A Matrix Formulation for Small-x RG Improved Evolution

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- Some “historical” physical problems
  - Reliable description of rising “hard” cross sections and structure functions at high energies
  - Precise determination of parton splitting functions at small-x while keeping their well known behaviour at larger-x;
  - Providing a small-x resummation in matrix form: quarks and gluons are treated on the same ground and in a collinear factorization scheme as close as possible to $\overline{\mathrm{MS}}$

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- Outline
  - Generalizing BFKL and DGLAP evolutions
  - Criteria and mechanism of matrix kernel construction
  - Resummed results and partonic splitting function matrix
  - Conclusions

Generalizing BFKL and DGLAP eqs

- The BFKL equation (1976) predicts rising cross-sections but
  - Leading log predictions overestimate the hard Pomeron exponent, while NLL corrections are large, negative, and may make it ill-defined (Fadin, Lipatov; Camici, Ciafaloni: 1998)
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  - Need to reconcile BFKL and DGLAP approaches
- Collinear + small-x Resummations
  - In the last decade, various (doubly) resummed approaches (CCS + CCSS; Altarelli, Ball, Forte; Thorne, White ...)
  - Main idea: to incorporate RG constraints in the BFKL kernel Output: effective (resummed) BFKL eigenvalue $\chi_{eff}(\gamma)$ or the “dual” DGLAP anomalous dimension $\Gamma_{eff}(\omega)$ (+ running α_s)
  - So far, only the gluon channel is treated self-consistently; the quark channel is added by k-factorization of the $q-\bar{q}$ dipole

- The matrix approach
  - Generalizes DGLAP self-consistent evolution for quarks and gluons in k-factorized matrix form, so as to be consistent, at small x, with BFKL gluon evolution
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- Main construction criteria for the matrix kernel
  - Should incorporate exactly NLO DGLAP matrix evolution and the NLx BFKL kernel
  - Should satisfy RG constraints in both ordered and antiordered collinear regions, and thus the $\gamma \leftrightarrow 1 - \gamma + \omega$ symmetry (below)
  - Is assumed to satisfy the Minimal-pole Assumption in the γ- and ω- expansions (see below)

BFKL vs. DGLAP evolution

- Recall: DGLAP is evolution equation for PDF $f_a(Q^2)$in hard scale Q^2 and defines the anomalous dimension matrix $\Gamma(\omega)$, with the moment index $\omega = \partial/\partial Y$ conjugated to $Y = \log 1/x$

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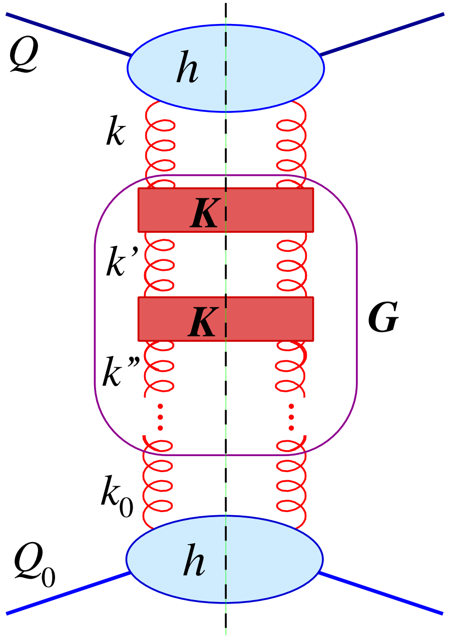
- If $\boldsymbol{k}$-factorization is used, DGLAP evolution of the Green's function G corresponds to either the ordered $k \gg k' \gg ... k_0$ or the antiordered momenta, while BFKL incorporates all possible orderings



