Firenze, 3 October '07

Progress on Singlet QCD Evolution at Small-x

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Based on G.A., R. Ball, S.Forte:

hep-ph/9911273 (NPB 575,313)
hep-ph/0001157 (lectures)
hep-ph/0011270 (NPB 599,383)
hep-ph/0104246

More specifically on

hep-ph/0109178 (NPB 621,359)
hep-ph/0306156 (NPB 674,459),
hep-ph/0310016

and finally, on our most recent works: hep-ph/0512237 (NPB 742,1,2006), hep-ph/0606323 and R. Ball, hep-ph/0708.1277 + work in progress)

Related work (same physics, same conclusion, different techniques): Ciafaloni, Colferai, Salam, Stasto; Thorne&White [see also Schmidt, Forshaw, D. Ross, Sabio Vera, Bartels, Kancheli, K. Ellis, Hautmann....]

Part 1

Short summary of previous results (very fast)

Part 2

New results Paper nearly completed

The problem is clear:

- At HERA & LHC at small x the terms in $(\alpha_s \log 1/x)^n$ cannot be neglected in the singlet splitting function
- BFKL have computed all coeff.s of $(\alpha_s \log 1/x)^n$ (LO BFKL)
- Just adding the sequel of $(\alpha_s \log 1/x)^n$ terms leads to a dramatic increase of scaling violations which is not observed (a too strong peaking of F_2 and of gluons is predicted)
- The inclusion of running coupling effects in BFKL was an issue
- Later, also all coeff.s of $\alpha_s(\alpha_s \log 1/x)^n$ (NLO BFKL) have been calculated
- (Fortunately) they completely destroy the LO BFKL prediction

⊕ The problem is to find the correct description at small x

The goal is to construct a relatively simple, closed form, improved anomalous dimension $\gamma_I(\alpha,N)$ or splitting function $P_I(\alpha,x)$

$P_I(\alpha,x)$ should

- reproduce the perturbative results at large x
- based on physical insight resum BFKL corrections at small x
- properly include running coupling effects
- be sufficiently simple to be included in fitting codes

The comparison of the result with the data provides a qualitatitevely new test of the theory

Moments

$$\xi = \log\frac{1}{x}; \qquad t = \log\frac{Q^2}{\mu^2}$$

$$G(x, Q^2) \equiv G(\xi, t) = x[g(x, Q^2) + k\Sigma(x, Q^2)]$$

Singlet quark

For each moment: singlet eigenvector with largest anomalous dimension eigenvalue

$$G(N, t) = \int_0^1 x^{N-1} G(x, Q^2)dx = \int_0^\infty e^{-N\xi} G(\xi, t)d\xi$$

Mellin transf. (MT)

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{N\xi} G(N, t)\frac{dN}{2\pi i}$$

Inverse MT ($\xi>0$)

t-evolution eq.n

$$\frac{d}{dt}G(N, t) = \gamma(N, \alpha(t))G(N, t)$$

γ: anom. dim

$$\gamma(N, \alpha) = \alpha\gamma_{1l}(N) + \alpha^2\gamma_{2l}(N) + \alpha^3\gamma_{3l}(N) + \ldots$$

Pert. Th.: LO NLO NNLO

known

Moch, Vermaseren, Vogt '04

Recall:

$$\gamma(N) = \int_0^1 x^N P(x)dx$$

$$P(x) = \frac{\alpha}{x}\left(\alpha \log\frac{1}{x}\right)^n \Leftrightarrow \gamma(N) = n!\left(\frac{\alpha}{N}\right)^{n+1}$$

splitting function anomalous dimension

At 1-loop:

$$\alpha \cdot \gamma_{1l}(N) = \alpha \cdot \left[\frac{1}{N} - A(N)\right]$$

This corresponds to the "double scaling" behavior at small x:

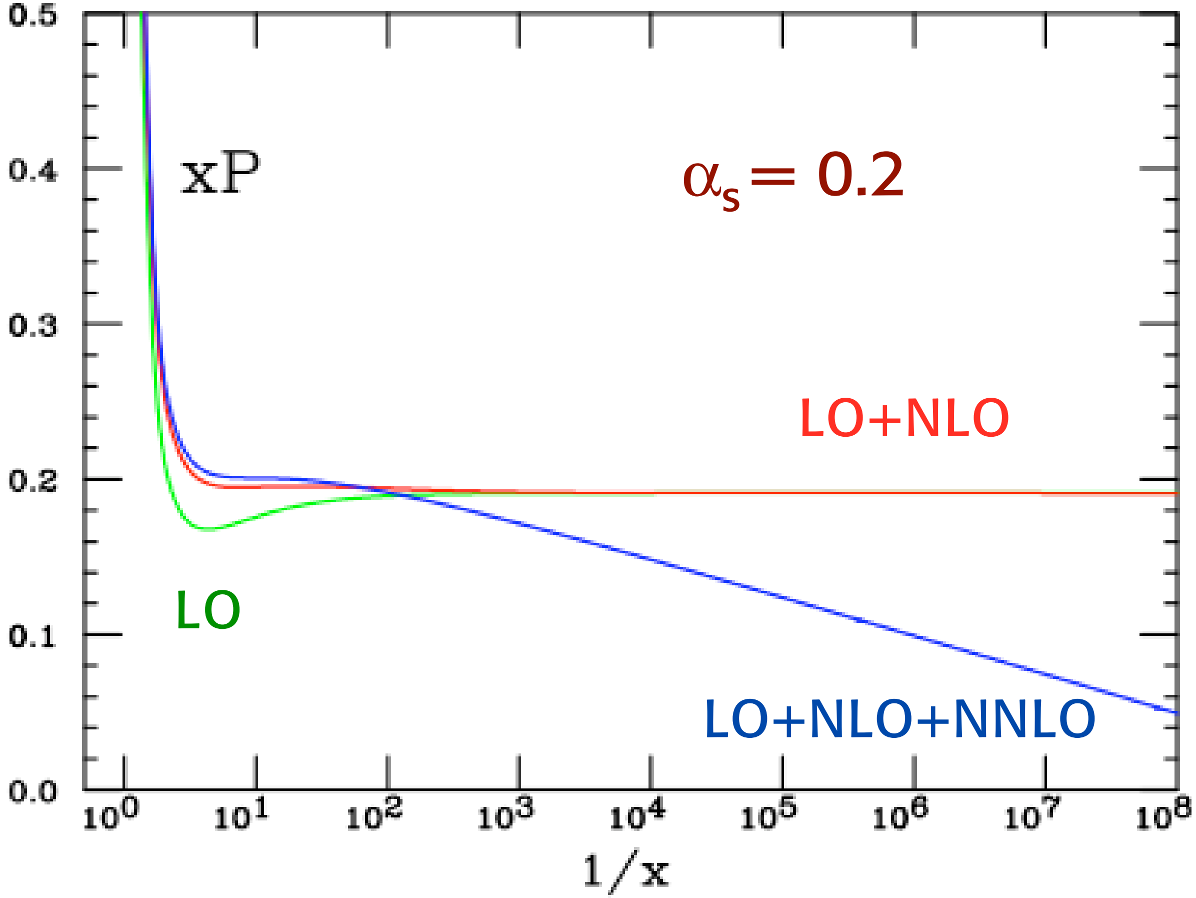
$$G(\xi, t) \sim \exp\left[\sqrt{\frac{4n_C}{\pi\beta_0} \cdot \xi \cdot \frac{\log Q^2/\Lambda^2}{\log \mu^2/\Lambda^2}}\right]$$

$$\beta(\alpha) = -\beta_0\alpha^2 + \ldots$$

A. De Rujula et al '74/Ball, Forte '94

Amazingly supported by the data

The singlet splitting function in perturbation theory



due to $\alpha_s^3 \log 1/x$ term

$$\alpha x P_{1l} + \alpha^2 x P_{2l} + \alpha^3 x P_{3l} + \ldots \sim \alpha + \alpha^2 \log 1/x + \alpha^2 + \alpha^3 (\log 1/x)^2 + \\ + \alpha^3 (\log 1/x) + \ldots$$

accidentally missing

In principle the BFKL approach provides a tool to control $(\alpha/N)^n$ corrections to $\gamma(N, \alpha)$, that is $(\alpha \log 1/x)^n$ to $xP(x,Q^2)/\alpha$

Define t- Mellin transf.:

$$G(\xi, M) = \int_{-\infty}^{+\infty} e^{-Mt} G(\xi, t) dt$$

with inverse:

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{Mt} G(\xi, M) \frac{dM}{2\pi i}$$

ξ-evolution eq.n (BFKL) [at fixed α]:

$$\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha) G(\xi, M)$$

with $\chi(M, \alpha) = \alpha \cdot \chi_0(M) + \alpha^2 \cdot \chi_1(M) + \ldots$

known

Bad behaviour, bad convergence

χ_0, χ_1 contain all info on $(\alpha \log 1/x)^n$ ⊕ and $\alpha(\alpha \log 1/x)^n$

The minimum value of $\alpha\chi_0$ at M=1/2 is the Lipatov intercept:

$$\lambda_0 = \alpha\chi_0\left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi} 4\ln 2 = \alpha c_0 \sim 2.65\alpha \sim 0.5$$

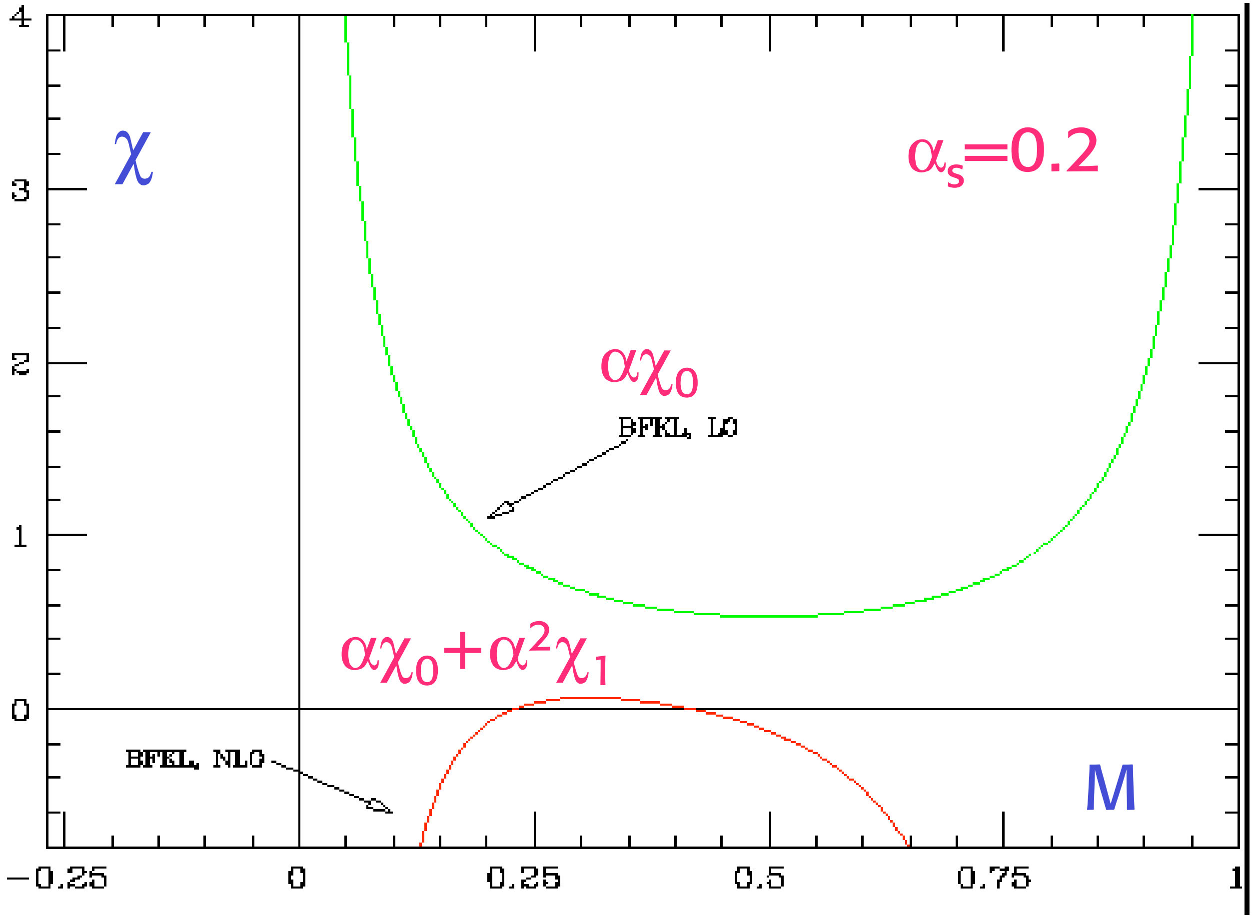
It corresponds to (for x->0, Q^2 fixed):

$xP(x) \sim \alpha x^{-\lambda_0}$

Too hard, not supported by data

But the NLO terms are very large

χ_1 totally overwhelms χ_0!!



Basic ingredients of the resummation procedure

- Duality relation $\chi(\gamma(\alpha, N), \alpha) = N$

  from consistency of 1/x and Q^2 evolution

- Momentum conservation $\chi(0, \alpha) = 1$

  as $\gamma(\alpha,1) = 0$

- Symmetry properties of the BFKL kernel

- Running coupling effects

In the region of t and x where both

$$\frac{d}{dt}G(N,t) = \gamma(N,\alpha)G(N,t)$$

$$\frac{d}{d\xi}G(\xi,M) = \chi(M,\alpha)G(\xi,M)$$

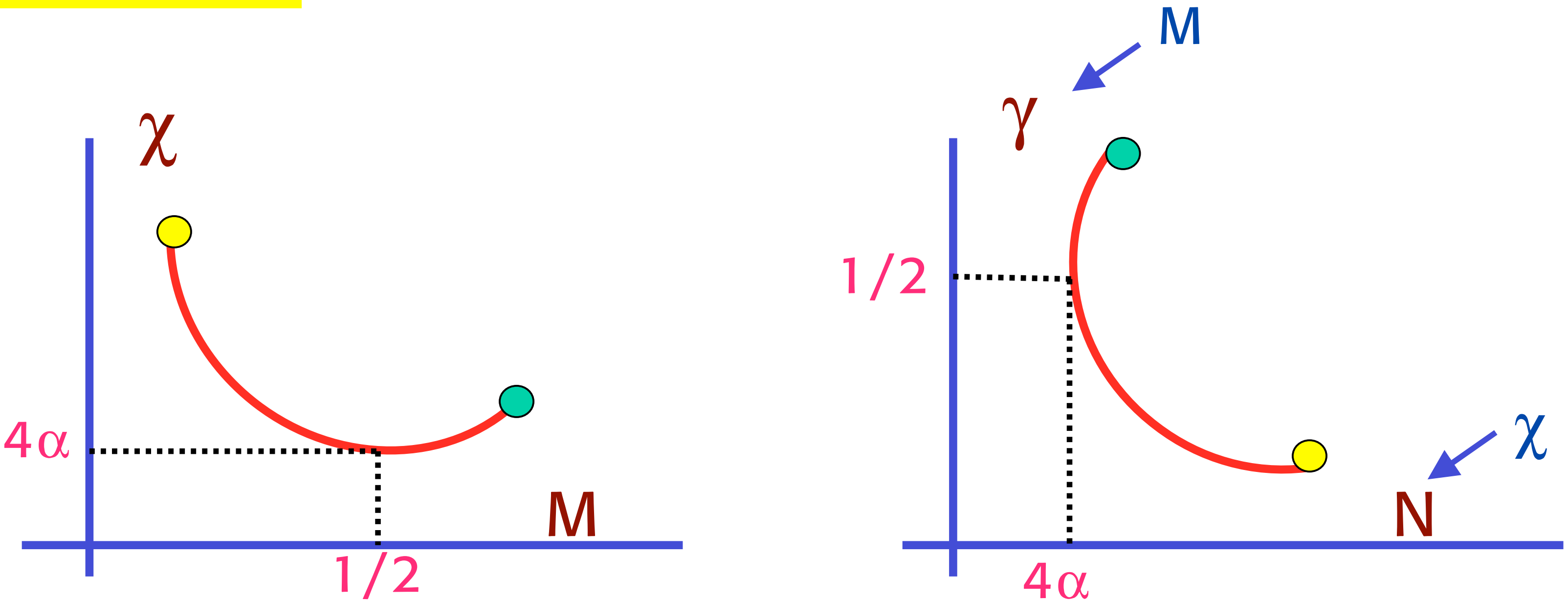
are approximately valid, the "duality" relation holds:

$$\chi(\gamma(\alpha,N),\alpha) = N$$

Note: γ is leading twist while χ is all twist.
Still the two perturbative exp.ns are related and improve each other.
Non perturbative terms in χ correspond to power or exp. suppressed terms in γ.

