{s}}} \right)$$

- μ-integration $\equiv$ Dimension-Shift

$$\begin{aligned}
\int \frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}} (\mu^2)^r f(\mu^2) &= \int d\Omega_{-1-2\varepsilon} \int \frac{d\mu^2}{2(2\pi)^{-2\varepsilon}} (\mu^2)^{-1-\varepsilon+r} f(\mu^2) = \frac{(2\pi)^{2r} \int d\Omega_{-1-2\varepsilon}}{\int d\Omega_{2r-1-2\varepsilon}} \int \frac{d^{2r-2\varepsilon}\mu}{(2\pi)^{2r-2\varepsilon}} f(\mu^2) \\
&= -\varepsilon(1-\varepsilon)(2-\varepsilon)\cdots(r-1-\varepsilon)(4\pi)^r \int \frac{d^{2r-2\varepsilon}\mu}{(2\pi)^{2r-2\varepsilon}} f(\mu^2)
\end{aligned}$$

Triple-Cut ⊕ Spinor-Integration

PM, *Phys. Lett.* B644 (2007) 272



$$
\begin{aligned}
N &= \chi(\varepsilon)\int d\mu^{-2\varepsilon}\,\Theta\,, \\
\Theta &= \int d^4\Phi_\mu\,\delta^{(+)}((L+K_3)^2-\mu^2)\;A_L^{\rm tree}\otimes A_M^{\rm tree}\otimes A_R^{\rm tree} \\
&= \int dz\,\delta(z-z_0)\sum_i\left\{\delta F_i^{(1)}+\delta F_i^{(2)}\right\}
\end{aligned}
$$

with

$$
\begin{aligned}
\delta F_i^{(1)} &\equiv \frac{1}{(2\pi i)}\left(F_i^{(1,+)}-F_i^{(1,-)}\right) = \int dx\,(1-x)^n\;G_i(\not{R}|\eta],z)\;\delta\Big((R^2)^{n+1}\Big) \\
\delta F_i^{(2)} &\equiv \frac{1}{(2\pi i)}\left(F_i^{(2,+)}-F_i^{(2,-)}\right) \\
&= \sum_j \lim_{\ell\to\ell_{ij}}\langle\ell\,\ell_{ij}\rangle\;G_i(|\ell\rangle,z))\;[\eta\,\ell]^{n+1}\int dx\,(1-x)^n\;\delta\Big(\langle\ell_{ij}|R|\ell_{ij}]^{n+1}\langle\ell_{ij}|R|\eta]\Big)
\end{aligned}
$$

The integration over the Feynman parameter is frozen.

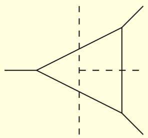
▷ *Cuts* in Feynman Parameters

$$\frac{1}{(ax+b)+i0} \quad \to \quad K_1(x) = \frac{1}{a}\delta(x-x_0)$$

$$\frac{1}{(ax^2+bx+c)+i0} \quad \to \quad K_2(x) = \frac{1}{a\,|x_1-x_2|}\Big(\delta(x-x_1)+\delta(x-x_2)\Big)$$

where $x_{0,1,2}$ are the *zeroes* of the corresponding denominators.

- I_3^{3m}

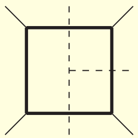


$$= \quad \cdots = \frac{1}{(2\pi i)}\int dx \left\{\frac{1}{R^2+i0} - \frac{1}{R^2-i0}\right\} = \int dx\, \delta(R^2) = \int dx\, K_2(x) = \frac{(-2)}{\sqrt{\Lambda}}$$