Matrix Kernel Construction

At frozen α_s, our RG-improved matrix kernel is expanded in the form $\mathcal{K}(\bar{\alpha}_s, \gamma, \omega) = \bar{\alpha}_s \mathcal{K}_0(\gamma, \omega) + \bar{\alpha}_s^2 \mathcal{K}_1(\gamma, \omega)$ and satisfies the minimal-pole assumption in the γ- and ω- expansions ($\gamma = 0 \leftrightarrow$ ordered $\boldsymbol{k}$'s)

$$\begin{aligned}\mathcal{K}(\bar{\alpha}_s, \gamma, \omega) &= (1/\gamma)\, \mathcal{K}^{(0)}(\bar{\alpha}_s, \omega) + \mathcal{K}^{(1)}(\bar{\alpha}_s, \omega) + O(\gamma) \\ &= (1/\omega)\, {}_0\mathcal{K}(\bar{\alpha}_s, \gamma) + {}_1\mathcal{K}(\bar{\alpha}_s, \gamma) + O(\omega)\end{aligned}$$

from which DGLAP anomalous dimension matrix Γ and BFKL kernel χ:

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- Such expressions used to constrain $\mathcal{K}_0$ and $\mathcal{K}_1$ iteratively to yield the known NLO/NLx evolution, and approximate momentum conservation
- RG constraints in both ordered and antiordered collinear regions are met by the $\gamma \leftrightarrow 1 + \omega - \gamma$ symmetry of the kernel.

The Matrix Kernel

$$\mathcal{K}_0 = \begin{pmatrix} \Gamma^0_{qq}(\omega)\chi^\omega_c(\gamma) & \Gamma^0_{qg}(\omega)\chi^\omega_c(\gamma) \\ \Gamma^0_{gq}(\omega)\chi^\omega_c(\gamma) & \left[\Gamma^0_{gg}(\omega) - \frac{1}{\omega}\right]\chi^\omega_c(\gamma) + \frac{1}{\omega}\chi^\omega_0(\gamma) \end{pmatrix}$$

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- $\mathcal{K}_0$ has simple poles in γ (in χ^ω_c and χ^ω_0) and simple poles in ω in the gluon row
- No ω-poles are present in the quark row, consistently with LO DGLAP and reggeization of the quark at $\omega = -1$. We keep this structure also in $\mathcal{K}_1$

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$$\mathcal{K}_0 = \begin{pmatrix} \Gamma^0_{qq}(\omega)\chi^\omega_c(\gamma) & \Gamma^0_{qg}(\omega)\chi^\omega_c(\gamma)+\Delta_{qg}(\gamma,\omega) \\ \Gamma^0_{gq}(\omega)\chi^\omega_c(\gamma) & [\Gamma^0_{gg}(\omega)-\frac{1}{\omega}]\chi^\omega_c(\gamma)+\frac{1}{\omega}\chi^\omega_0(\gamma) \end{pmatrix}$$

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- At NLO Γ^1_{qq} and Γ^1_{qg} contain $\frac{\bar{\alpha}_s^2}{\omega}$. Instead of adding such terms in $\mathcal{K}_1$ (see above) we add a proper non-singular $\Delta_{qg}(\gamma,\omega)$ term
- $\mathcal{K}_1$ is obtained by adding NLO DGLAP matrix Γ_1 and NLx BFKL kernel χ_1 (in $\mathcal{K}_{1,gg}$) with the subtractions due to the γ- and ω- expansions explained before

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- In $(\boldsymbol{k}, x)$ space one has the $\boldsymbol{k} \leftrightarrow \boldsymbol{k}'$ and $x \leftrightarrow xk^2/k'^2$ symmetry of the matrix elements and running coupling is introduced

$$\mathcal{K}(\boldsymbol{k},\boldsymbol{k}';x) = \bar{\alpha}_s(\boldsymbol{k}^2_>)\mathcal{K}_0(\boldsymbol{k},\boldsymbol{k}';x) + \bar{\alpha}_s^2(\boldsymbol{k}^2_>)\mathcal{K}_1(\boldsymbol{k},\boldsymbol{k}';x)$$