$\chi(\gamma(N)) = N$

Graphically duality is a reflection



Example: if $\chi(M,\alpha) = \alpha\left[\frac{1}{M} + \frac{1}{1-M}\right]$ ⟹

⟹ $\alpha\left[\frac{1}{\gamma} + \frac{1}{1-\gamma}\right] = N$ ⟹ $\gamma = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4\alpha}{N}}\right]$

Note: γ contains $(\alpha/N)^n$ terms

For example at 1-loop:

$\chi_0(\gamma_s(\alpha,N))=N/\alpha$

χ_0 improves γ by adding a series of terms in $(\alpha/N)^n$:

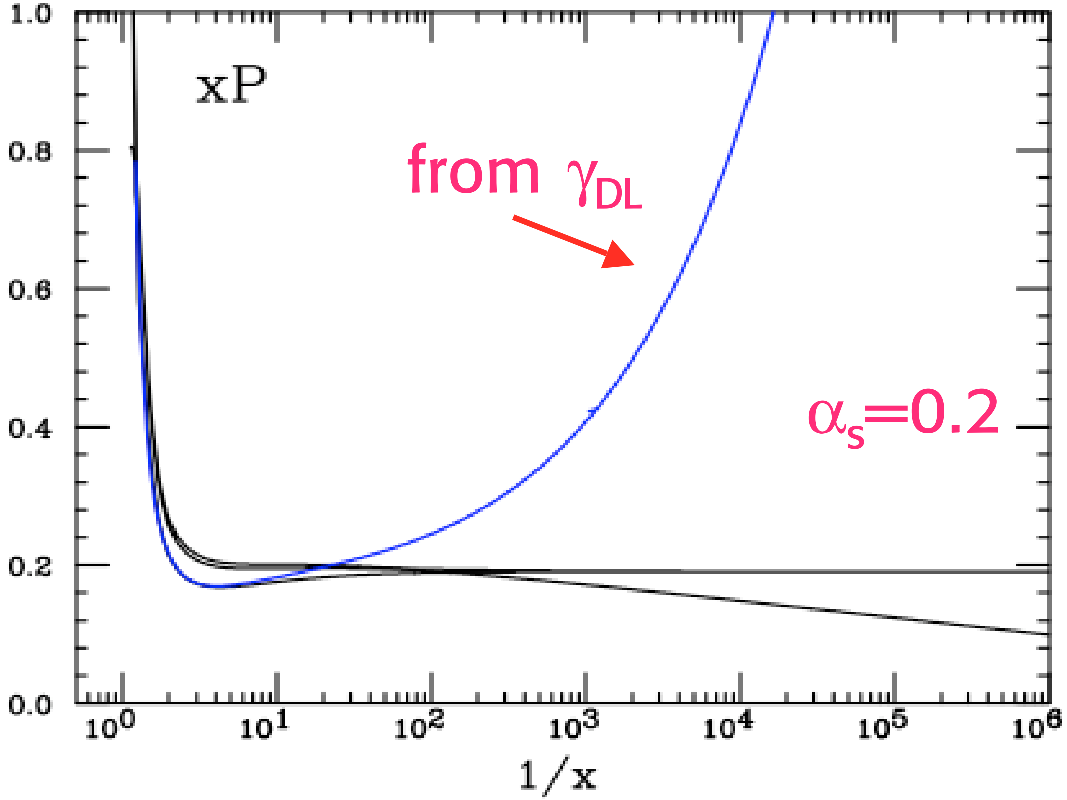
χ_0 -> $\gamma_s\left(\frac{\alpha}{N}\right)$

$$\gamma_s\left(\frac{\alpha}{N}\right) = \sum_k c_k\left(\frac{\alpha}{N}\right)^k$$

$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \ldots \text{ -double count.}$$

γ_{DL} is the naive result from GLAP+(LO)BFKL

The data discard such a large raise at small x



Similarly it is very important to improve χ by using $\gamma_{1l.}$

Near M=0, $\chi_0 \sim 1/M$, $\chi_1 \sim -1/M^2$.......

Duality + momentum cons. ($\gamma(\alpha,N=1)=0$)

$$\chi(\gamma(\alpha, N), \alpha) = N \longrightarrow \chi(0, \alpha) = 1$$

$$\lim_{M\text{->}0} \chi(M, \alpha) \approx \frac{\alpha}{M+\alpha}$$

$$\begin{cases} \gamma(\chi(M)) = M \rightarrow \gamma_{1l} \Rightarrow \chi_s\left(\frac{\alpha}{M}\right) \\ \chi_s\left(\frac{\alpha}{M}\right) = \sum_k d_k \left(\frac{\alpha}{M}\right)^k \end{cases}$$

$$\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$

Double Leading Expansion

$$\gamma(N,\alpha) = \alpha\cdot\gamma_{1l}(N) + \ldots \sim \alpha\cdot\left[\frac{1}{N} - A(N)\right]$$

Momentum conservation: $\gamma(1,\alpha)=0$ → $A(1)=1$

Duality: $\gamma(\chi(M)) = M$ → $\alpha\cdot\left[\frac{1}{\chi} - A(\chi)\right] = M$ →

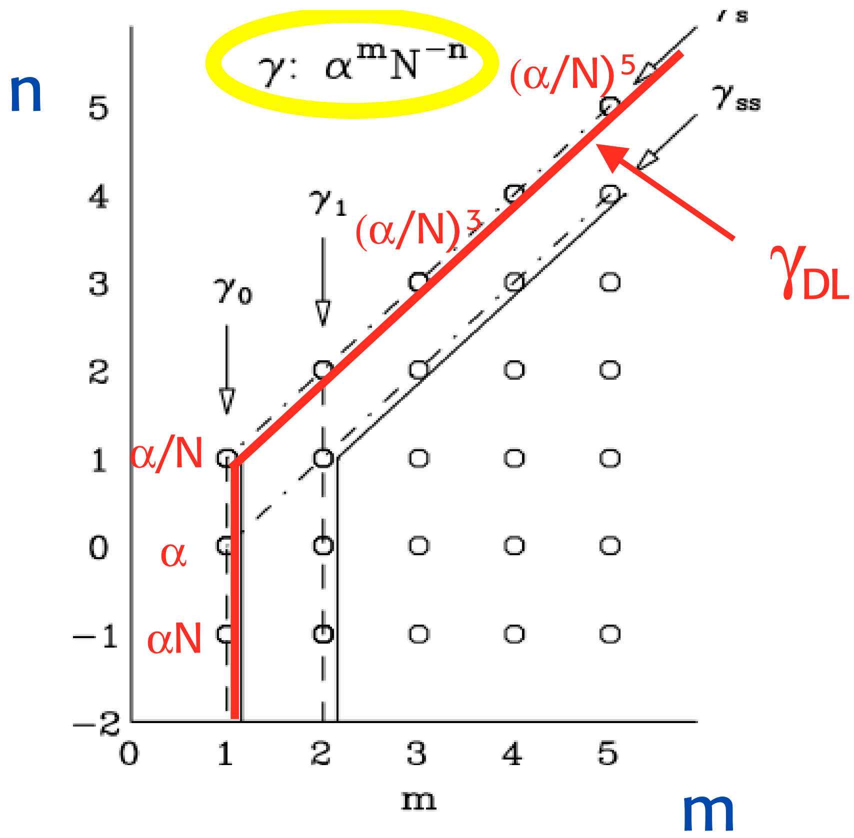
→ $\chi = \frac{\alpha}{M+\alpha A(\chi)}$ → $\chi(M\sim 0) \sim \frac{\alpha}{M+\alpha A(1)} \sim \frac{\alpha}{M+\alpha}$

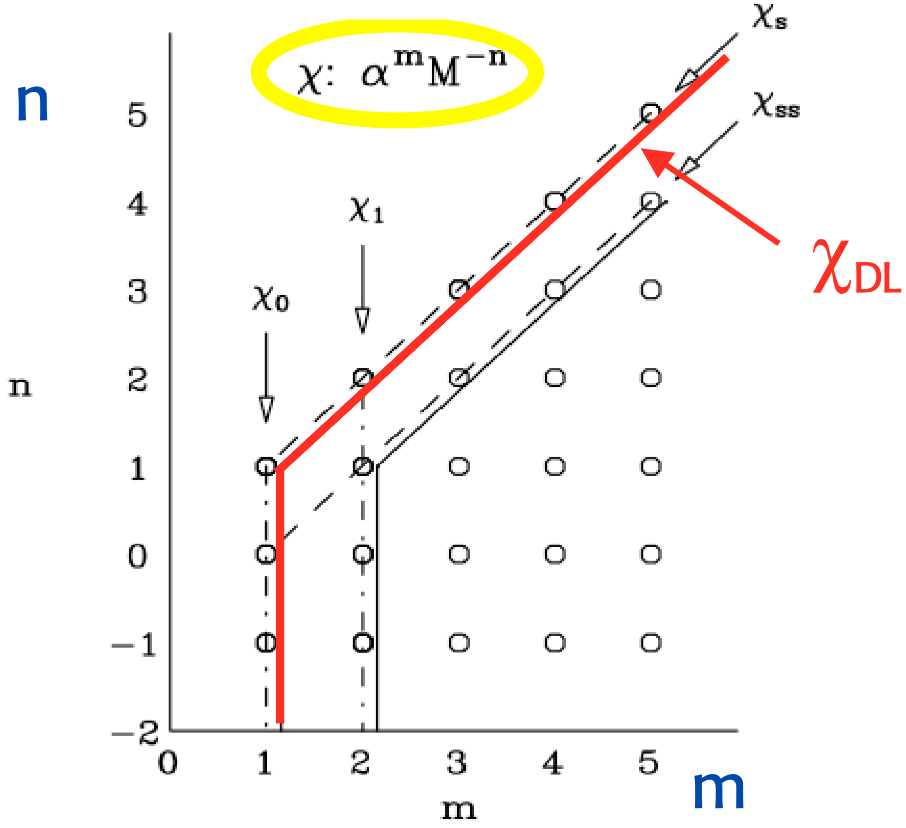
$$\chi_{DL}(M,\alpha) = \alpha\cdot\chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$

$$\chi_0(M) = \alpha\cdot\left[\frac{1}{M} + 0(M^2)\right]$$

$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \ldots \text{ -double count.}$$

$$\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$

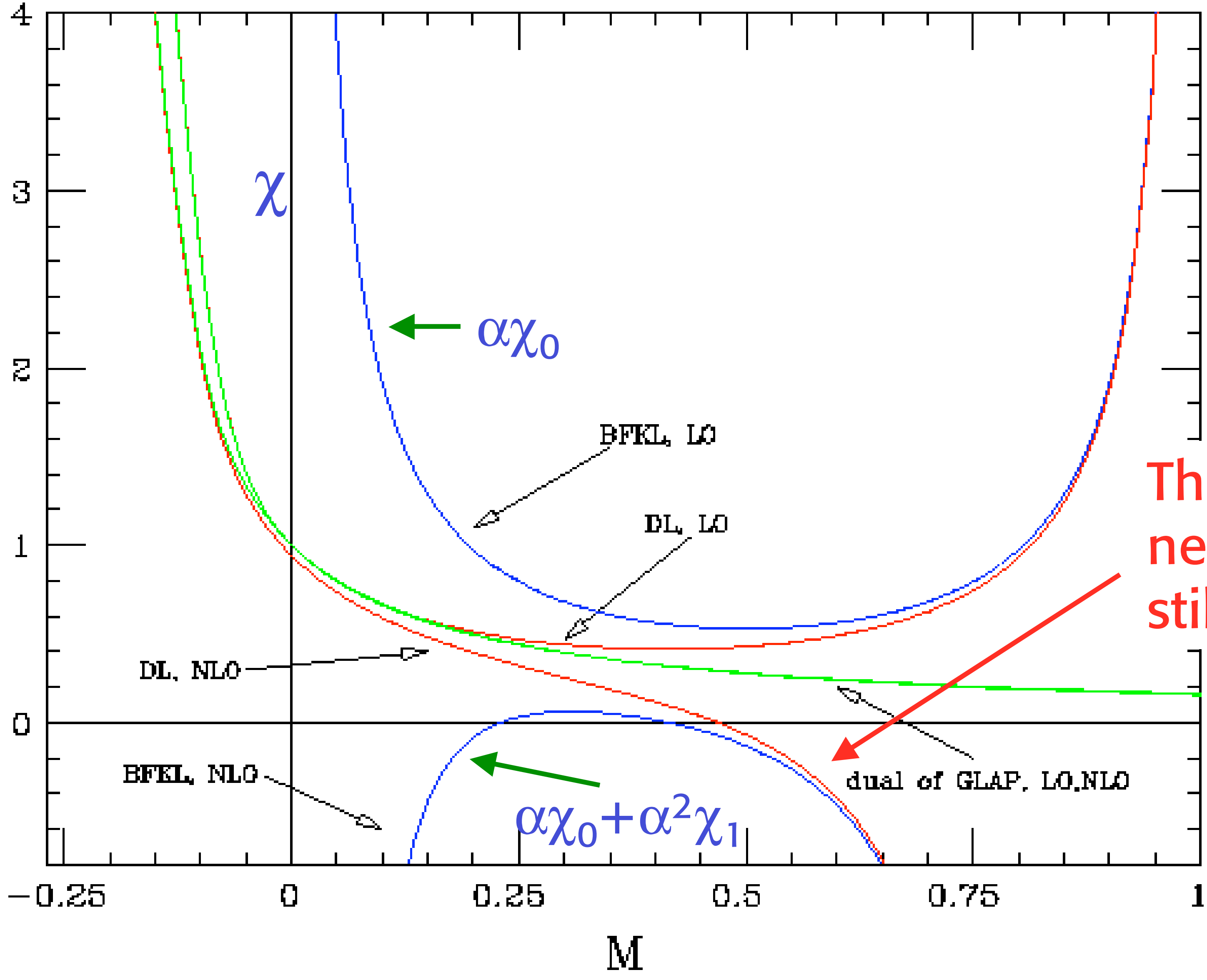




In the DL expansion one sums over "frames" rather than over vertical lines like in ordinary perturb. theory

DL, LO: $\chi_{DL}(M,\alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right)$ +... -double count.

