Recurrence relations
in the large space-time dimension limit

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Recurrence relations (RR):

$$R(I^-, I^+, d) F(n_1, ..., n_k, d) = 0$$

Result of the reduction procedure:

$$F(\underline{n}, d) = C_1(\underline{n}, d) F_1 + ... + C_k(\underline{n}, d) F_k$$

$C_i(\underline{n}, d)$ **are rational in** d **and obey the RR:**

$$R(I^-, I^+) C_i(\underline{n}, d) = 0$$

$C_i(\underline{n}, d) = 0$ **if some "hard"** $n_l \leq 0$ **(for lines in** F_i**)**

C_i **are much simpler then** F

Can we calculate C_i **directly (without reduction)?**

Expand C_i in $1/d \to 0$

Calculate sufficiently many coefficients

Reconstruct exact rational d dependence

Expansion

$$R(I^-, I^+) = R^{(0)} + \frac{1}{d} R^{(1)}$$

$$C_i = C_i^{(0)} + \frac{1}{d} C_i^{(1)} + \frac{1}{d^2} C_i^{(2)} + \ldots$$

$$0 = R(I^-, I^+) C_i \qquad \Rightarrow$$

$$0 = R^{(0)} C_i^{(0)}$$

$$0 = R^{(0)} C_i^{(1)} + R^{(1)} C_i^{(0)}$$

$$\ldots$$

$$0 = R^{(0)} C_i^{(k)} + R^{(1)} C_i^{(k-1)}$$

$$\ldots$$

1-dimentional example

$$f_n = \int_{-\infty}^{\infty} (x^2 + 2x + 2)^d / x^n \; dx$$

$$n f_{n+1} = (d + 1 - n) f_n + (d + 1 - n/2) f_{n-1}$$

Reduction is trivial, but let's try $1/d$

$$0 = \underbrace{(f_n + f_{n-1})}_{R^{(0)} f} + \; 1/d \; \underbrace{((1 - n/2) f_{n-1} + (1 - n) f_n - n f_{n+1})}_{R^{(1)} f}$$

$$0 = \; R^{(0)} f^{(0)} = f_n^{(0)} + f_{n-1}^{(0)} \quad \Rightarrow \; f_n^{(0)} = (-1)^n$$

$$0 = \quad R^{(0)} f^{(1)} + R^{(1)} f^{(0)}$$

$$= f_n^{(1)} + f_{n-1}^{(1)} + n/2 (-1)^n \; \Rightarrow \; f_n^{(1)} = 1/4 (n^2 + n)(-1)^n$$

$\boldsymbol{R(I^-)C_i = 0}$ **can be solved in multi-dimentional case**

$$C_i(\underline{n}) = \Pi_a\, r_a^{-n_a}, \quad \text{where } R(r_a) = 0$$

$$R(I_a^-)\ C_i \ = \ R(r_a)\ C_i \ = \ 0$$

$\boldsymbol{R(I^-)}$ **vs.** $\boldsymbol{R(I^-, I^+)}$ **like**

algebraic vs. differential equations

It is convinient to choose $\boldsymbol{R^{(0)} = R(I^-)}$

It can be done in case of Feynman integrals

Feynman integrals

$$F(\underline{n}, d) = \int d^d p_1..d^d p_L/(E_1^{n_1} \cdots E_a^{n_a})$$

$$E_a = A_a^{ik}(p_i p_k) + m_a^2$$

IBP:

$$0 = \int d^d p_1..d^d p_L \; \partial_{p_i}(p_k \cdots)$$

$$\partial_{p_i} \; (p_k \; \cdot) = d \; \delta_k^i + p_k(\partial_{p_i} \; \cdot) = d \; \delta_k^i + (AA)_b^a \; E_a \; (\partial_{E_b} \; \cdot)$$

RR:

$$0 = d \; \delta_k^i \; F + (AA)_b^a \; E_a \; \partial_{E_b} \; F$$

$$0 = d\ \delta^i_k\ F + (AA^i_k)^a_b\ E_a\ \partial_{E_b}\ F$$

$$R^{(0)}? \qquad R^{(1)}?$$

$$0 = R^{(0)} C^{(0)} = \delta^i_k\ C^{(0)}_n \quad \Rightarrow \quad C^{(0)}_n = 0 \quad ???$$

1/d does not work ?

No solutions like $C_i = C_i^{(0)} + \frac{1}{d} C_i^{(1)} + ...$

Indeed, $C_k \approx d^{-S(\underline{n})}$, $\quad S(\underline{n}) = \Sigma\ (\text{"hard"} n_i)$

We apply $1/d$ expansion to the subcase "hard" $n_i = 1$

$n_i > 1$ can be reduced to $n_i = 1$ by direct recursion

Modified IBP

$$0 = \int d^d p_1 .. d^d p_L \ \partial_{p_i} (p_k \Pi^{ik}(E_a) \cdots)$$

With some polinomials Π^{ik} we come to diagonalized RR

$$0 = \underbrace{\partial_{E_a}(P(E)\ F)}_{R^{(1)}} - \underbrace{(d - L - 1)/2}_{\text{large}}\ \underbrace{(\partial_{E_a} P(E))\ F}_{R^{(0)}}$$

Equations with "hard" ∂_{E_a}: $\quad n_a \to 1$

Equations with "soft" ∂_{E_a} ("hard" $E_a = 0$)

$$0 = R^{(0)} F^{(0)} = (\partial_{E_a} P(E)) F^{(0)} \quad \Rightarrow \quad F^{(0)}(n) = \Pi\ r_a^{n_a}$$

$$r_a : \ \partial_{E_a} P(E)|_{E_a = r_a} = 0$$

What is the optimal way to calculate $C_i^{(k)}$?

One can to obtain $C_i^{(k)}(\underline{n})$ as polinomials in $\underline{n}$

One can construct recurrent procedure for $C_i^{(k)}(\underline{n})$

More efficient is to expand in $1/d \to 0$ auxiliary integrals

$$C_i^{(k)}(\underline{n}) = \int dx_1 .. \, dx_a / (x_1^{n_1} .. \, x_a^{n_a}) P(\underline{x})^{(d-L-1)/2}$$

In $1/d \to 0$ they expand to Gaussian type integrals

$$\int dx_1 .. \, dx_a \; x_1^{k_1} .. \, x_a^{k_a} \exp(-A^{ik} x_i x_k)$$

Possible applications   Pro and Contra

Massless 0-scale problems

$1/d$ coefficients are pure numbers

Relatively small set of master integrals

Very convinient

4-loop propagators are fine

Few calendar months for $R(s, N_f = 3)$

1-mass 0-scale problems (bubbles)

$1/d$ coefficients are pure numbers

But many master integrals $\Rightarrow$

Many contributions to calculate

Calculational efforts (setup + CPU)

comparable to other approaches

(Laporta, Smirnov's, direct recursion)

Multi-scale problems

$1/d$ coefficients are multi-scale rationals $\Rightarrow$

Reduction is possible, but difficult

Many master integrals, difficult to calculate

Summary

When d is large RR for FI become "algebraic"

$\Rightarrow$ systematic reduction is possible

$1/d$ coefficients demands big amount of CPU

But massless 4-loop propagators are reachable