Kac-Moody Symmetries in Reductions to Two Dimensions

String and M-Theory Approaches to Particle Physics and Cosmology

Galileo Institute, Firenze, 20'th June 2007

Plan

- Toroidal Reduction of Eleven-Dimensional Supergravity to $3 \leq D \leq 10$
- Borel-Gauge Description of the Scalar Coset Manifolds
- Reduction of Pure $D = 4$ Gravity to $D = 2$
- Reduction of Eleven-Dimensional Supergravity to $D = 2$
- Construction of Infinity of Conserved Currents

The description of the reductions to $3 \leq D \leq 10$ and the Borel gauge coset construction summarises results in hep-th/9710119, Cremmer, Julia, Lü and Pope.

The approach described here to understanding the Kac-Moody symmetries of the reductions to $D = 2$ is work in progress by Lü, Perry, Pope and Stelle–

"Demystification of Kac-Moody Symmetries in $D = 2$"

Kaluza-Klein Reduction on S^1

Reduction on T^n can be broken up into a step-by-step reduction on a sequence of circles. Consider the reduction of gravity and a p-form potential,

$$\hat{\mathcal{L}} = \hat{R}\,\hat{*}\mathbf{1} - \tfrac{1}{2}\hat{*}\hat{F}_{(p+1)} \wedge \hat{F}_{(p+1)}$$

in the step from $D+1$ to D dimensions:

$$\begin{aligned} d\hat{s}^2 &= e^{2\alpha\phi}\, ds^2 + e^{2\beta\phi}(dz + \mathcal{A}_{(1)})^2 \\ \hat{A}_{(p)} &= A_{(p)} + A_{(p-1)} \wedge dz \end{aligned}$$

where all quantities on the RHS are independent of the circle coordinate z. The constants α and β are chosen such that

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\,\alpha$$

with the latter ensuring the lower-dimensional metric is in the Einstein frame, and the former fixing a canonical normalisation for the kinetic term of the KK scalar ϕ:

$$\begin{aligned} \mathcal{L} \;=\; & R * \mathbf{1} - \tfrac{1}{2}{*d\phi} \wedge d\phi - -\tfrac{1}{2}e^{-2(D-1)\alpha\phi} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} \\ & -\tfrac{1}{2}e^{-2p\alpha\phi} * F_{(p+1)} \wedge F_{(p+1)} - \tfrac{1}{2}e^{2(D-p-1)\alpha\phi} * F_{(p)} \wedge F_{(p)} \end{aligned}$$

Here $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$, $F_{(p+1)} = dA_{(p)} - dA_{(p-1)} \wedge \mathcal{A}_{(1)}$ and $F_{(p)} = dA_{(p-1)}$.

Kaluza-Klein Reduction of Pure Gravity on T^n

At each successive step of reduction on S^1, the metric gives rise to a metric and a new KK scalar (dilaton) and KK vector. Each existing p-form potential gives rise to a p-form and a $(p-1)$-form potential. Note that a 1-form potential (such as an already existing KK vector) gives a 1-form and a 0-form potential, and that the latter is an axionic scalar.

Pure gravity reduced on T^n will therefore have a set of n 1-forms $\mathcal{A}^i_{(1)}$; a set of $\frac{1}{2}n(n-1)$ axionic scalars $\mathcal{A}^i_{(0)j}$ (with $j > i$) from the reductions of the 1-forms at subsequent steps; and a set of n dilatonic scalars $\vec{\phi} = (\phi_1, \phi_2, \cdots, \phi_n)$.

The kinetic term for each form field will have an exponential prefactor of the form $e^{\vec{b}\cdot\vec{\phi}}$, where the constant "dilaton vector" $\vec{b}$ characterises the coupling of the dilatonic scalars to that particular form field:

$$\mathcal{L}_{\text{grav}} = R\,{*\mathbf{1}} - \tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}\sum_i e^{\vec{b}_i\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(2)}} \wedge \mathcal{F}^i_{(2)} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(1)j}} \wedge \mathcal{F}^i_{(1)j}$$