BFKL, LO



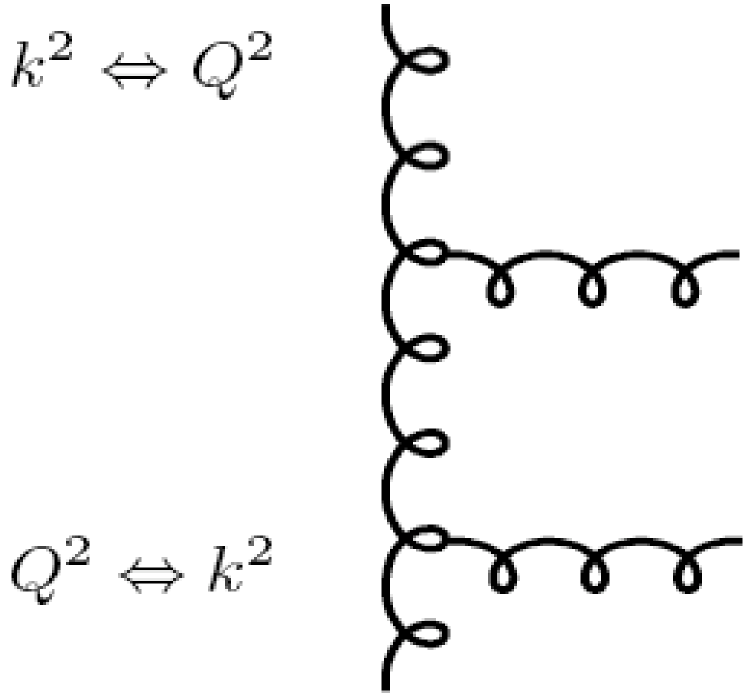
The NLO-DL is good near M=0, but it is still bad near M=1

Can be fixed by symmetrization

Symmetrization

G. Salam '98

The BFKL kernel is symmetric under exchange of the external gluons



This implies a symmetry under M <--> 1- M for $\chi(\alpha,M)$ broken by two effects:

- Running coupling effects ($\alpha(Q^2)$ breaks the symmetry)
- The change of scale from the BFKL symm. scale $\xi=\ln(s/Qk)$ to the DIS scale $\xi=\ln(s/Q^2)$

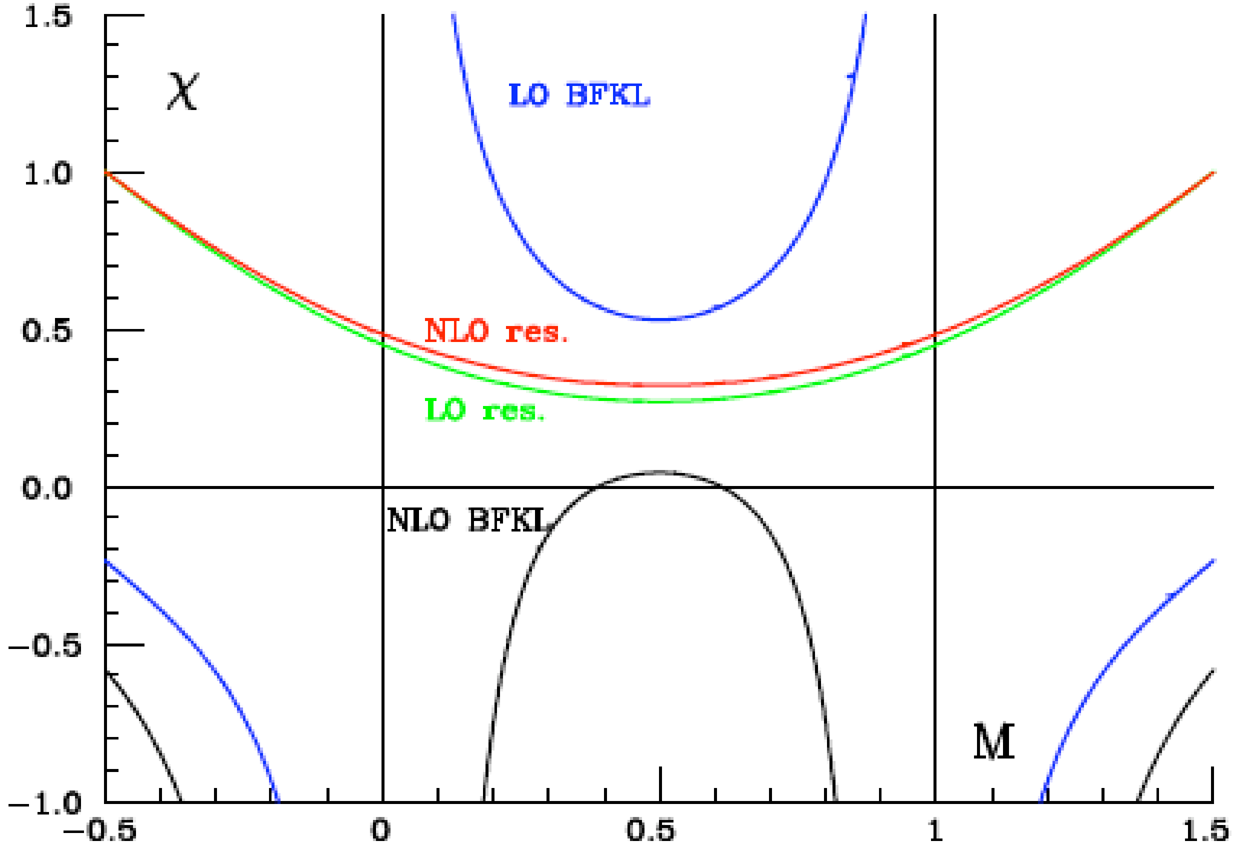
$$\chi_{DIS}\left(M+\frac{\chi_{SYMM}(M)}{2}\right)=\chi_{SYMM}(M)$$

$$\chi=\chi_{DIS}$$

Symmetrization makes χ regular at M=0 AND M=1

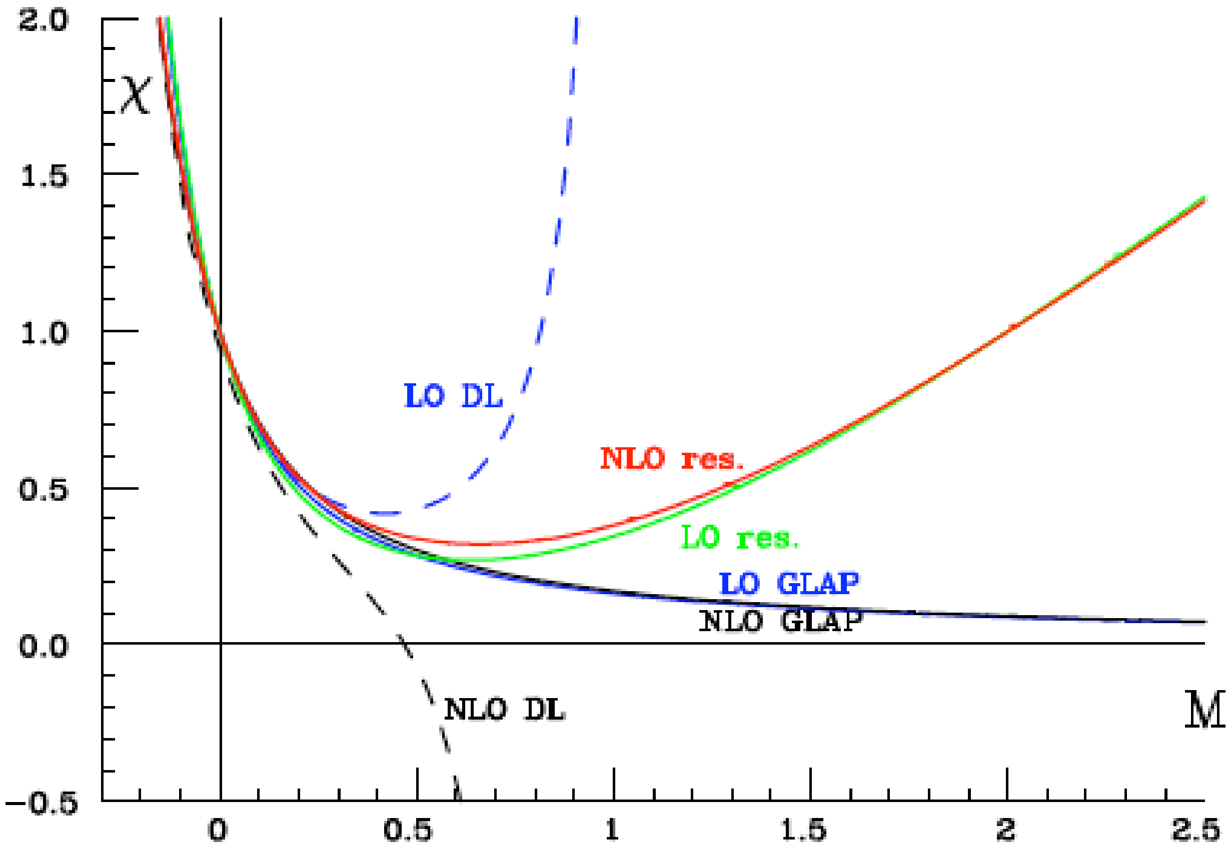
In symmetric variables:

fixed coupling: α=0.2



Note how the symmetrized LO DL and NLO DL are very close!

The same now in DIS variables



All χ curves have a minimum and follow GLAP closer.
The remaining ingredient is the running of the coupling.

A considerable further improvement is obtained by including running coupling effects

Recall that the x-evolution equation was at fixed α

$$\frac{d}{d\xi}G(\xi, M)= \chi(M, \alpha)G(\xi, M)$$

The implementation of running coupling in BFKL is not simple. In fact in M-space α becomes an operator

$$\alpha(t) = \frac{\alpha}{1+\beta_0\alpha t} \Rightarrow \frac{\alpha}{1-\beta_0\alpha\frac{d}{dM}}$$

In leading approximation:

$$\frac{d}{d\xi}G(\xi, M)= \chi(M, \alpha)G(\xi, M)$$

$$\frac{d}{d\xi}G(\xi, M)= \frac{\alpha}{1-\beta_0\alpha\frac{d}{dM}}\chi_0(M)G(\xi, M)$$

By taking a second MT the equation can be written as [F(M) is a boundary condition]

$$\left(1-\beta_0\alpha\frac{d}{dM}\right)NG(N,M)+F(M) = \alpha\chi_0(M)G(N,M)$$

It can be solved iteratively

$$G(N,M) = \frac{F(M)}{N-\alpha\chi_0(M)}+\frac{\alpha\beta_0}{N-\alpha\chi_0(M)}\frac{d}{dM}\frac{F(M)}{N-\alpha\chi_0(M)}+\ldots$$

or in closed form:

$$G(N,M) = H(N,M)+$$

$$+\int_{M_0}^{M}dM'\exp\left[\frac{M-M'}{\beta_0\alpha}-\frac{1}{\beta_0 N}\int_{M'}^{M}\chi_0(M'')dM''\right]\frac{F(M')}{\beta_0\alpha N}$$

H(N,M) is a homogeneous eq. sol. that vanishes faster than all pert. terms and can be dropped.

The small x behaviour is controlled by the minimum of $\chi(M)$

We make a quadratic expansion of $\chi(M)$ near the minimum.

$$\chi_q(\hat{\alpha}_s, M) = [c(\hat{\alpha}_s) + \frac{1}{2}\kappa(\hat{\alpha}_s)(M - M_s)^2]$$

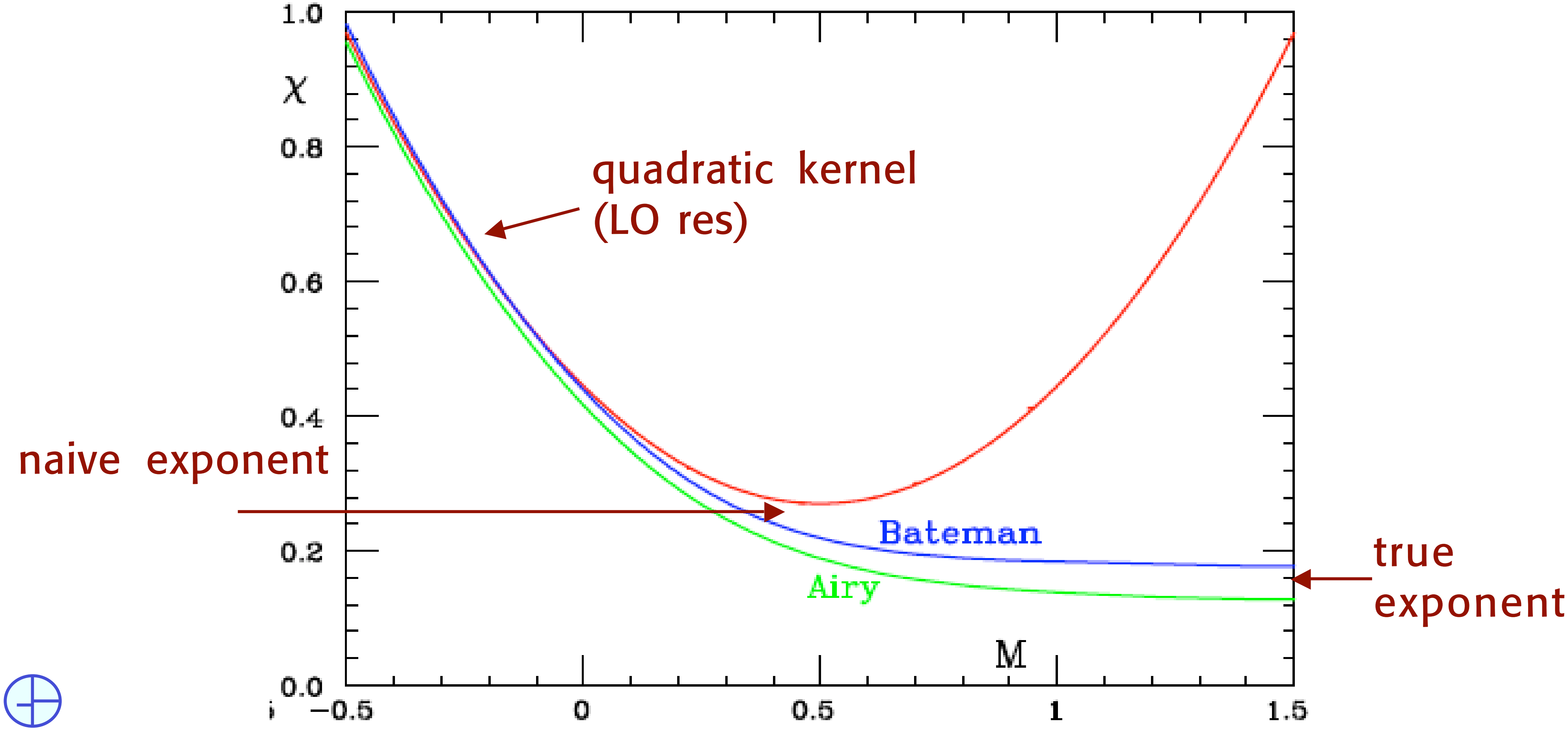
We can solve the equation exactly:

For c, k proportional to α : the solution is an Airy function
For example, if we take $\chi(\alpha, M) \sim \alpha\, \chi_0(M)$

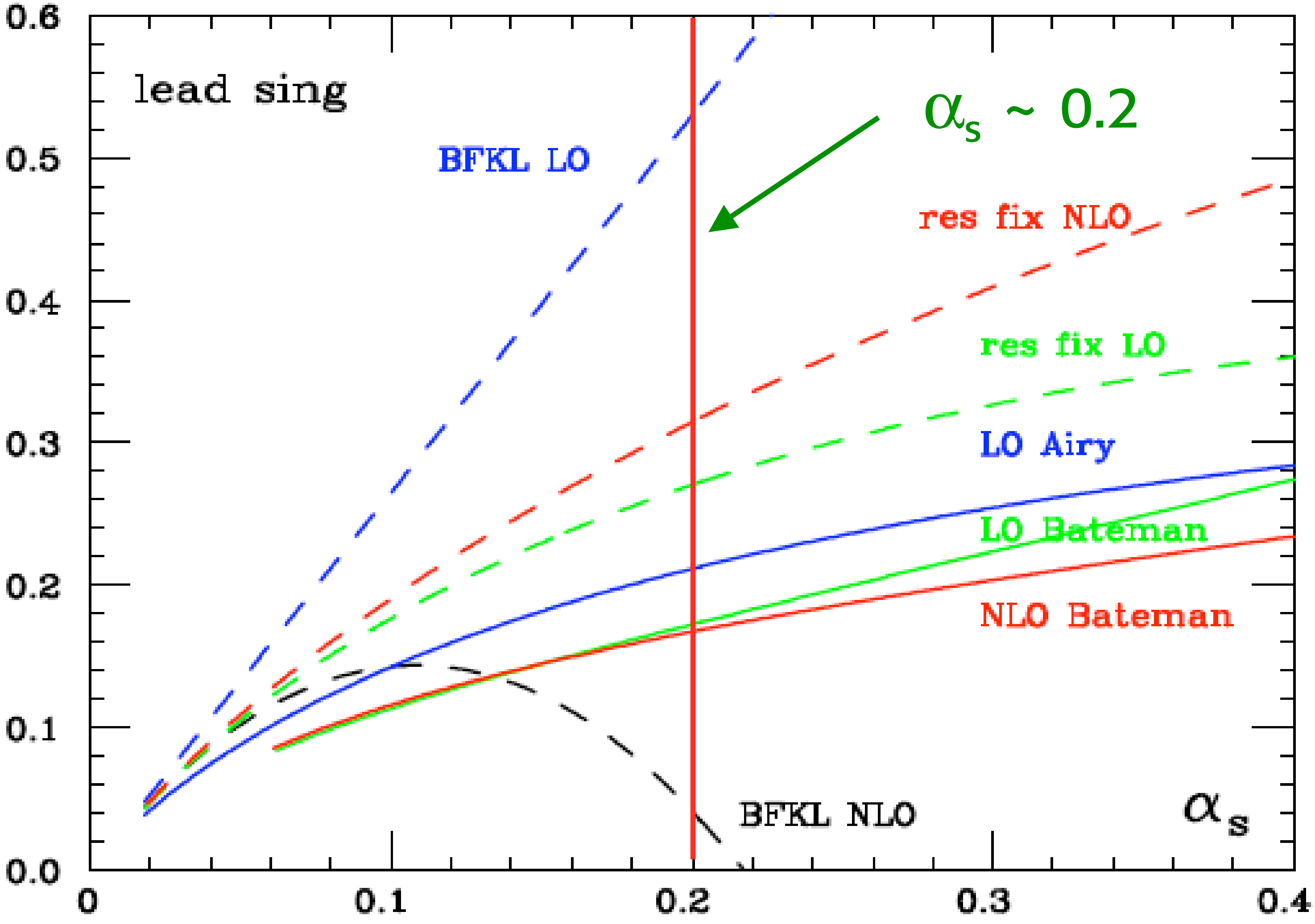
For general $c(\alpha), \kappa(\alpha)$, to the required accuracy, it is sufficient to make a linear expansion in $\hat{\alpha} - \alpha$
the solution is a Bateman function.