(the scale $\boldsymbol{k}^2_> \equiv \max(\boldsymbol{k}^2, \boldsymbol{k}'^2)$ is replaced by $(\boldsymbol{k}-\boldsymbol{k}')^2$ in front of the BFKL kernel χ^ω_0)

Remarks

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- We choose the NLO/NLx scheme: incorporates exact $\overline{\mathrm{MS}}$ anomalous dimension up to NLO and high-energy NLx BFKL kernel for the gluon channel

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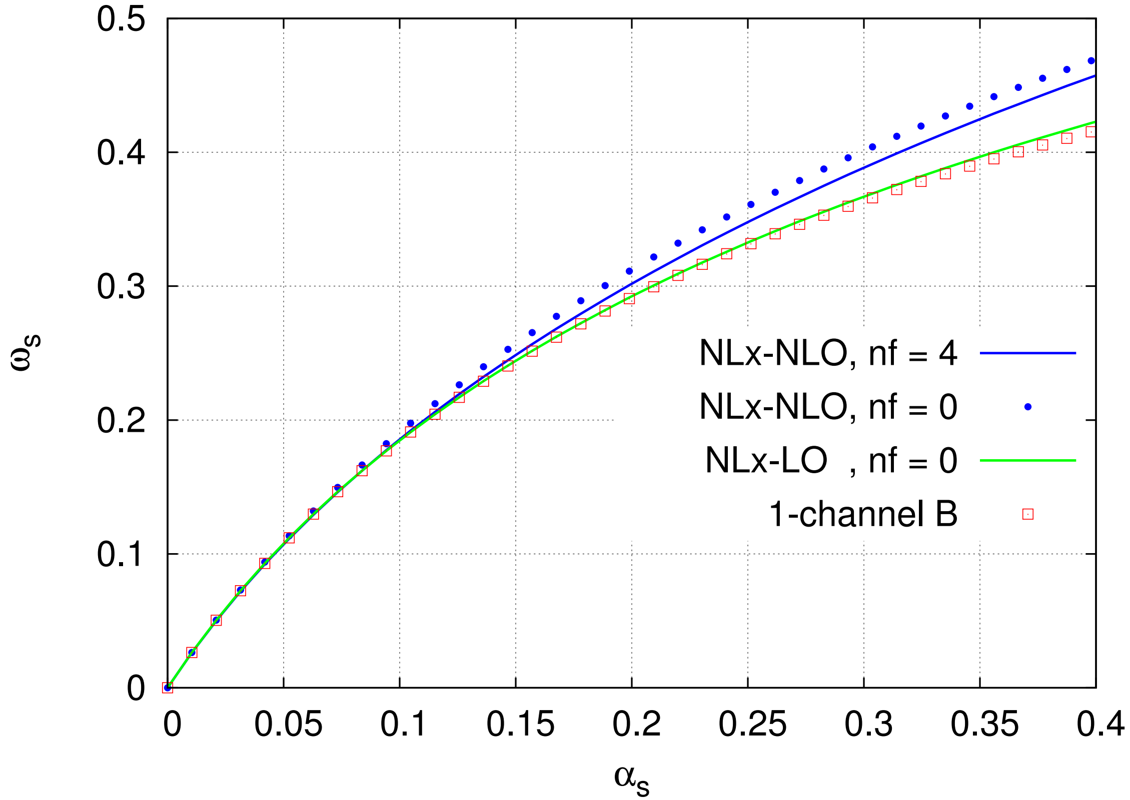
- Frozen coupling results are partly analytical, running coupling splitting functions obtained by a numerical deconvolution method.

