A Matrix Formulation for Small-$x$ RG Improved Evolution

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Some “historical” physical problems

- Reliable description of rising “hard” cross sections and structure functions at high energies
- Precise determination of parton splitting functions at small-$x$ while keeping their well known behaviour at larger-$x$;
- Providing a small-$x$ resummation in matrix form: quarks and gluons are treated on the same ground and in a collinear factorization scheme as close as possible to $\overline{\text{MS}}$. 
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Outline

- Generalizing BFKL and DGLAP evolutions
- Criteria and mechanism of matrix kernel construction
- Resummed results and partonic splitting function matrix
- Conclusions
Generalizing BFKL and DGLAP eqs

- The BFKL equation (1976) predicts rising cross-sections but
- Leading log predictions overestimate the hard Pomeron exponent, while NLL corrections are large, negative, and may make it ill-defined (Fadin, Lipatov; Camici, Ciafaloni: 1998)
- Low order DGLAP evolution is consistent with rise of HERA SF, with marginal problems (hints of negative gluon density)
- Need to reconcile BFKL and DGLAP approaches
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Collinear + small-\(x\) Resummations

- In the last decade, various (doubly) resummed approaches (CCS + CCSS; Altarelli, Ball, Forte; Thorne, White ...)
- Main idea: to incorporate RG constraints in the BFKL kernel
Output: effective (resummed) BFKL eigenvalue \(\chi_{eff}(\gamma)\) or the “dual” DGLAP anomalous dimension \(\Gamma_{eff}(\omega)\) (+ running \(\alpha_s\))
- So far, only the gluon channel is treated self-consistently; the quark channel is added by \(k\)-factorization of the \(q - \bar{q}\) dipole
The matrix approach

- Generalizes DGLAP self-consistent evolution for quarks and gluons in $k$-factorized matrix form, so as to be consistent, at small $x$, with BFKL gluon evolution.
- Defines, by construction, some unintegrated partonic densities at any $x$, and provides the resummed Hard Pomeron exponent and the Splitting Functions matrix.
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Main construction criteria for the matrix kernel

- Should incorporate exactly NLO DGLAP matrix evolution and the NL\( x \) BFKL kernel.
- Should satisfy RG constraints in both ordered and antiordered collinear regions, and thus the \( \gamma \leftrightarrow 1 - \gamma + \omega \) symmetry (below).
- Is assumed to satisfy the Minimal-pole Assumption in the \( \gamma \)- and \( \omega \)- expansions (see below).
BFKL vs. DGLAP evolution

Recall: DGLAP is evolution equation for PDF $f_a(Q^2)$ in hard scale $Q^2$ and defines the anomalous dimension matrix $\Gamma(\omega)$, with the moment index $\omega = \partial / \partial Y$ conjugated to $Y = \log 1/x$

$$\frac{\partial}{\partial t} f_a = \frac{\partial}{\partial \log Q^2} f_a = [\Gamma(\omega)]_{ab} f_b$$
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If $k$-factorization is used, DGLAP evolution of the Green’s function $G$ corresponds to either the ordered $k \gg k' \gg \ldots k_0$ or the antiordered momenta, while BFKL incorporates all possible orderings
Matrix Kernel Construction

At frozen $\alpha_s$, our RG-improved matrix kernel is expanded in the form

$$K(\bar{\alpha}_s, \gamma, \omega) = \bar{\alpha}_s K_0(\gamma, \omega) + \bar{\alpha}_s^2 K_1(\gamma, \omega)$$

and satisfies the minimal-pole assumption in the $\gamma$- and $\omega$- expansions ($\gamma = 0 \leftrightarrow$ ordered $k$’s)

$$K(\bar{\alpha}_s, \gamma, \omega) = \frac{1}{\gamma} K^{(0)}(\bar{\alpha}_s, \omega) + K^{(1)}(\bar{\alpha}_s, \omega) + O(\gamma)$$

$$= (1/\omega) \; _0K(\bar{\alpha}_s, \gamma) + _1K(\bar{\alpha}_s, \gamma) + O(\omega)$$

from which DGLAP anomalous dimension matrix $\Gamma$ and BFKL kernel $\chi$:

$$\Gamma_0 = K_0^{(0)}(\omega); \quad \Gamma_1 = K_1^{(0)}(\omega) + K_1^{(1)}(\omega) \Gamma_0(\omega); \ldots$$

$$\chi_0 = [K_0(\gamma)]_{gg}; \quad \chi_1 = [K_1(\gamma) + K_0(\gamma) \; _1K_0(\gamma)]_{gg}; \ldots$$
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$$\chi_0 = [\mathcal{K}_0(\gamma)]_{gg}; \quad \chi_1 = [\mathcal{K}_1(\gamma) + \mathcal{K}_0(\gamma) \mathcal{K}_0(\gamma)]_{gg}; \ldots$$