The asymptotic small x behaviour is considerably softened by the running!

Note that the running effect is not replacing $\alpha \to \alpha(Q^2)$ in the naive exponent

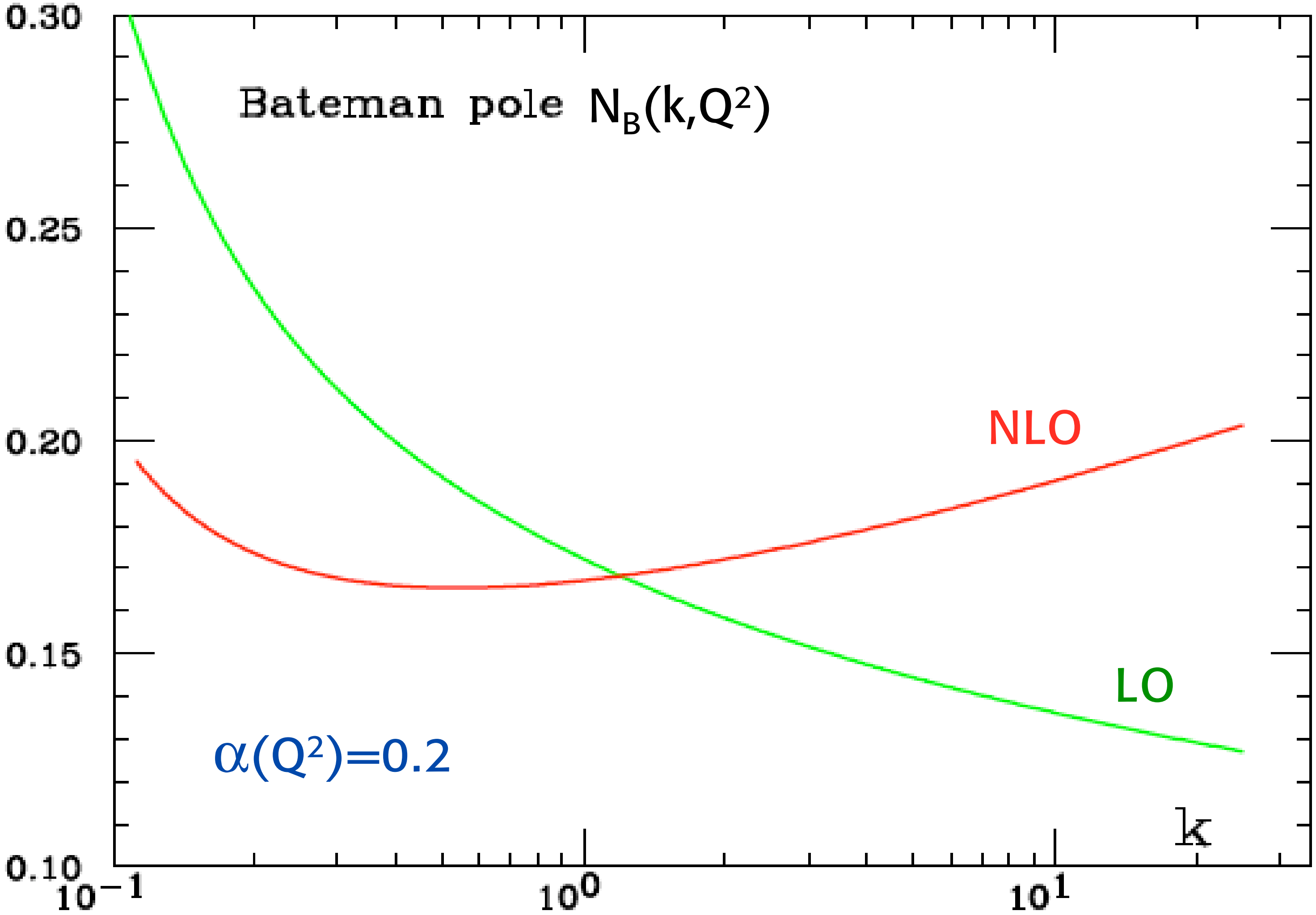


DL resummation with symmetrization and running coupling effects progressively soften the small x behaviour

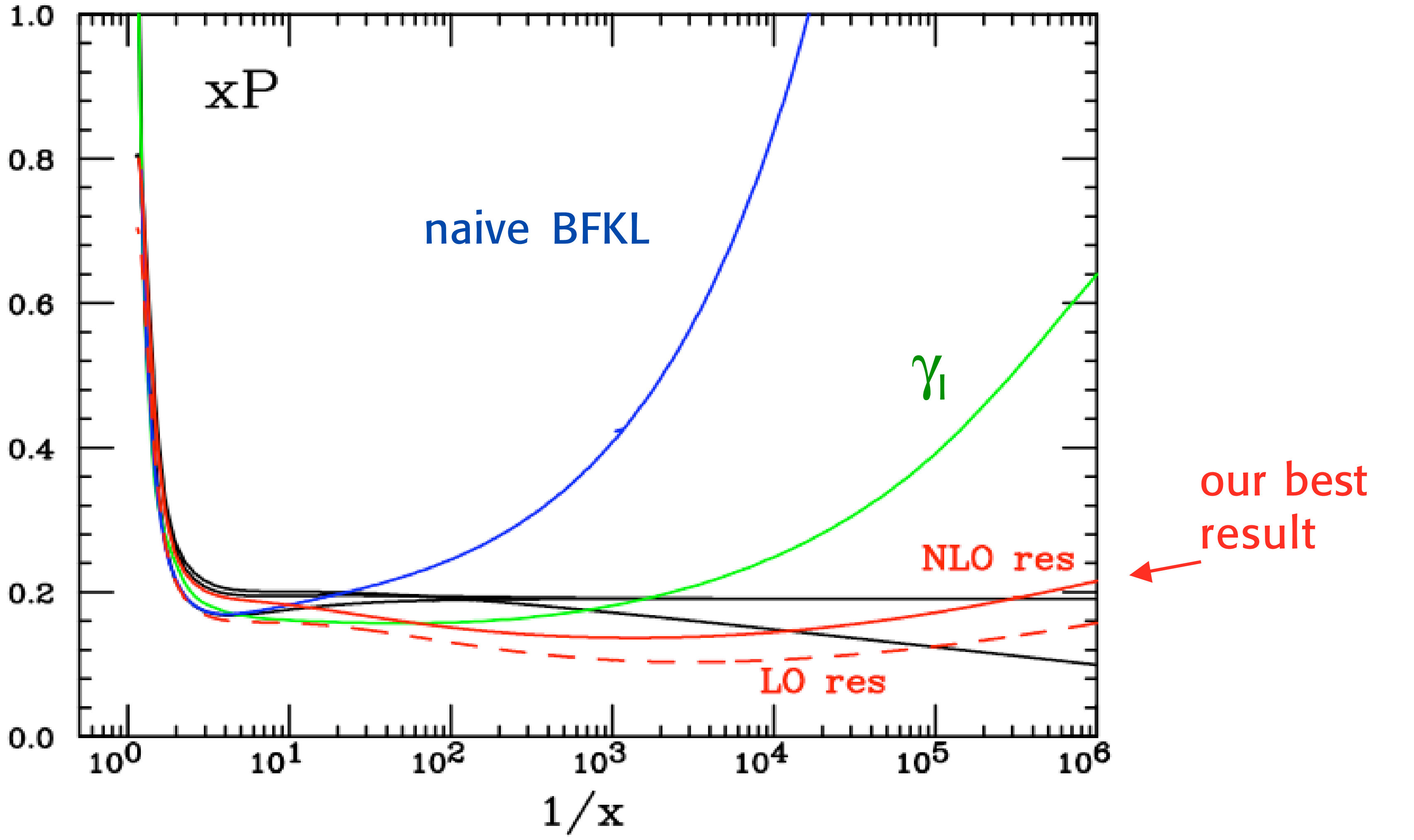


The scale dependence of the leading exponent at small x is reduced at NLO

$$N_B(k,Q^2) = N_B(\alpha_s(kQ^2)) + \beta_0 \ln k\, \alpha_s^2(Q^2) \frac{\partial}{\partial \alpha_s} N_B(\alpha_s(Q^2))$$

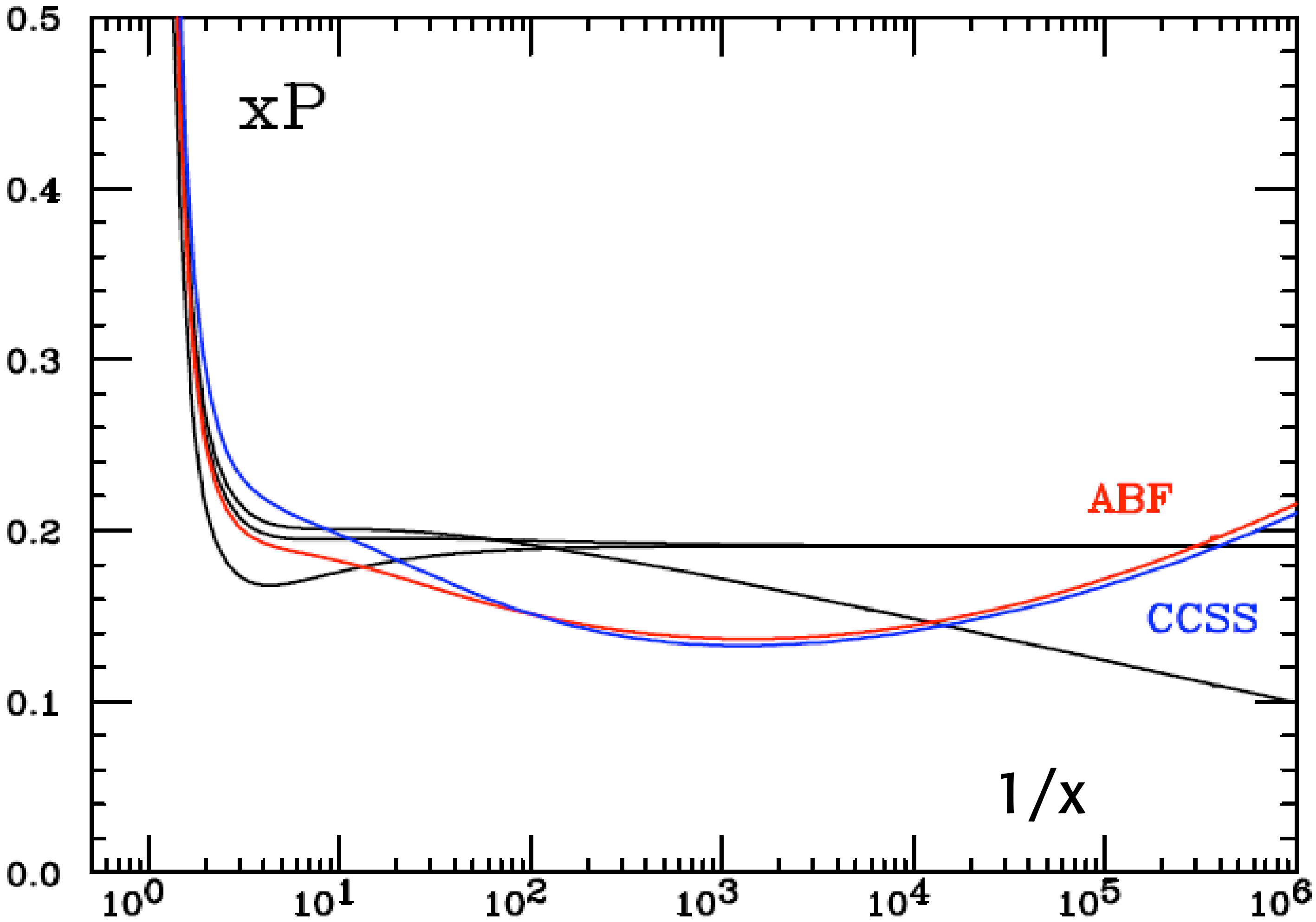


Here are the complete results using the DL resummation, symmetry and running coupling effects at LO and NLO



The comparison with Ciafaloni et al (CCSS) is simply too good not to be in part accidental (given the theory ambiguities in each method)

The main diff. with CCSS is that they solve numerically the running coupling eqn. (no quadratic expansion near minimum). They do not include NLO GLAP