Results: Hard Pomeron Exponent

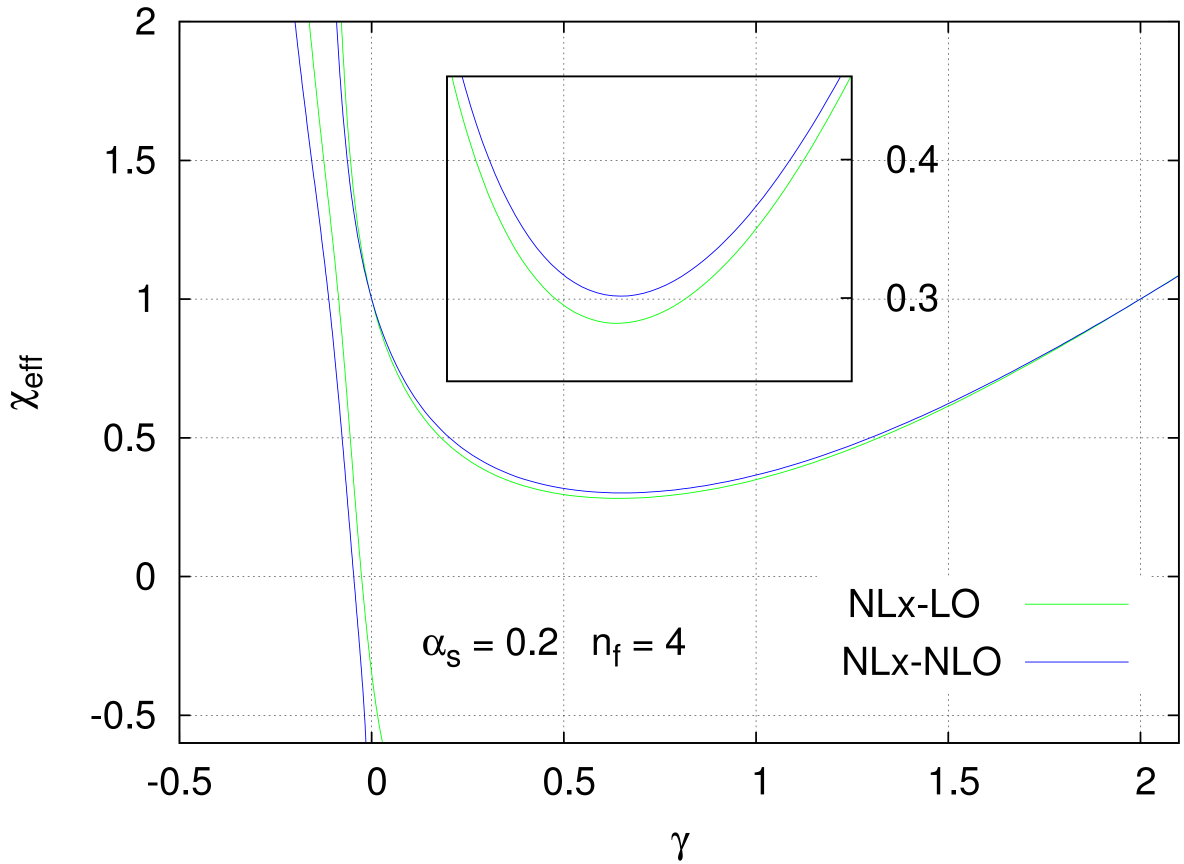
Frozen-α_s exponent $\omega_s(\alpha_s)$. LO/NLx scheme has only gg entry in $\mathcal{K}_1$



- Modest decrease from n_f-dependence (running α_s not included)
- LO/NLx scheme joins smoothly the gluon-channel limit at $n_f = 0$