Reduction of $D=11$ Supergravity on T^n to $D=11-n$

$D=11$ supergravity $\hat{\mathcal{L}} = \hat{R}\,\hat{*}\mathbf{1} - \frac{1}{2}\hat{*}\hat{F}_{(4)} \wedge \hat{F}_{(4)} + \frac{1}{6}\hat{F}_4 \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}$
reduced on T^n then gives

$$\begin{aligned}
\mathcal{L} \;=\;& R*\mathbf{1} - \tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \sum_i e^{\vec{b}_i\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(2)}} \wedge \mathcal{F}^i_{(2)} - \tfrac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,{*\mathcal{F}^i_{(1)j}} \wedge \mathcal{F}^i_{(1)j} \\
& -\tfrac{1}{2}e^{\vec{a}\cdot\vec{\phi}}\,{*F_{(4)}} \wedge F_{(4)} - \tfrac{1}{2}\sum_i e^{\vec{a}_i\cdot\vec{\phi}}\,{*F_{(3)i}} \wedge F_{(3)i} \\
& -\tfrac{1}{2}\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\,{*F_{(2)ij}} \wedge F_{(2)ij} - \tfrac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,{*F_{(1)ijk}} \wedge F_{(1)ijk} + \mathcal{L}_{FFA}
\end{aligned}$$

where the dilaton vectors are given by

	$\hat{F}_{(4)}$	Metric
4 − form :	$\vec{a}$	
3 − forms :	$\vec{a}_i = \vec{a} - \vec{b}_i$	
2 − forms :	$\vec{a}_{ij} = \vec{a} - \vec{b}_i - \vec{b}_j$	$\vec{b}_i$
1 − forms :	$\vec{a}_{ijk} = \vec{a} - \vec{b}_i - \vec{b}_j - \vec{b}_k$	$\vec{b}_{ij} = \vec{b}_i - \vec{b}_j$

$$\vec{b}_i \cdot \vec{b}_j = 2\delta_{ij} + \frac{2}{D-2}, \qquad \vec{a} = \tfrac{1}{3}\sum_i \vec{b}_i$$

Global Symmetry of Toroidally-Reduced Theory

Any theory including gravity, reduced on T^n, will have at least an $SL(n,\mathbb{R})$ global symmetry that acts "internally" (i.e. it leaves the lower-dimensional Einstein-frame metric invariant).

It corresponds to the subgroup of general coordinate transformations of the original theory comprising rigid $SL(n,\mathbb{R})$ transformations in the torus T^n:

$$\delta x^\mu = 0\,, \qquad \delta y^i = \Lambda^i{}_j\, y^j$$

If the original theory has an overall scaling symmetry ("trombone symmetry"), such as pure gravity or $D=11$ supergravity:

$$\hat g_{MN} \longrightarrow \lambda^2 \hat g_{MN}\,, \quad \hat A_{MNP} \longrightarrow \lambda^3 A_{MNP}\,, \quad \Rightarrow \quad \hat{\mathcal{L}} \longrightarrow \lambda^9 \hat{\mathcal{L}}\,,$$

then volume-changing transformations are included too and this global internal symmetry becomes $GL(n,\mathbb{R})$.

If there are other form fields in the higher-dimensional theory, the global symmetry may be enhanced further.

The global symmetry $\mathcal{G}$ is non-linearly realised on the scalar fields (dilatons plus axions) in the reduced theory. These scalars lie in a coset space $\mathcal{K} = \mathcal{G}/\mathcal{H}$. The group $\mathcal{G}$ acts linearly on the other form fields.

Global Symmetry of Toroidally-Reduced Pure Gravity

The global symmetry can therefore conveniently be studied by first focusing on the scalar sector. Consider first the reduction of pure gravity from $D+n$ to $D \geq 4$. The scalar sector comprises n dilatons $\vec{\phi}$ and $\frac{1}{2}n(n-1)$ axions $\mathcal{A}^i_{(0)j}$ with dilaton vectors $\vec{b}_{ij} = \vec{b}_i - \vec{b}_j$, where $\vec{b}_i \cdot \vec{b}_j = 2\delta_{ij} + 2/(D-2)$.

These dilaton vectors are in one-to-one correspondence with the positive roots of the $A_{n-1} = SL(n, \mathbb{R})$ algebra. The simple roots are $\vec{b}_{i,i+1}$, for $1 \leq i \leq n-1$:

If reduced to $D=3$, the KK 1-forms $\mathcal{A}^i_{(1)}$, (with dilaton vectors $\vec{b}_i$), can be dualised to give an additional n axions, with dilaton vectors $-\vec{b}_i$. The symmetry enhances to $A_n = SL(n+1, \mathbb{R})$, with $\{\vec{b}_{ij}, -\vec{b}_i\}$ as positive roots, and $-\vec{b}_1$ the extra simple root:

Global Symmetry of T^n-Reduced $D = 11$ Supergravity

In a reduction on T^n to $D = 11-n$, we have n dilatons $\vec{\phi}$, $\frac{1}{2}n(n-1)$ axions $\mathcal{A}^i_{(0)j}$ from the metric and $\frac{1}{6}n(n-1)(n-2)$ axions $A_{(0)ijk}$ from the 3-form $\hat{A}_{(3)}$. These have dilatons vectors $\vec{b}_{ij} = \vec{b}_i - \vec{b}_j$ and $\vec{a}_{ijk} = \vec{a} - \vec{b}_i - \vec{b}_j - \vec{b}_k$ respectively ($\vec{a} = \frac{1}{3}\sum_\ell \vec{b}_\ell$).