Part 2

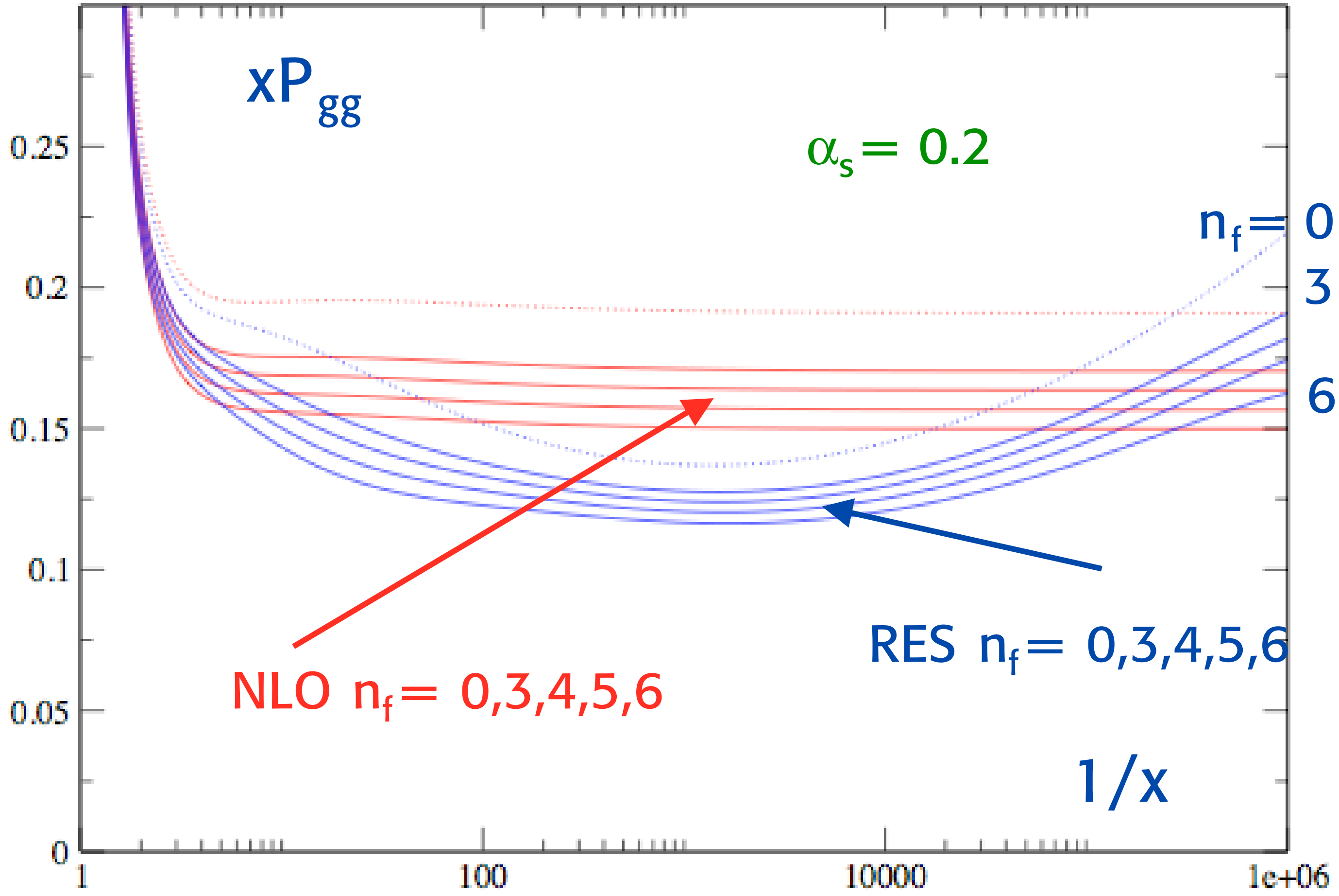
New results

Paper nearly completed

The previous curves are for $n_f = 0$

At finite n_f the diagonalization of xP is more complicated

The n_f dependence is not negligible



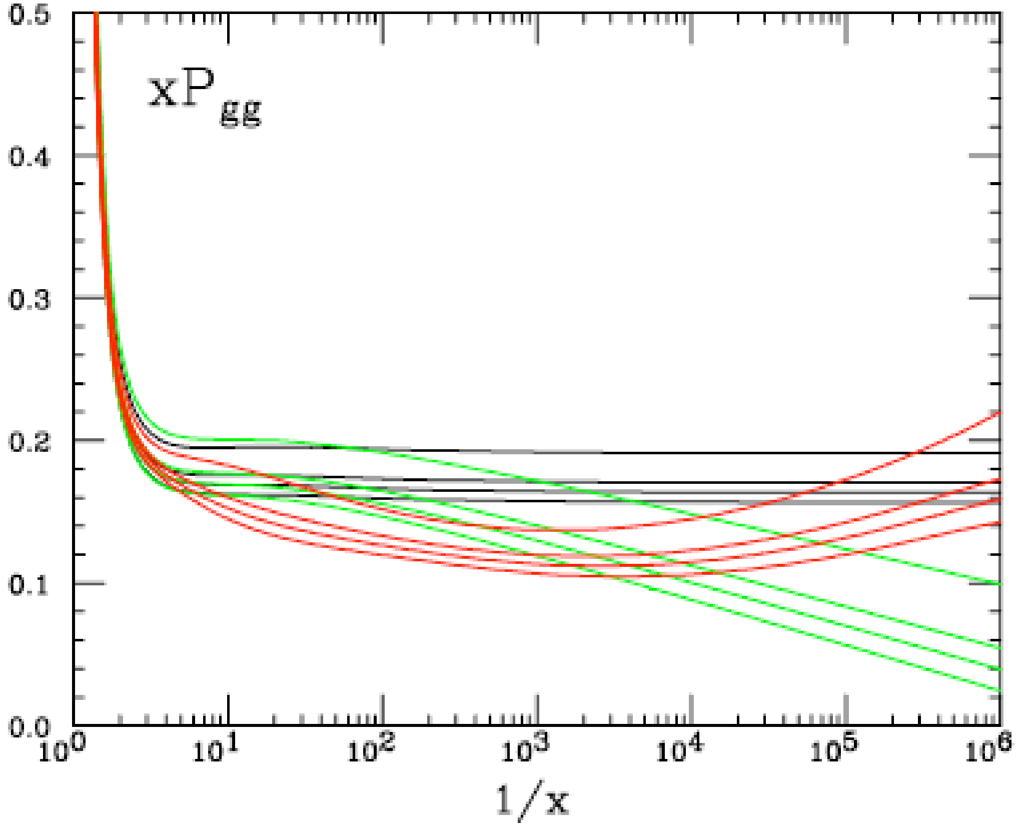
Prel. presented at the HERA-LHC Workshop, DESY, March '07

$n_f \neq 0$: THE GLUON SECTOR

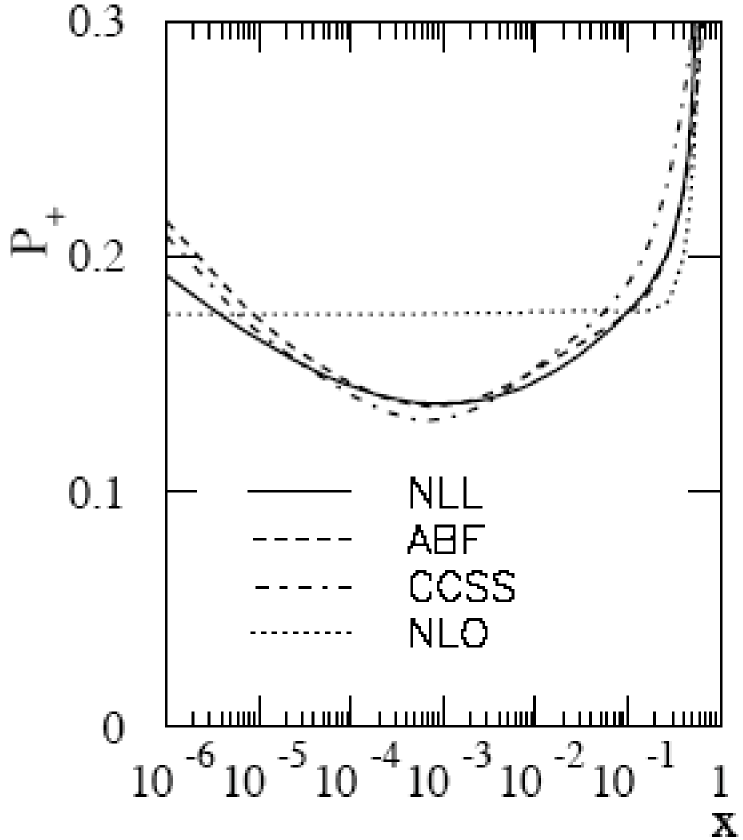
CAN COMPARE WITH THORNE & WHITE: AGREEMENT DETERIORATES AS $n_f \neq 0$

P_{gg}, $n_f = 0, 3, 4, 5$ (top to bottom)

NLO, NNLO, RESUMMED

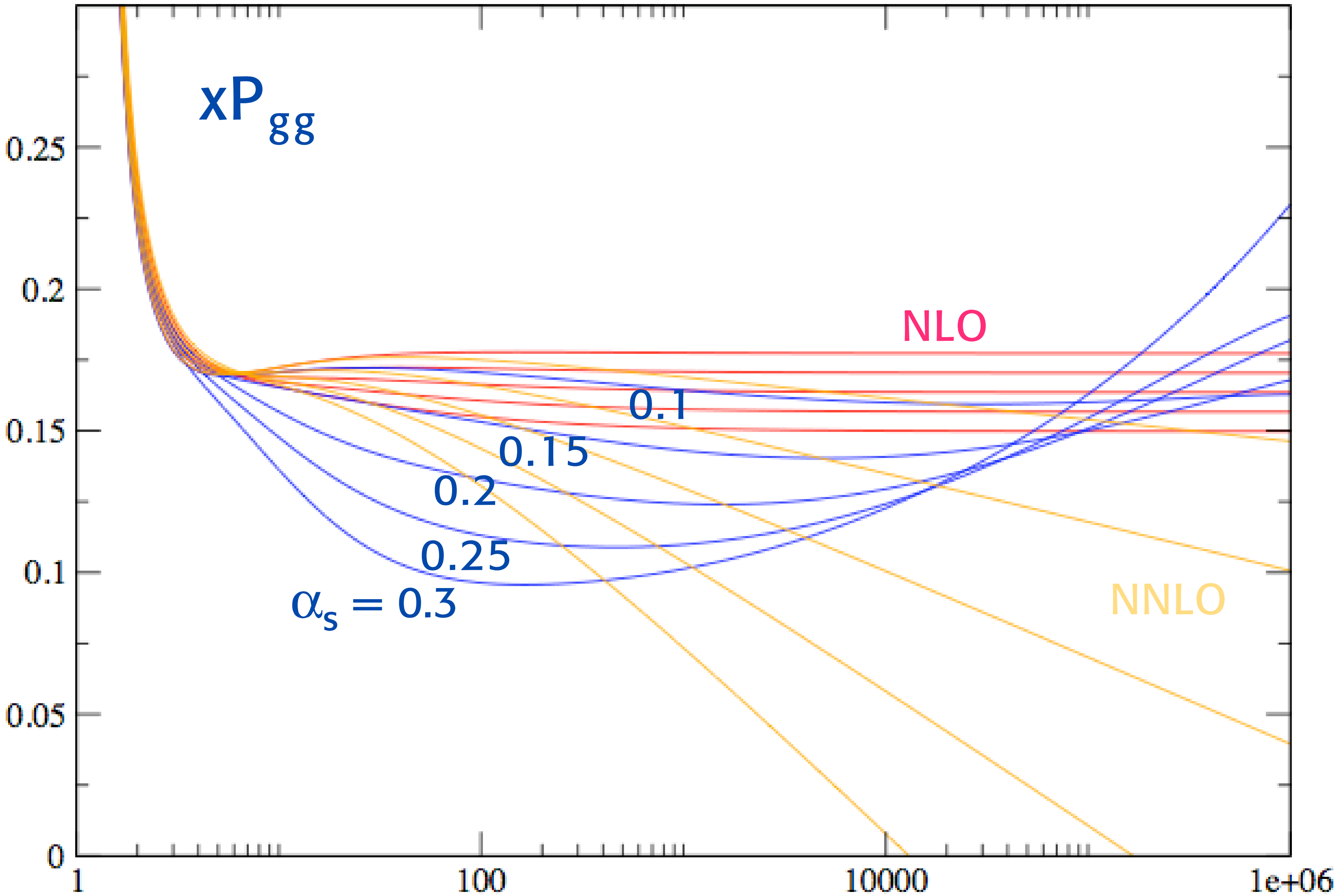


THORNE & WHITE



Forte: HERA-LHC Workshop '07

α_s dependence $(n_f = 4)$

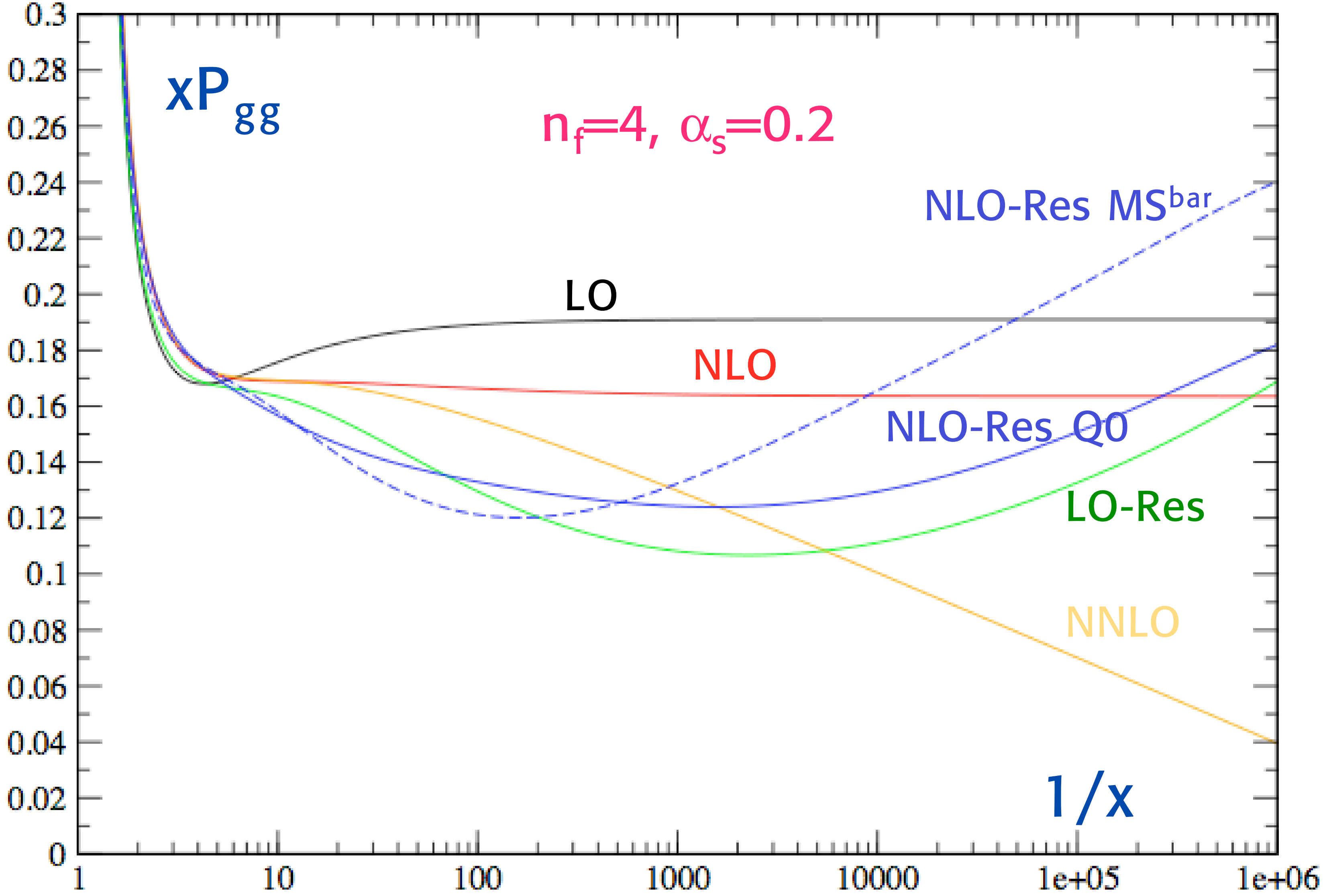


The xP_{ab} shown so far are in the Q0 scheme because in MS^{bar} there is a singularity in M=1/2 both in the splitting function and the coefficient which in Q0 is absorbed in the pdf's (in pert. theory Q0 and MSbar coincide up to and including NNLO but differ at higher orders)

Catani, Hautmann '93; Ciafaloni '95....;
Ball, Forte '99.......;
Ciafaloni, Colferai, Salam, Stasto '06

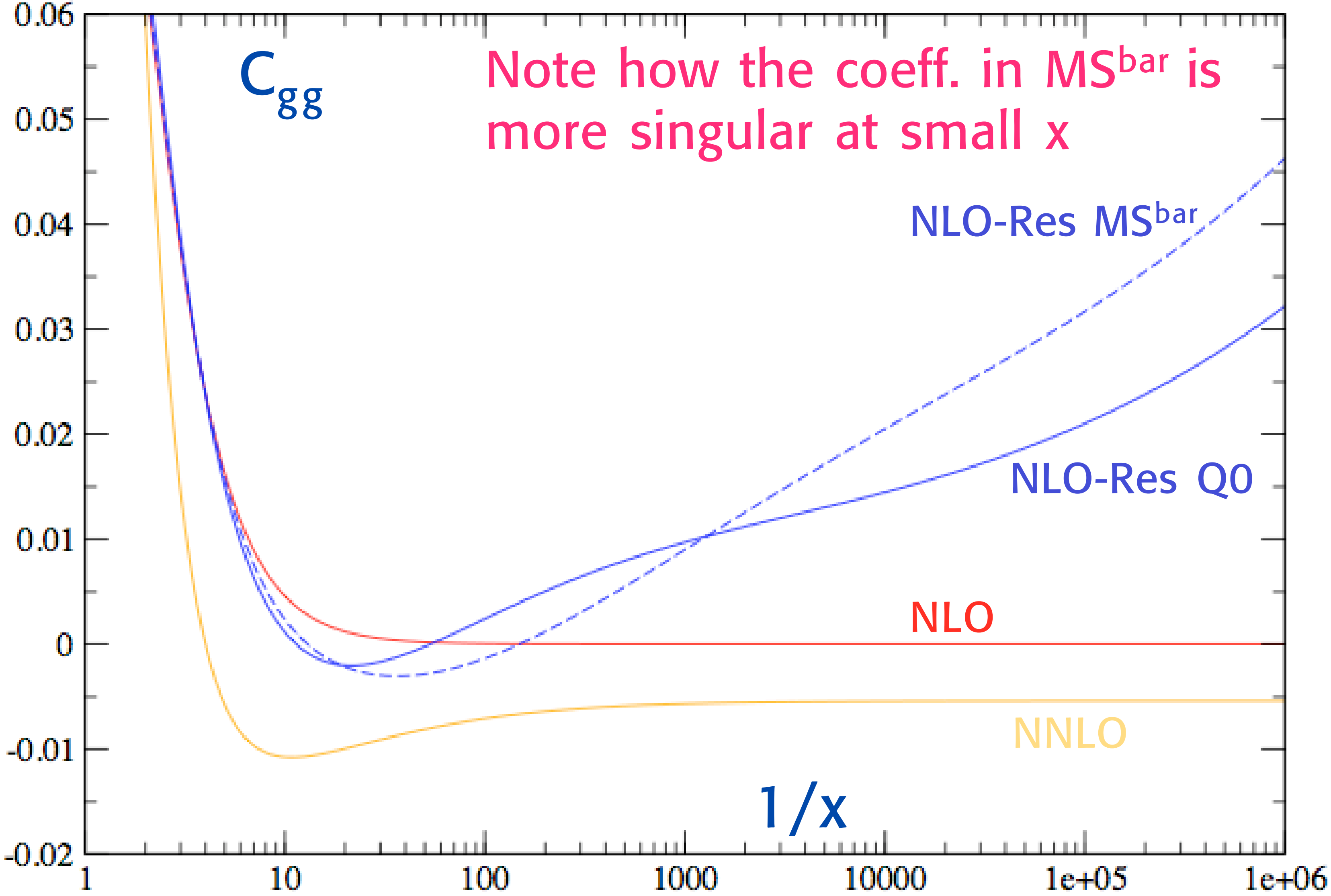
An important progress we have accomplished is the calculation of coefficients and splitting functions (by using running coupling duality) in both Q0 and MS^{bar} schemes (complete control of scheme change: could also have DIS or....)