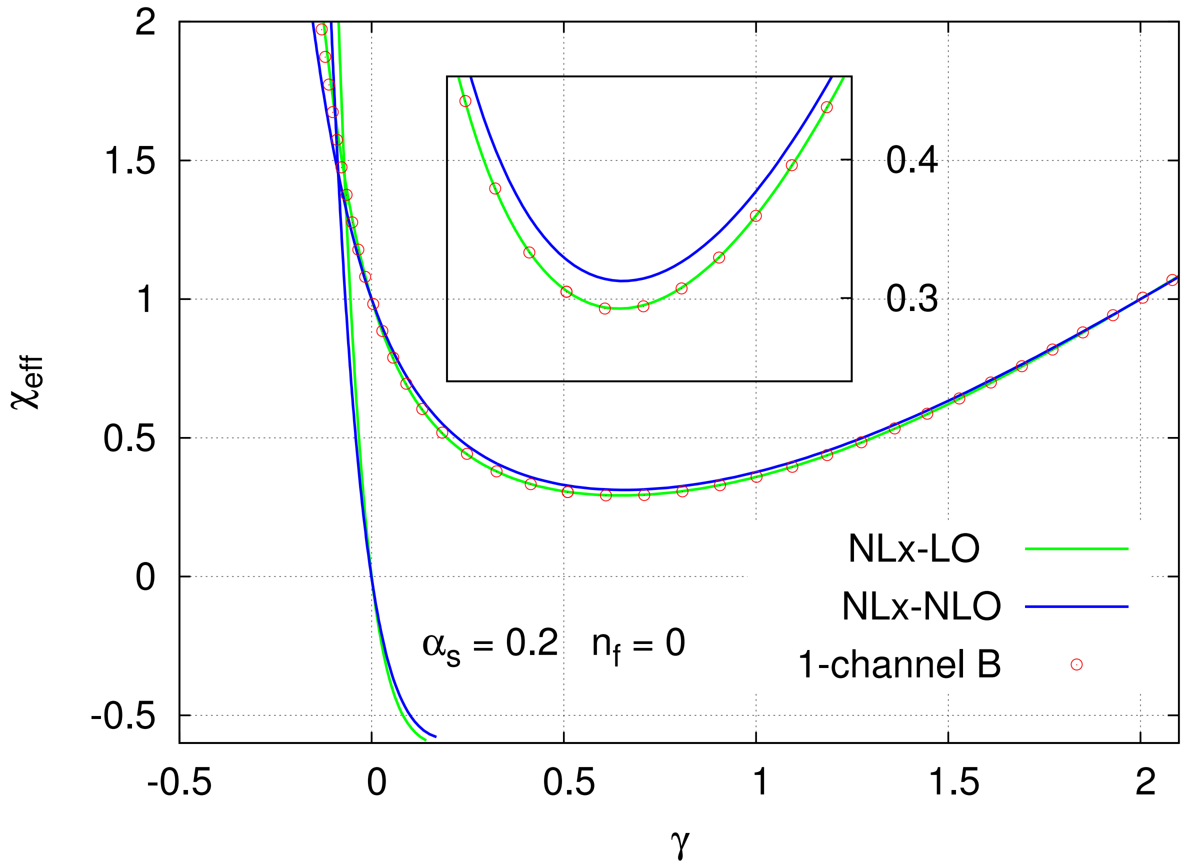
Effective Eigenvalue Functions ($n_f = 4$)

There are two, frozen α_s, resummed eigenvalue functions: $\omega = \chi_\pm(\alpha_s, \gamma)$



Fixed points at $\gamma = 0, 2$ and $\omega = 1 \Rightarrow$ momentum conservation
in both collinear and anti-collinear limits.
New subleading eigenvalue χ_-

Effective Eigenvalue Functions ($n_f = 0$)