Such expressions used to constrain $\mathcal{K}_0$ and $\mathcal{K}_1$ iteratively to yield the known NLO/NLx evolution, and approximate momentum conservation.
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Such expressions used to constrain $K_0$ and $K_1$ iteratively to yield the known NLO/NLx evolution, and approximate momentum conservation.

RG constraints in both ordered and antiordered collinear regions are met by the $\gamma \leftrightarrow 1 + \omega - \gamma$ symmetry of the kernel.
The Matrix Kernel

\[ \mathcal{K}_0 = \begin{pmatrix}
\Gamma_{qq}^0(\omega)\chi^\omega_c(\gamma) & \Gamma_{gg}^0(\omega)\chi^\omega_c(\gamma) \\
\Gamma_{qq}^0(\omega)\chi^\omega_c(\gamma) & \left[\Gamma_{gg}^0(\omega) - \frac{1}{\omega}\right]\chi^\omega_c(\gamma) + \frac{1}{\omega}\chi^\omega_0(\gamma)
\end{pmatrix} \]
The Matrix Kernel

\[ \mathcal{K}_0 = \begin{pmatrix}
\Gamma^0_{qq}(\omega) \chi_c^\omega(\gamma) & \Gamma^0_{qg}(\omega) \chi_c^\omega(\gamma) \\
\Gamma^0_{gq}(\omega) \chi_c^\omega(\gamma) & \left[ \Gamma^0_{gg}(\omega) - \frac{1}{\omega} \right] \chi_c^\omega(\gamma) + \frac{1}{\omega} \chi_0^\omega(\gamma)
\end{pmatrix} \]

- \( \mathcal{K}_0 \) has simple poles in \( \gamma \) (in \( \chi_c^\omega \) and \( \chi_0^\omega \)) and simple poles in \( \omega \) in the gluon row.
- No \( \omega \)-poles are present in the quark row, consistently with LO DGLAP and reggeization of the quark at \( \omega = -1 \). We keep this structure also in \( \mathcal{K}_1 \).
The Matrix Kernel

\[ \mathcal{K}_0 = \begin{pmatrix} 
\Gamma_{qq}^0(\omega)\chi_c^\omega(\gamma) & \Gamma_{qg}^0(\omega)\chi_c^\omega(\gamma) + \Delta_{qg}(\gamma, \omega) \\
\Gamma_{gq}^0(\omega)\chi_c^\omega(\gamma) & \left[ \Gamma_{gg}^0(\omega) - \frac{1}{\omega} \right] \chi_c^\omega(\gamma) + \frac{1}{\omega} \chi_0^\omega(\gamma) 
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- At NLO \( \Gamma_{qq}^1 \) and \( \Gamma_{qg}^1 \) contain \( \frac{\alpha_s^2}{\omega} \). Instead of adding such terms in \( \mathcal{K}_1 \) (see above) we add a proper non-singular \( \Delta_{qg}(\gamma, \omega) \) term.
- \( \mathcal{K}_1 \) is obtained by adding NLO DGLAP matrix \( \Gamma_1 \) and NLx BFKL kernel \( \chi_1 \) (in \( \mathcal{K}_{1,gg} \)) with the subtractions due to the \( \gamma \)- and \( \omega \)- expansions explained before.
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\[ \mathcal{K}_0 = \begin{pmatrix} \Gamma_{qq}^0(\omega)\chi_c^\omega(\gamma) & \Gamma_{qg}^0(\omega)\chi_c^\omega(\gamma) + \Delta_{qg}(\gamma, \omega) \\ \Gamma_{gq}^0(\omega)\chi_c^\omega(\gamma) & [\Gamma_{gg}^0(\omega) - \frac{1}{\omega}]\chi_c^\omega(\gamma) + \frac{1}{\omega}\chi_0^\omega(\gamma) \end{pmatrix} \]