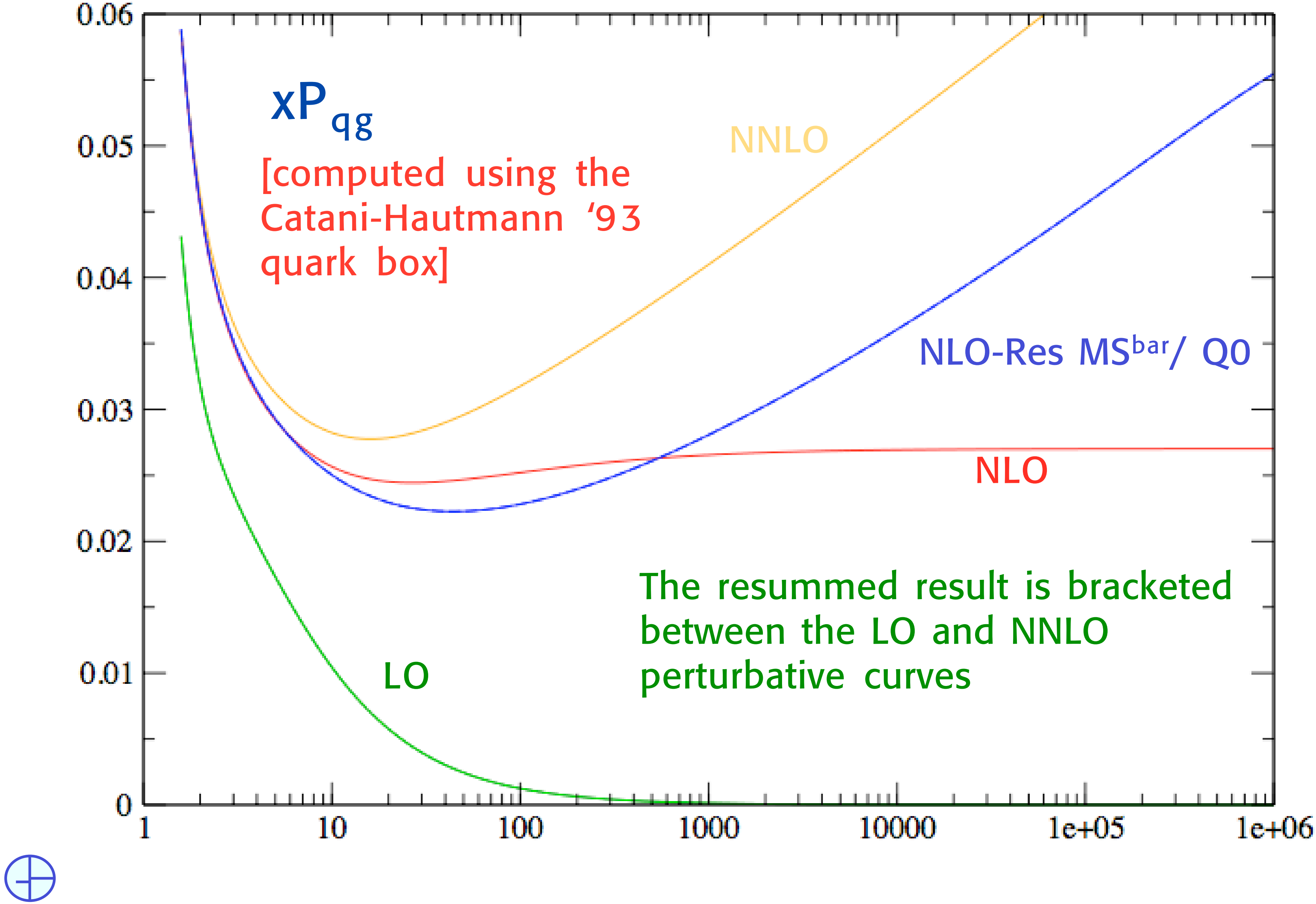
Combining splitting functions and coefficients in the same scheme is needed to obtain the evolution of pdf's and structure functions



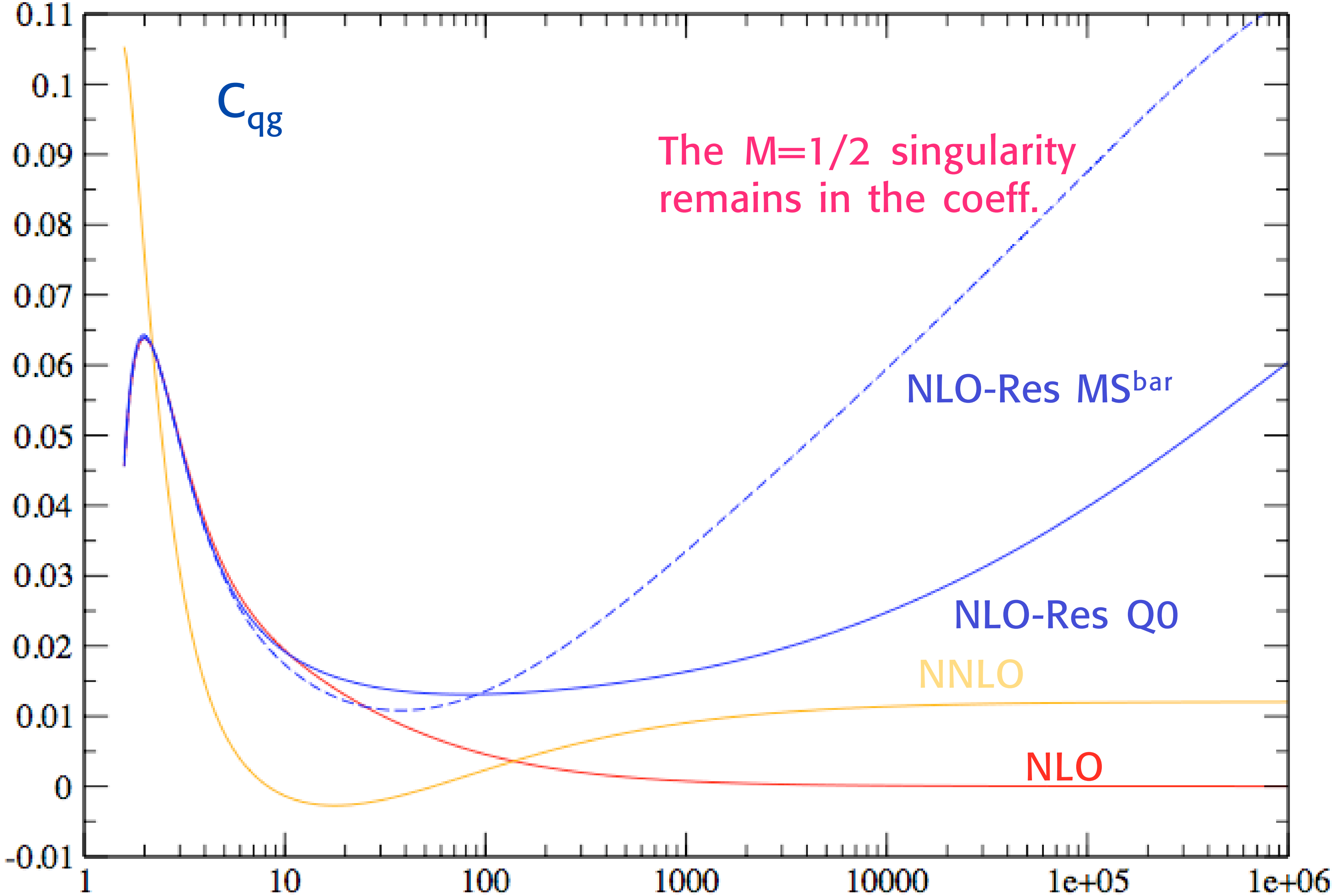
xP$_{gg}$
n_f=4, α_s=0.2
NLO-Res MS^{bar}
LO
NLO
NLO-Res Q0
LO-Res
NNLO
1/x
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1
10
100
1000
10000
1e+05
1e+06

This is the gg coefficient

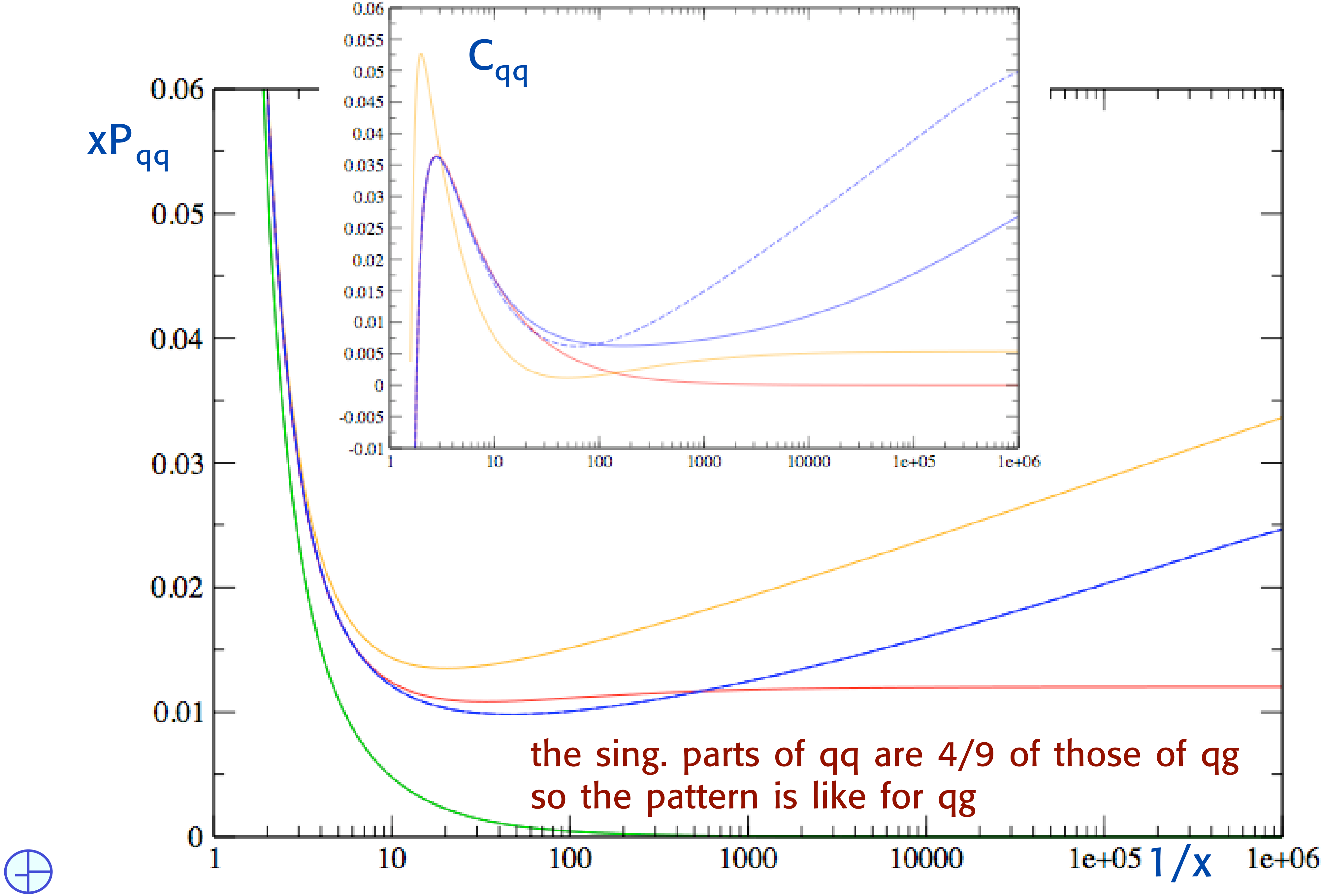




xP_qg
[computed using the
Catani-Hautmann '93
quark box]
NNLO
NLO-Res MSbar/ Q0
NLO
LO
The resummed result is bracketed
between the LO and NNLO
perturbative curves
0.06
0.05
0.04
0.03
0.02
0.01
0
1
10
100
1000
10000
1e+05
1e+06

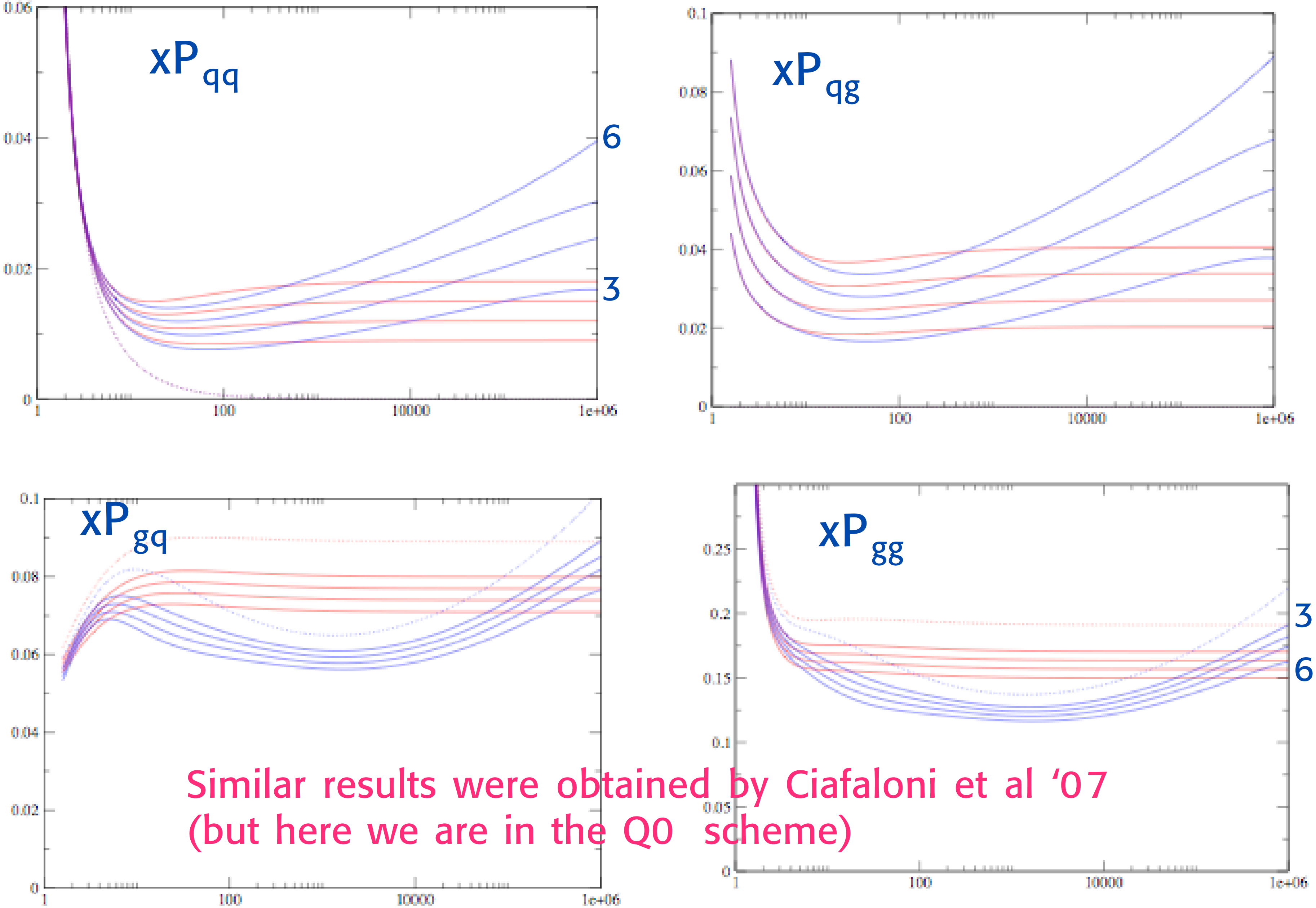


C_{qg}
The M=1/2 singularity
remains in the coeff.
NLO-Res MS^bar
NLO-Res Q0
NNLO
NLO
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
1
10
100
1000
10000
1e+05
1e+06