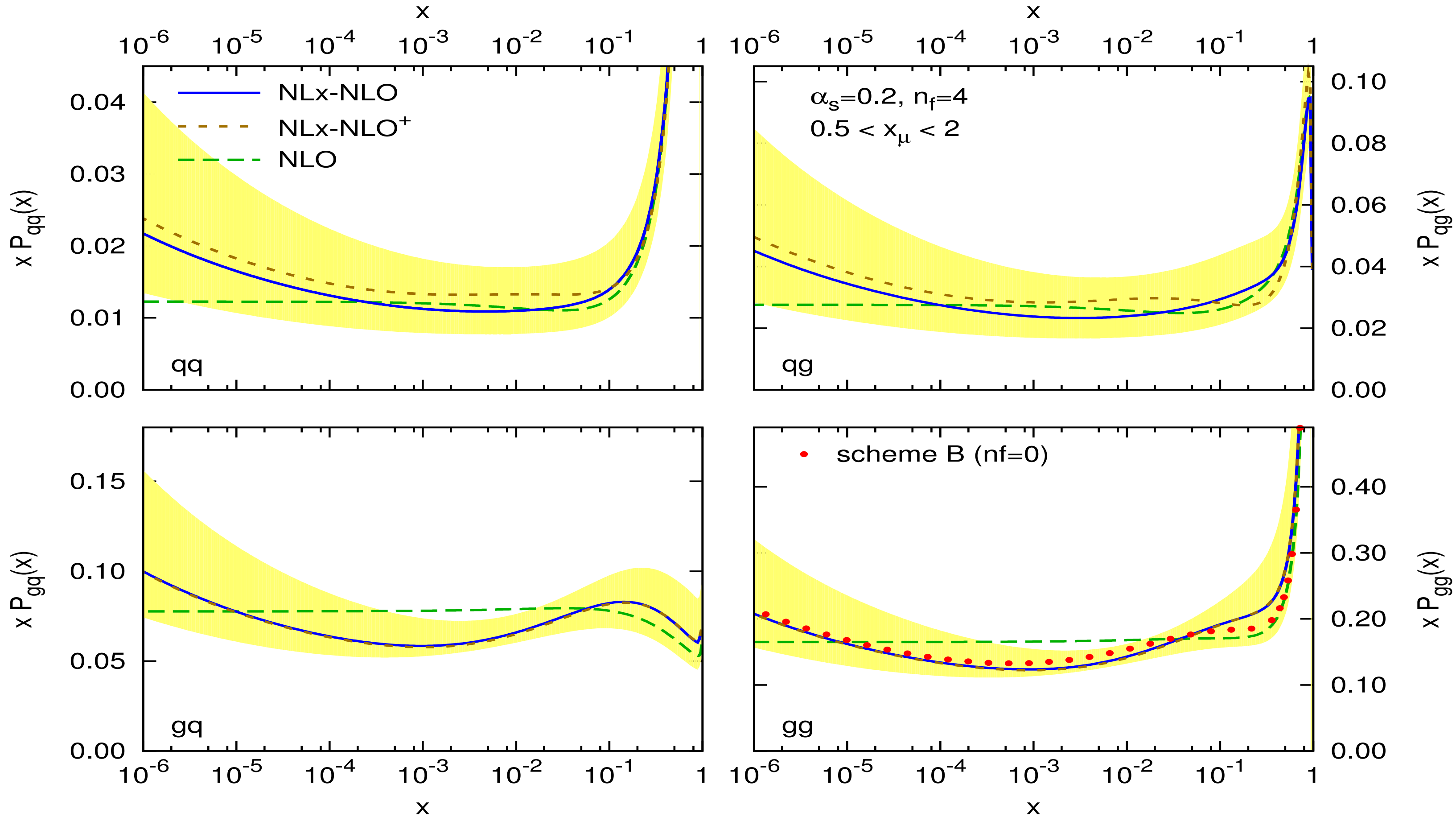


Modest n_f-dependence of $\chi_+(\alpha_s, \gamma)$. NLx-LO scheme recovers the known gluon-channel result (in agreement with ABF) at $n_f = 0$.