with

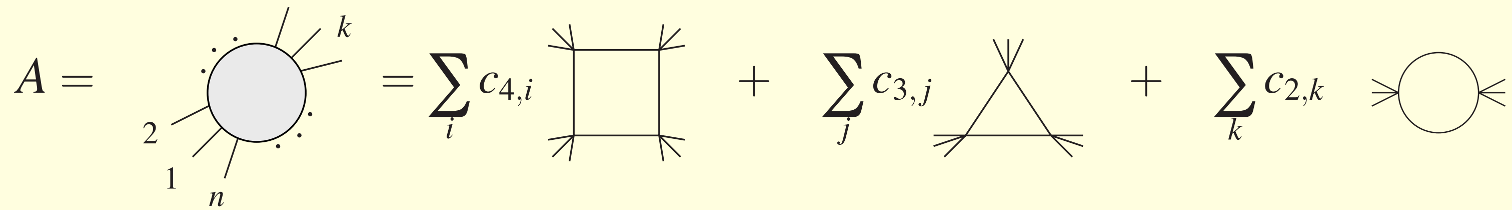
$$R^2 \quad = \quad ax^2+2bx+c\,, \qquad x_{1,2} = \frac{-b\pm\sqrt{\Lambda}}{a}\,, \qquad \Lambda = \text{Källen func'n}$$

- massive-I_4^{0m}

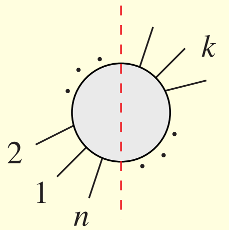


$$= \frac{(-2)}{st\sqrt{1-4\frac{(s+t)}{s}\mu^2}}$$

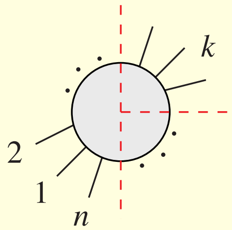
Cut-Construction of One-Loop Amplitudes



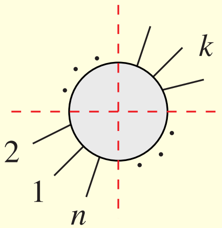
$c_{2,i} \Leftarrow$



$c_{3,i} \Leftarrow$



$c_{4,i} \Leftarrow$



On-Shell Complex Momenta enable the *fulfillment* of the cut-constraints!

Master Formulae

Schouten identity to reduce $|\ell]$

$$\frac{[\ell\, a]}{[\ell\, b]\, [\ell\, c]} = \frac{[b\, a]}{[b\, c]}\frac{1}{[\ell\, b]} + \frac{[c\, b]}{[c\, b]}\frac{1}{[\ell\, c]} \qquad (1)$$

Integration-by-Parts in $|\ell]$

$$[\ell\; d\ell]\frac{[\eta\, \ell]^n}{\langle \ell|P|\ell]^{n+2}} = \frac{[d\ell\; \partial_{|\ell]}]}{(n+1)}\frac{[\eta\, \ell]^{n+1}}{\langle \ell|P|\ell]^{n+1}\langle \ell|P|\eta]}\;. \qquad (2)$$

Cauchy's Residue Theorem in $|\ell\rangle$,

$$[d\ell\; \partial_{|\ell]}]\frac{1}{\langle \ell\, x\rangle} = 2\pi\delta\Big(\langle \ell\, x\rangle\Big)\;, \qquad \int \langle \ell\; d\ell\rangle\; \delta\Big(\langle \ell\, x\rangle\Big)\; f(|\ell\rangle, |\ell]) = f(|x\rangle, |x]) \qquad (3)$$

Residues in Feynman parameters, at the *zeroes* of the Standard Quadratic Function.

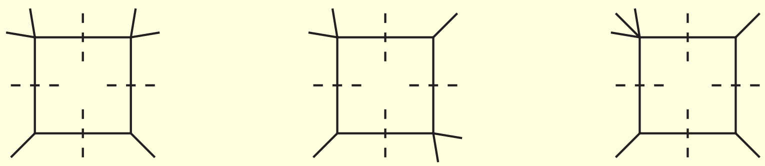
These *zeroes* are the signature of the Master Integrals: they correspond to branch points, therefore determining the polylogarithmic structure.