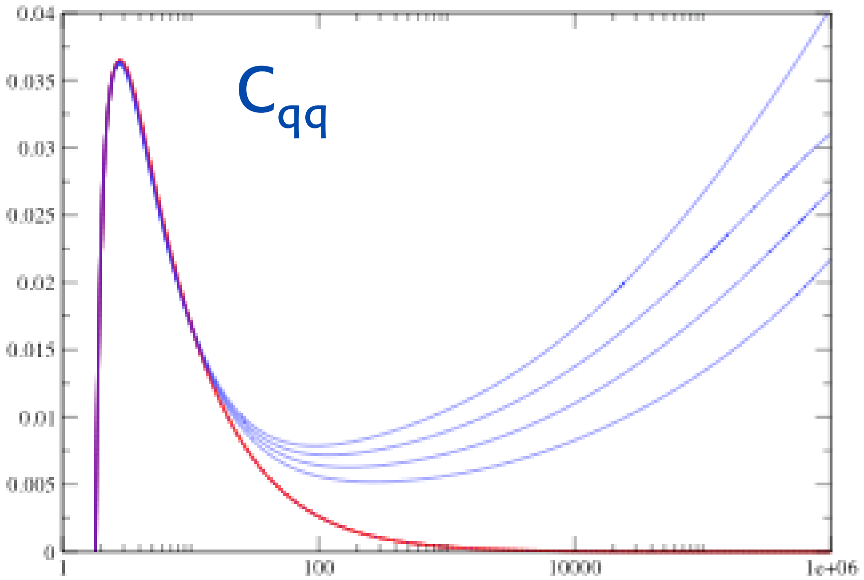
the sing. parts of qq are 4/9 of those of qg
so the pattern is like for qg

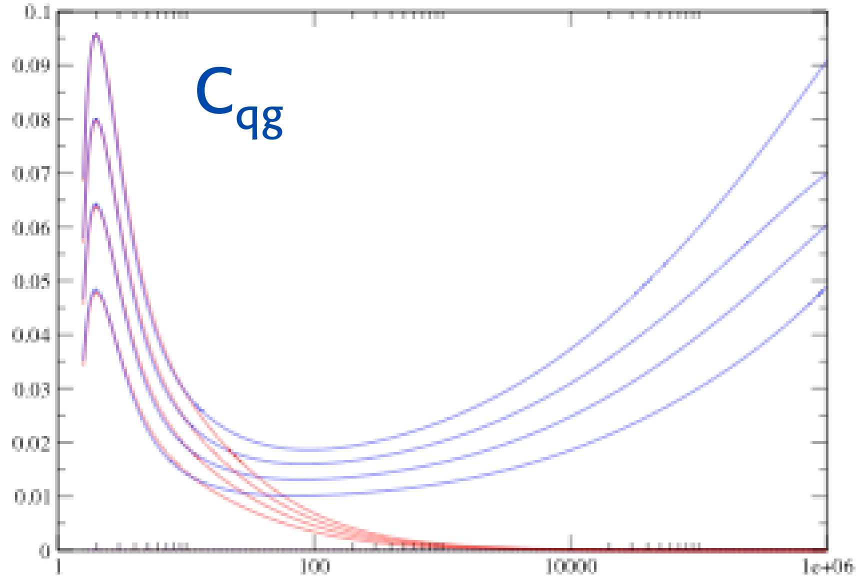
Here is the n_f dep. splitting function matrix

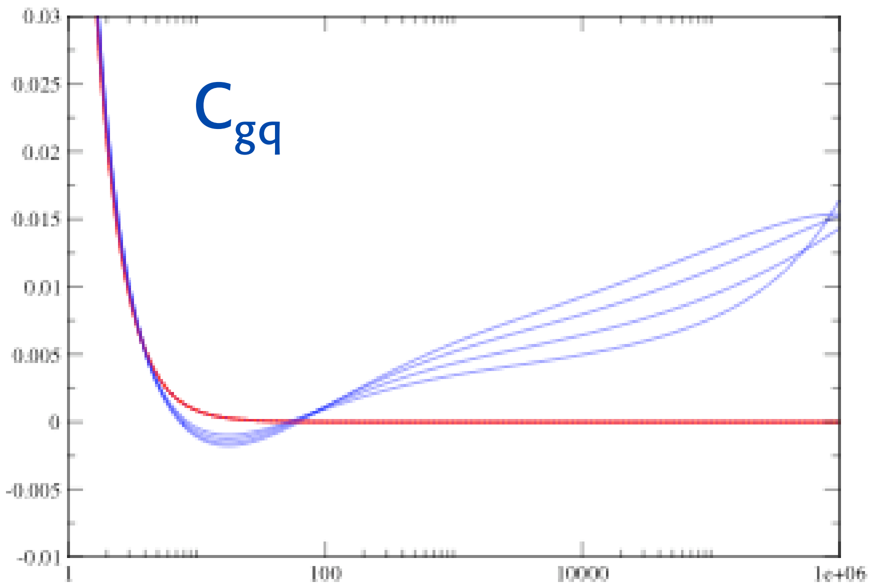


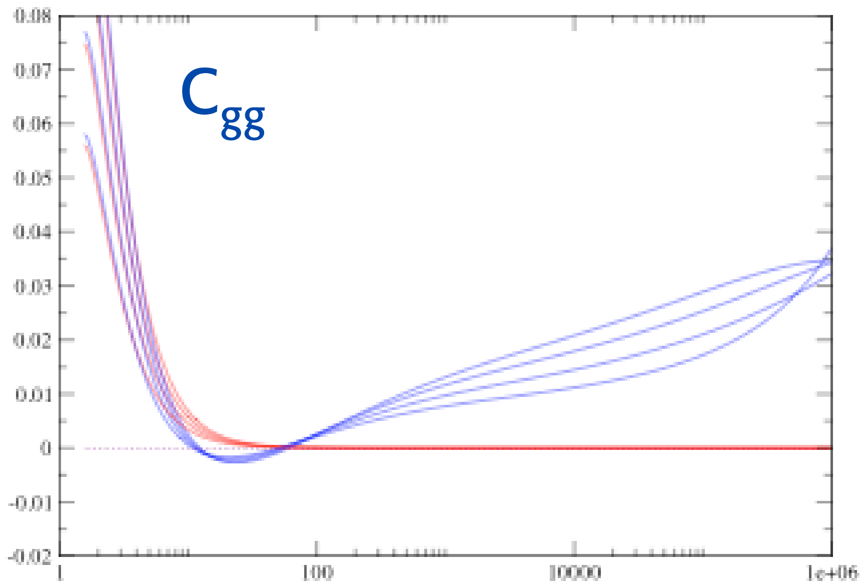
Similar results were obtained by Ciafaloni et al '07
(but here we are in the Q0 scheme)

....and the nf dep. coefficient matrix





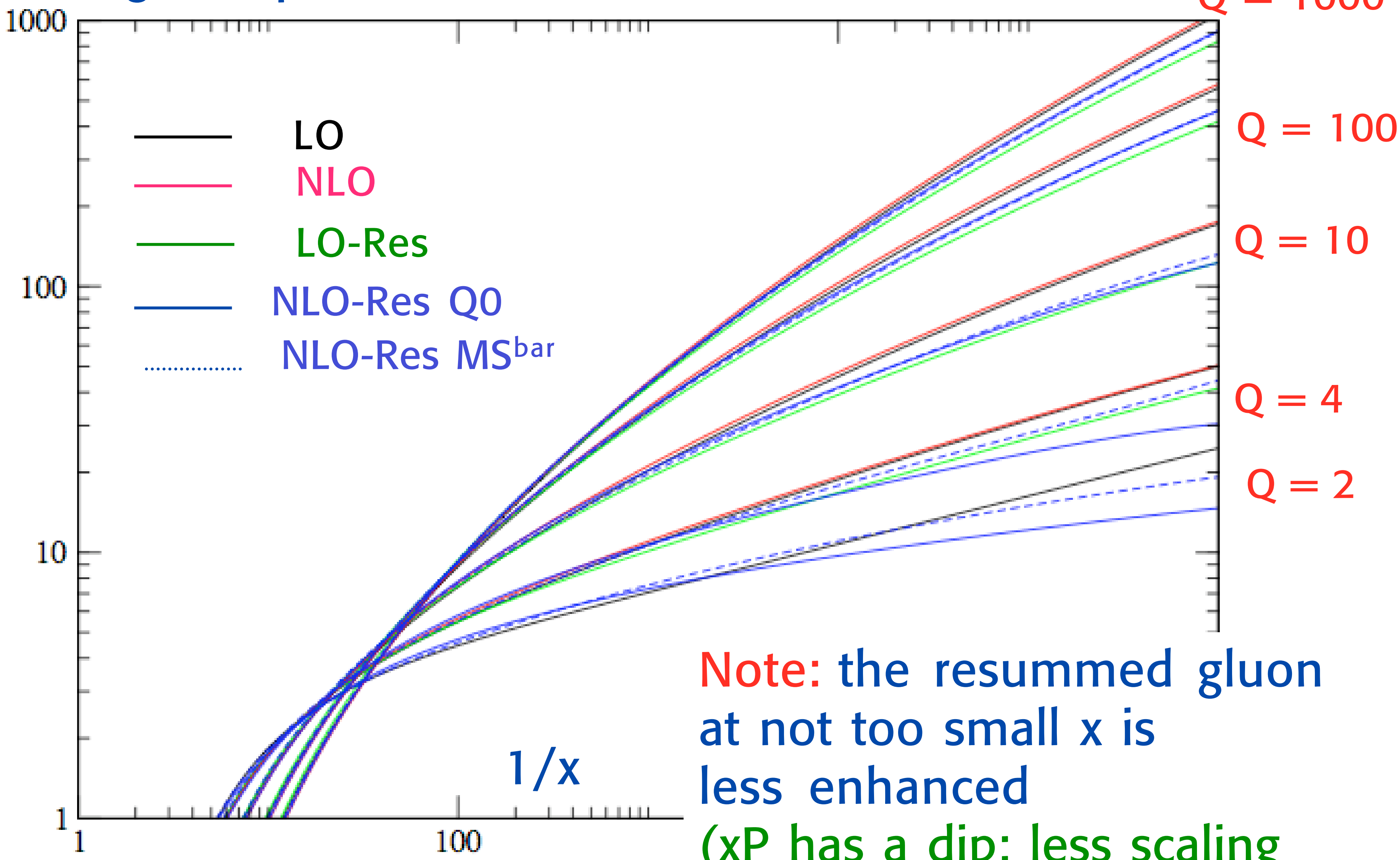




We are finally ready to applications to pdf's and structure functions

At the starting point Q=2 GeV we start with some model for valence, sea and gluon pdf's. Then, going from perturbative to resummed formulae, the pdf's are readjusted such that the initial structure functions (the physical objects!) are the same and then compare their evolution with or without resummation