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- \( \mathcal{K}_1 \) is obtained by adding NLO DGLAP matrix \( \Gamma_1 \) and NL\( x \) BFKL kernel \( \chi_1 \) (in \( \mathcal{K}_{1,gg} \)) with the subtractions due to the \( \gamma \)- and \( \omega \)-expansions explained before.
- In \( (k, x) \) space one has the \( k \leftrightarrow k' \) and \( x \leftrightarrow x k^2 / k'^2 \) symmetry of the matrix elements and running coupling is introduced

\[ \mathcal{K}(k, k'; x) = \bar{\alpha}_s(k^2)\mathcal{K}_0(k, k'; x) + \bar{\alpha}_s^2(k^2)\mathcal{K}_1(k, k'; x) \]

(the scale \( k^2 > \equiv \max(k^2, k'^2) \) is replaced by \( (k - k')^2 \) in front of the BFKL kernel \( \chi_0^\omega \)).
Remarks

Reproducing both low order DGLAP and BFKL evolutions provides novel Consistency Relations between the matrix $k$-factorization scheme and $\overline{\text{MS}}$. They are satisfied at NLO/NLx accuracy.
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- A small violation would appear at NNLO: the simple-pole assumption in $\omega$-space implies that $[\Gamma_2]_{gg} = (C_F/C_A)[\Gamma_2]_{gg}$ at order $\alpha_s^3/\omega^2$, violated by $(n_f/N_c^2)$-suppressed terms ($\leq 0.5\%$ for $n_f \leq 6$) in $\overline{\text{MS}}$ (taken from Moch, Vermaseren, Vogt 2004).
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- Note a source of ambiguity: integrated PDF are defined at $\gamma \sim 0$, all $\omega$; but unintegrated ones are well defined by $k$-factorization around different $\omega$ values: $\omega \sim 0$ (gluon) and $\omega \sim -1$ (quark).

- We choose the NLO/NLx scheme: incorporates exact $\overline{\text{MS}}$ anomalous dimension up to NLO and high-energy NLx BFKL kernel for the gluon channel.
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- Frozen coupling results are partly analytical, running coupling splitting functions obtained by a numerical deconvolution method.
Results: Hard Pomeron Exponent

Frozen-$\alpha_s$ exponent $\omega_s(\alpha_s)$. LO/NL$x$ scheme has only $gg$ entry in $K_1$

- Modest decrease from $n_f$-dependence (running $\alpha_s$ not included)
- LO/NL$x$ scheme joins smoothly the gluon-channel limit at $n_f = 0$
Effective Eigenvalue Functions ($n_f = 4$)

There are two, frozen $\alpha_s$, resummed eigenvalue functions: $\omega = \chi_{\pm}(\alpha_s, \gamma)$

Fixed points at $\gamma = 0, 2$ and $\omega = 1 \Rightarrow$ momentum conservation in both collinear and anti-collinear limits.

New subleading eigenvalue $\chi_-$
Modest $n_f$-dependence of $\chi_+(\alpha_s, \gamma)$. NL$x$-LO scheme recovers the known gluon-channel result (in agreement with ABF) at $n_f = 0$. Level crossing of $\chi_-$ and $\chi_+$ in the $n_f = 0$ limit.
Resummed Splitting Function Matrix

$NLO^+$ scheme includes, besides NLO, also NNLO terms $\sim \alpha_s^3/\omega^2$

\[ x P_{qq}(x) \]

- $NLx-NLO$
- $NLx-NLO^+$
- NLO

\[ x P_{qg}(x) \]

- $\alpha_s=0.2$, $n_f=4$
- $0.5 < x_\mu < 2$

\[ x P_{gg}(x) \]

- scheme B ($n_f=0$)

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Resummed Splitting Function Matrix

\( NLO^+ \) scheme includes, besides NLO, also NNLO terms \( \sim \alpha_s^3/\omega^2 \)

- Infrared cutoff independence insures (matrix) collinear factorization
- At intermediate \( x \simeq 10^{-3} \) resummed \( P_{gg} \) and \( P_{gq} \) show a shallow dip
- Small-\( x \) rise of novel \( P_{gg} \) and \( P_{gq} \) delayed down to \( x \simeq 10^{-4} \)
- Scale uncertainty band (0.25<\( x_\mu^2 <4 \)) larger for the (small) \( P_{qa} \) entries
Conclusions

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- Hard Pomeron and leading eigenvalue function are stable, with modest $n_f$-dependence.
- New subleading eigenvalue is obtained
- Resummed splitting functions $P_{ga}$ show a shallow dip, small $x$ increase of $P_{qa}$ delayed to $x \simeq 10^{-4}$. Overall, gentle matching of low order with resummation
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- On the whole, it looks quite nice!