NLO 6-gluon Amplitude

- Numerical Result: Ellis, Giele, Zanderighi (2006)
- Analytical Result:

Amplitude	$N=4$	$N=1$	$N=0\|_{\mathrm{CC}}$	$N=0\|_{\mathrm{rat}}$
$(--++++)$	BDDK'94	BDDK'94	BDDK'94	BDK'05, KF'05
$(-+-+++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(-++-++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(---+++)$	BDDK'94	BBDD'04	BBDI'05, BFM'06	BBDFK'06
$(--+-++)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06
$(-+-+-+)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06

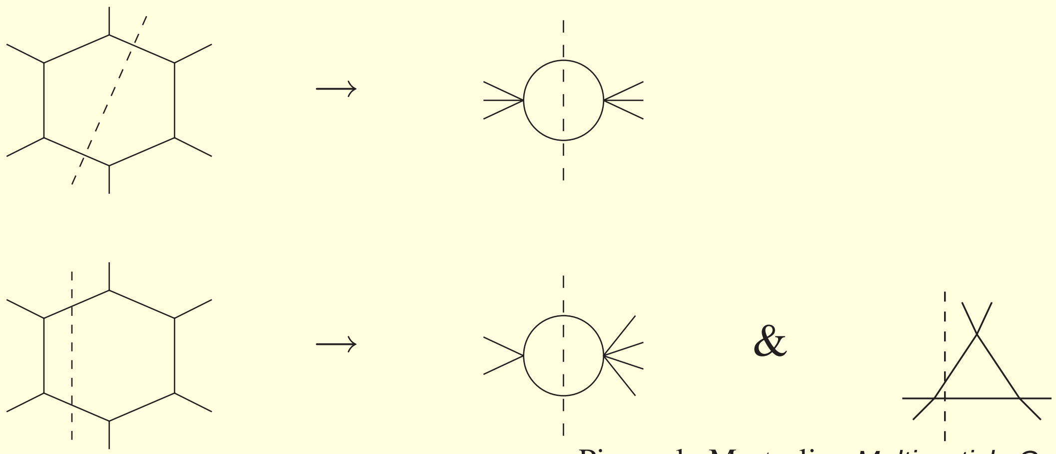
Quadruple Cuts



Bidder, Bjerrum-Bohr, Dunbar & Perkins (2005)

Double Cuts

Britto, Feng & PM (2006)



6-photon Amplitude

Mahlon (1996)

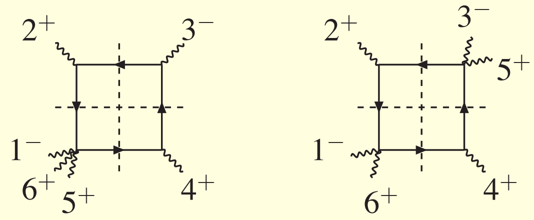
Nagy & Soper (2006)

Binoth, Guillet & Heinrich (2006)

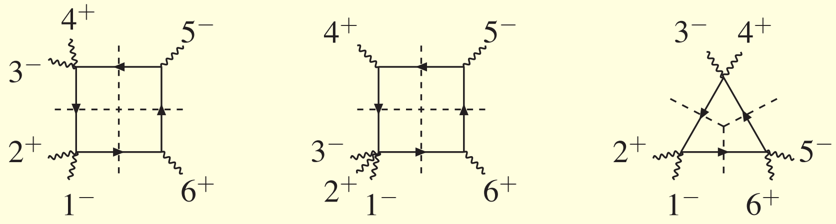
Binoth, Gehrmann, Heinrich & PM [hep-ph/0703311]

Ossola, Papadopoulous & Pittau (2007); Forde (2007)

- $(1^-, 2^+, 3^-, 4^+, 5^+, 6^+)$



- $(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$



NLO n-gluon⊕Higgs Amplitudes

- Heavy-top limit

- H + 4 partons Ellis, Giele, Zanderighi (2005)
- H + 5 partons Campbell, Ellis, Zanderighi (2006)