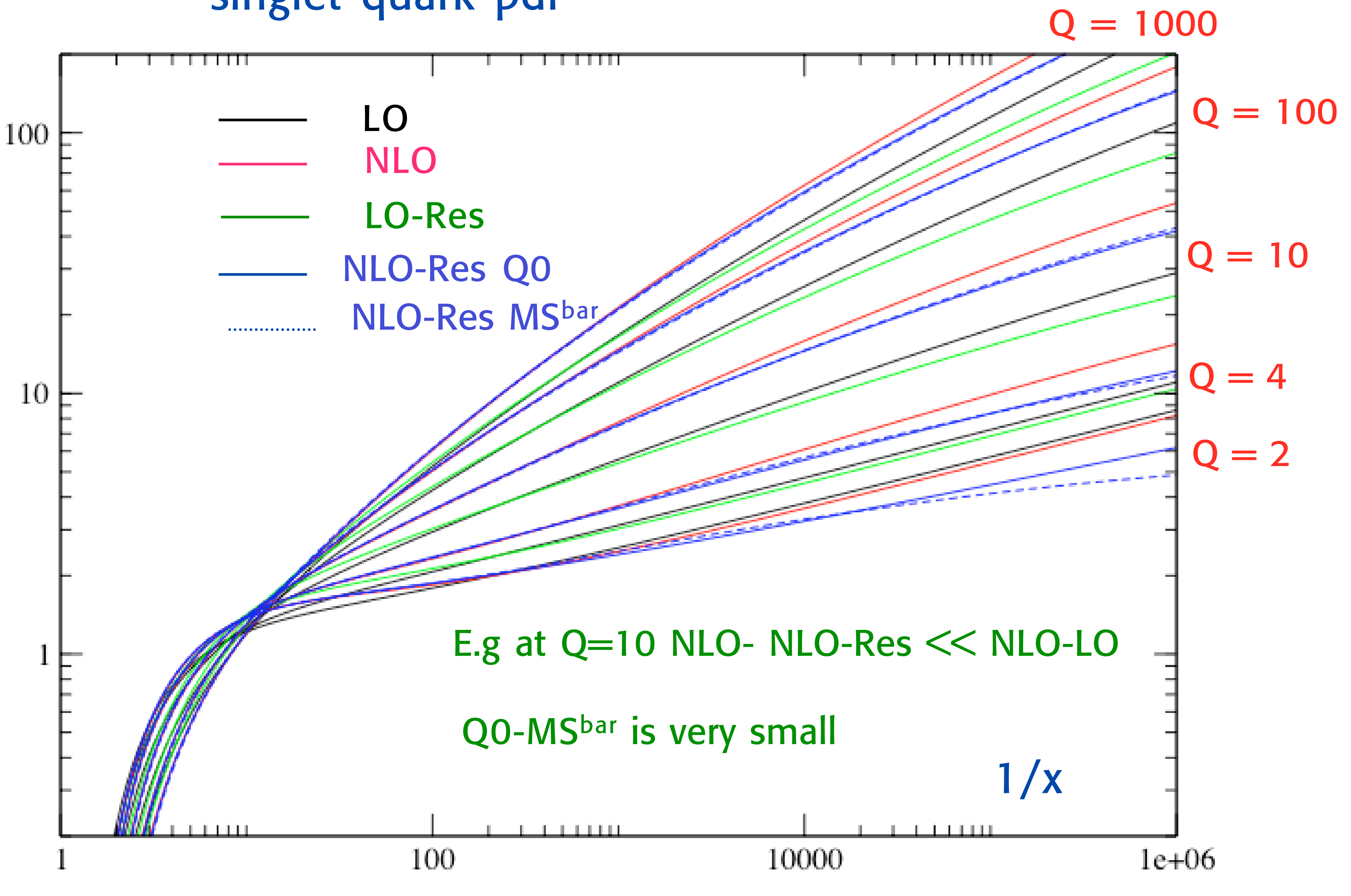
gluon pdf



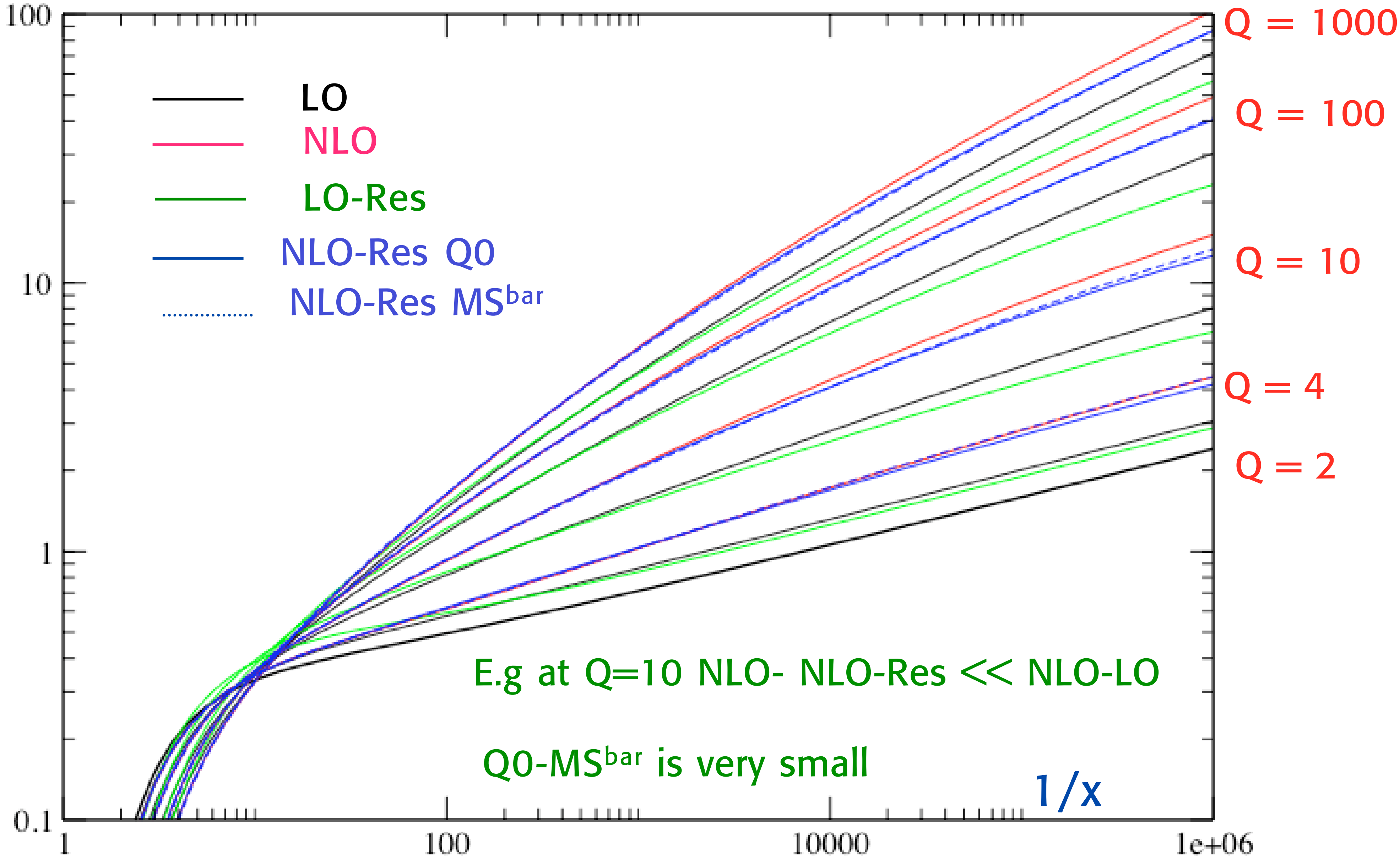
Note: the resummed gluon at not too small x is less enhanced

(xP has a dip: less scaling violations)

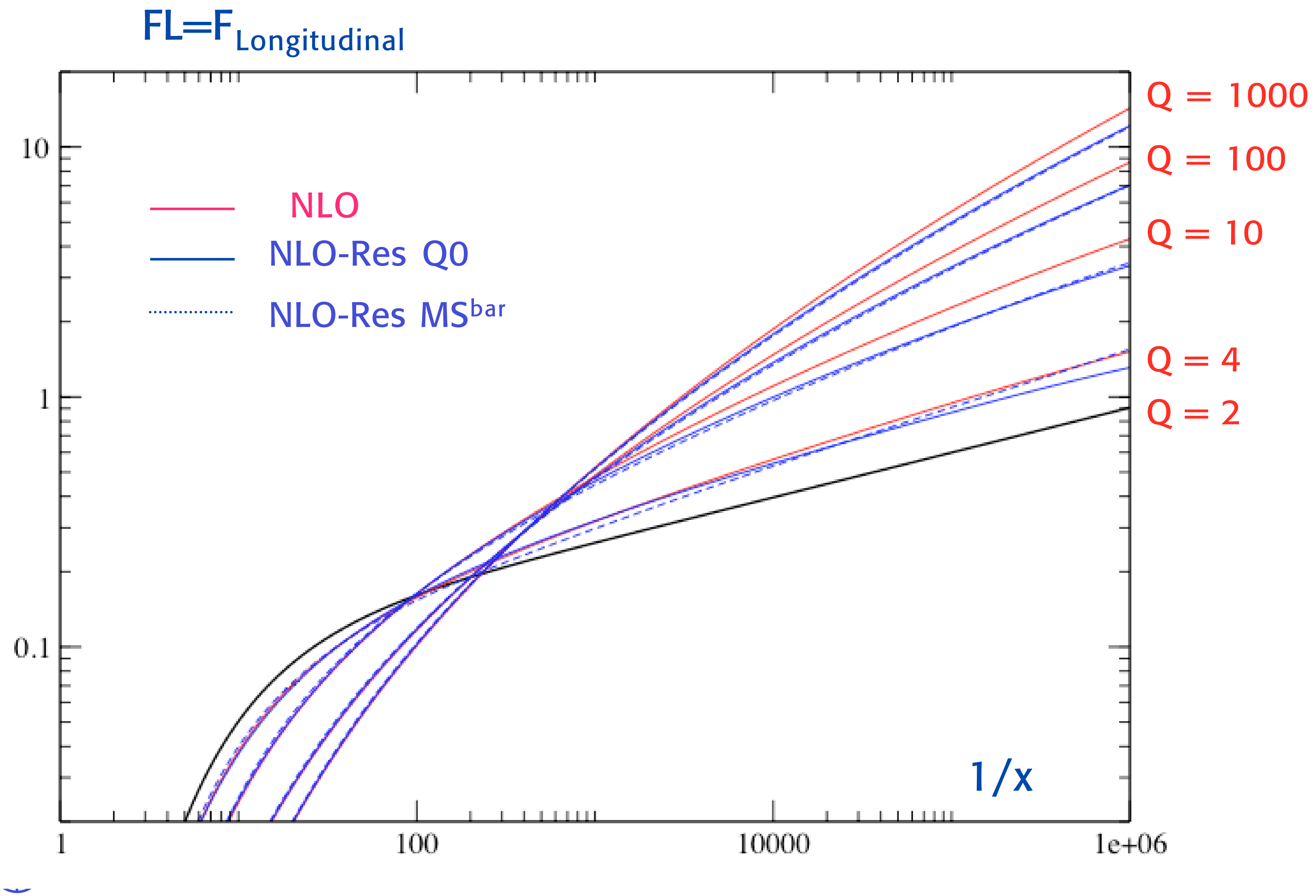
singlet quark pdf



F2 singlet

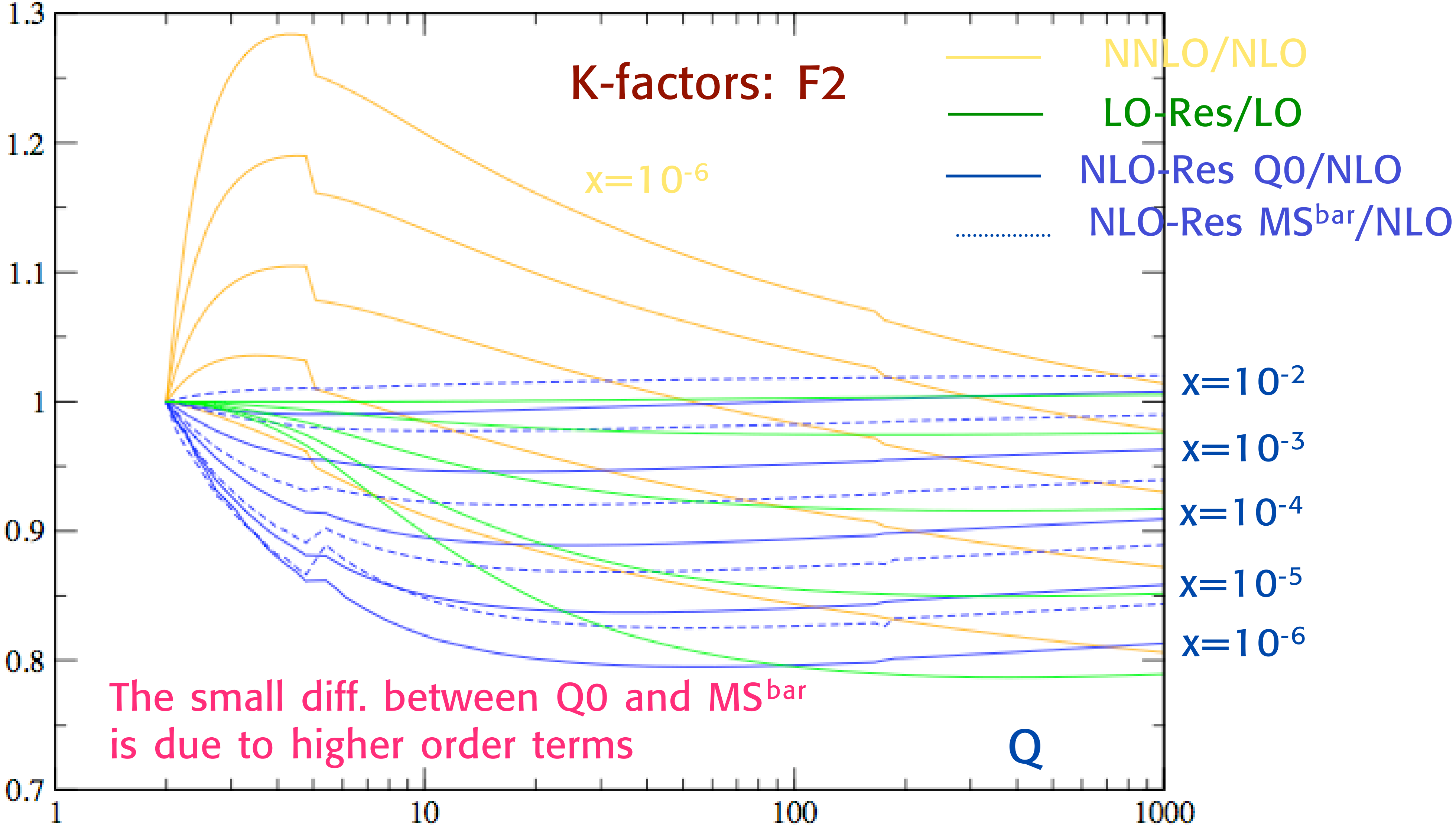


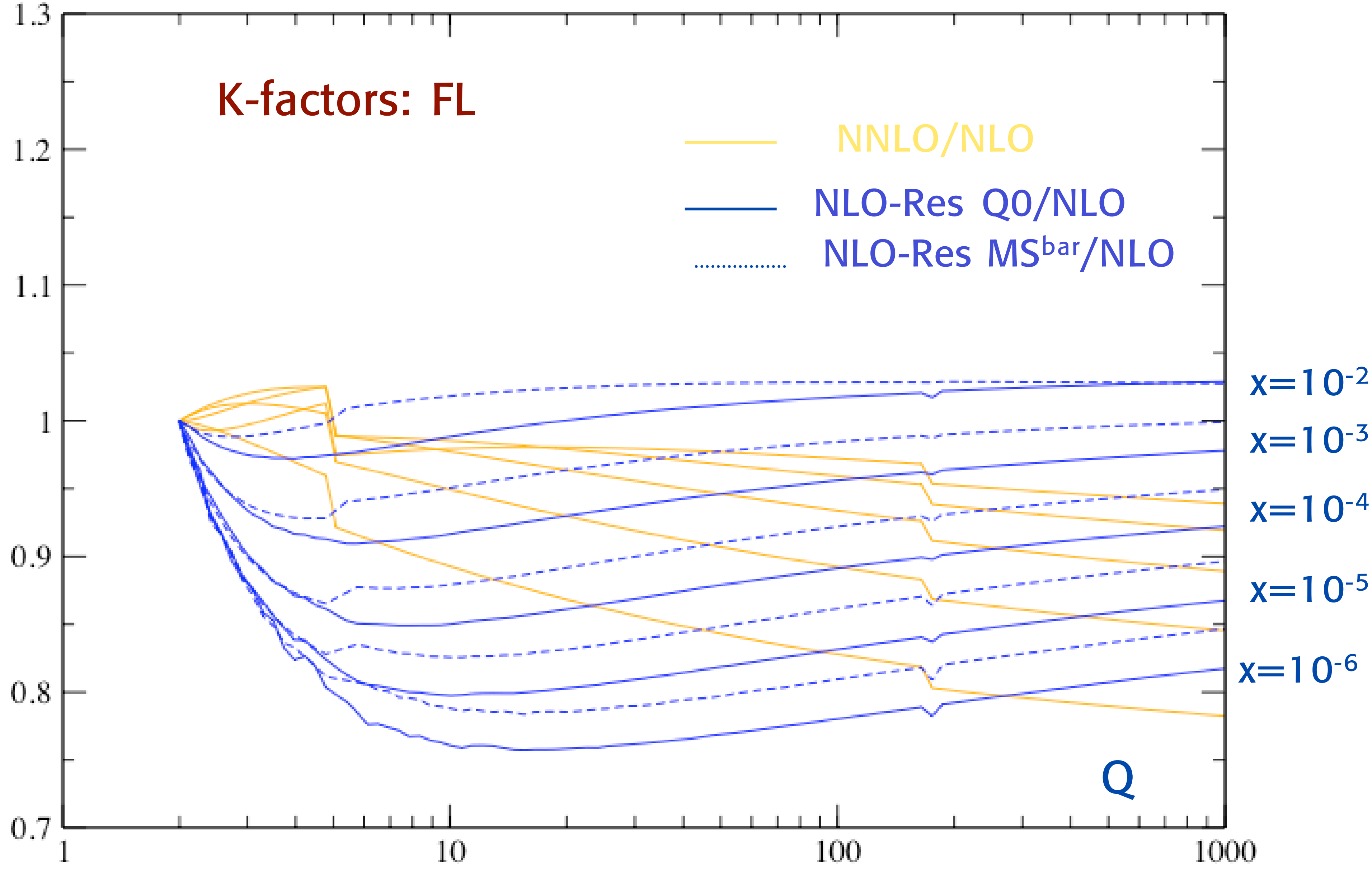
100
10
1
0.1
1
100
10000
1e+06
LO
NLO
LO-Res
NLO-Res Q0
NLO-Res MS^{bar}
Q = 1000
Q = 100
Q = 10
Q = 4
Q = 2
E.g at Q=10 NLO- NLO-Res << NLO-LO
Q0-MS^{bar} is very small
1/x


FL=F_Longitudinal
NLO
NLO-Res Q0
NLO-Res MSbar
Q = 1000
Q = 100
Q = 10
Q = 4
Q = 2
10
1
0.1
1
100
10000
1e+06
1/x

Application to physical quantities is now possible

Here the starting point is at Q= 2 GeV




K-factors: FL
NNLO/NLO
NLO-Res Q0/NLO
NLO-Res MSbar/NLO
x=10-2
x=10-3
x=10-4
x=10-5
x=10-6
Q
1.3
1.2
1.1
1
0.9
0.8
0.7
1
10
100
1000

Summary and Conclusion

- The matching of perturbative QCD evolution at large x and of BFKL at small x is now understood.
- Duality, momentum conserv., symm. under gluon exchange of the BFKL kernel and running coupling effects are essential
- The resulting asymptotic small x behaviour is much softened with respect to the naive BFKL, in agreement with the data.
- We have constructed splitting functions and coefficients that reduce to the pert. results at large x and incorporate BFKL with running coupling effects at small x.
- We have results expressed in the commonly used $\mathrm{MS}^{\mathrm{bar}}$ scheme, but can give them in any scheme.
- All formalism is ready for systematic phenomenology (e.g. at the LHC)