Level crossing of χ_- and χ_+ in the $n_f = 0$ limit

Resummed Splitting Function Matrix

NLO^+ scheme includes, besides NLO, also NNLO terms $\sim \alpha_s^3/\omega^2$



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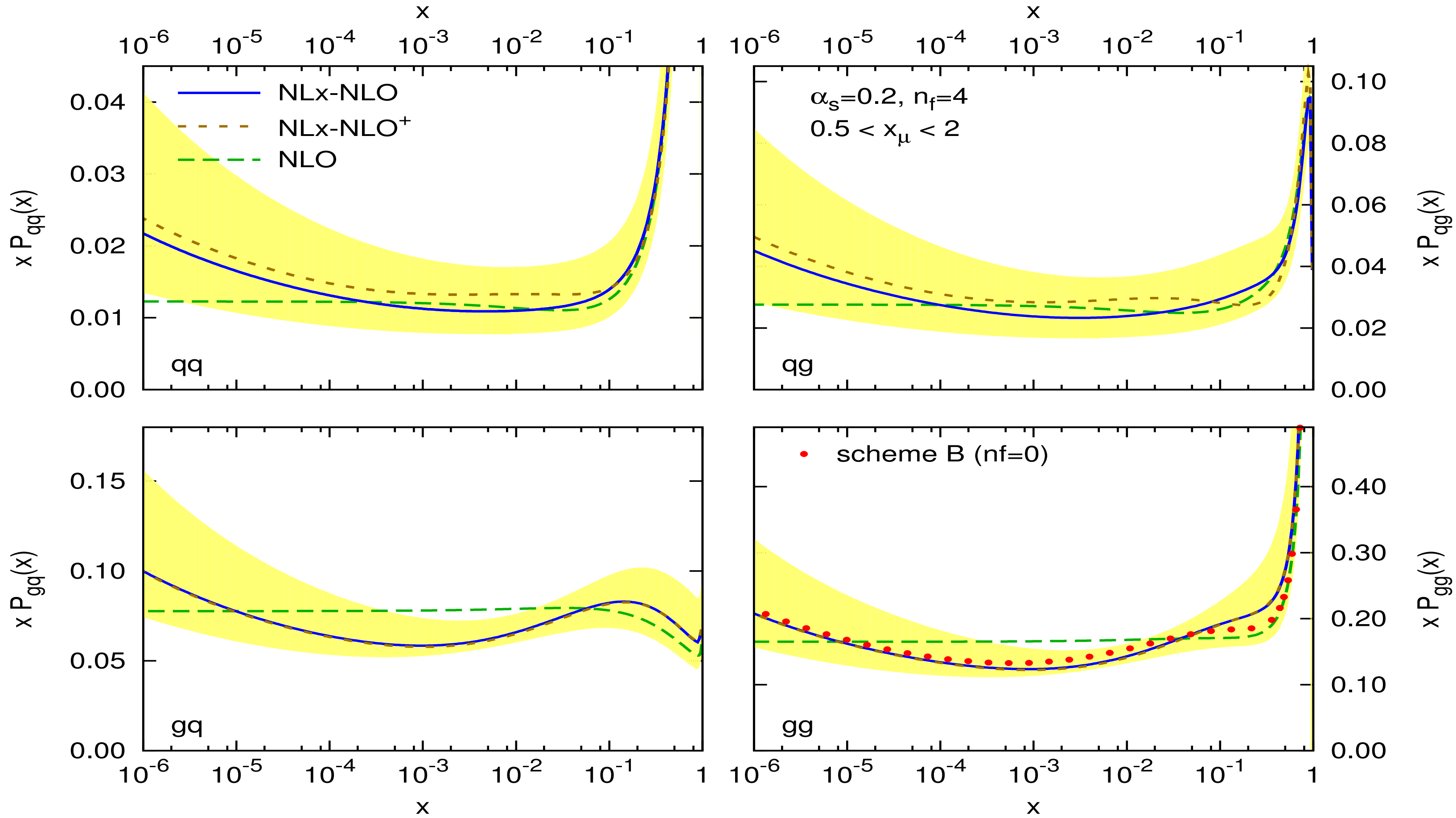

- Infrared cutoff independence insures (matrix) collinear factorization
- At intermediate $x \simeq 10^{-3}$ resummed P_{gg} and P_{gq} show a shallow dip
- Small-x rise of novel P_{qg} and P_{qq} delayed down to $x \simeq 10^{-4}$
- Scale uncertainty band (0.25<x_μ^2<4) larger for the (small) P_{qa} entries

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- On the whole, it looks quite nice!