- H + n-gluons

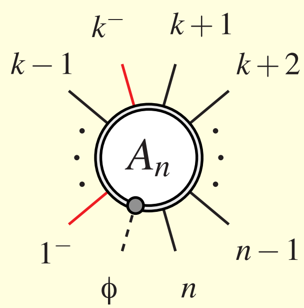
$$\phi = \frac{1}{2}(H + iA)$$

$$G^{\mu\nu}_{SD} = \frac{1}{2}(G^{\mu\nu} + \tilde{G}^{\mu\nu}) \; , \quad G^{\mu\nu}_{ASD} = \frac{1}{2}(G^{\mu\nu} - \tilde{G}^{\mu\nu}) \; , \quad \tilde{G}^{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$$

$$L_{\rm int} \propto H \, {\rm tr}G_{\mu\nu}G^{\mu\nu} + iA \, {\rm tr}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} \; = \; \phi \, {\rm tr}G_{SD,\mu\nu}G^{\mu\nu}_{SD} + \; \phi^\dagger \, {\rm tr}\tilde{G}_{ASD,\mu\nu}\tilde{G}^{\mu\nu}_{ASD} \; ,$$

- A(ϕ + n-gluons) $\longrightarrow$ A(n-gluons) w/out momentum conservation Dixon, Glover & Kohze

- ϕ-nite Berger, Del Duca, Dixon (2006)
- ϕ-MHV amplitudes (nearest neighbour minuses) Badger, Glover, Risager (2007)
- ϕ-MHV amplitudes (generic configuration) Glover, Williams, PM (wip)

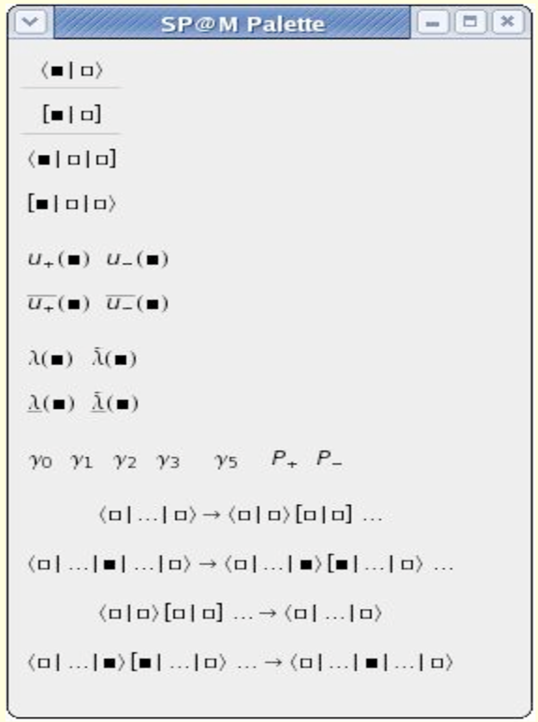


Outlook ...

- 5-point One-Loop Bhabha
- Gravity amplitudes [N=8 SuGra UV-behaviour]
- Generalised Unitarity ⇔ Iterated Cuts in Feynman Parameters Duplancic & PM (wip)
- Generalised Unitarity for Multi-loop

@ GGI Workshop

- ϕ-MHV amplitudes (generic configuration) Glover, Williams, PM
- S@M (Spinors @ MATHEMATICA) Maître & PM (to be released)
  i. spinor algebra
  ii. spinor shifts
  iii. numerics

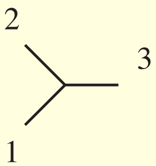


...& Summary

- Efficient technique for Generalised Unitarity

1. basic spinor algebra
2. spinor integration *via* holomorphic-δ
3. cuts in Feynman parameters: trivial parametric integrations frozen by δ's

- on-shell 3-point amplitude: $k_i^2 = 0$



$$0 = k_1^2 = (k_2 + k_3)^2 = 2k_2 \cdot k_3 = \langle 2\,3 \rangle [3\,2] \begin{cases} \langle 2\,3 \rangle \neq 0 \\ \\ |3]\ //\ |2] \end{cases} \qquad (k_3 \text{ on} - \text{shell \& complex})$$

The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and non-being. [...] there is something fishy about [...] imaginaries, but one can calculate with them because their form is correct. Leibniz