In $3 \le D \le 5$ we obtain further axions by dualising form fields:

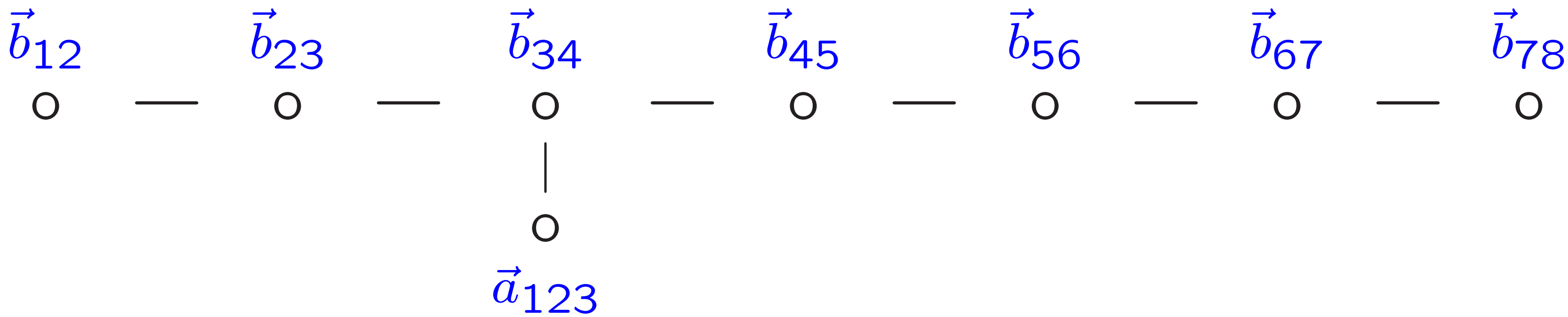
$D = 5:$	$*A_{(3)}$	Dilaton vector	$-\vec{a}$	1
$D = 4:$	$*A_{(2)i}$	Dilaton vectors	$-\vec{a}_i$	8
$D = 3:$	$(*\mathcal{A}^i_{(1)}\,, *A_{(1)ij})$	Dilaton vectors	$(-\vec{b}_i\,, -\vec{a}_{ij})$	$8 + 28$

In all dimensions $3 \le D \le 10$, the full set of axion dilaton vectors (including those coming from dualisation when $3 \le D \le 5$) are in one-to-one correspondence with the postive roots of E_n, where, for $n \le 5$ we have

$$\begin{aligned} E_1 &= \mathbb{R}\,, \quad E_2 = GL(2,\mathbb{R})\,, \quad E_3 = SL(3,\mathbb{R}) \times SL(2,\mathbb{R}) \\ E_4 &= SL(5,\mathbb{R})\,, \quad E_5 = O(5,5) \end{aligned} \tag{1}$$

The simple roots are $\vec{a}_{123}$ and $\vec{b}_{i,i+1}$ for $1 \le i \le n-1$.

The E_n Symmetry of $D = 11$ Supergravity on T^n



$\vec{b}_{i,i+1}$ with $i \le 7$ and $\vec{a}_{123}$ generate the E_8 Dynkin diagram

Vertices with indices exceeding n are to be deleted for $n < 8$.

We have exhibited the root structure of the dilaton vectors characterising the couplings of the dilatons $\vec{\phi}$ in the exponential prefactors of the axionic kinetic terms. We still need to show exactly why this implies that the scalars are described by the coset manifold $E_n/K(E_n)$, where $K(E_n)$ is the maximal compact subgroup of E_n.

The construction is extremely simple, by virtue of the fact that the step-by-step reduction scheme naturally leads to a parameterisation of the coset representative in the Borel gauge.

$SL(2,\mathbb{R})/O(2)$ Scalar Coset in Borel Gauge

First consider a toy model, namely an $SL(2,\mathbb{R})/O(2)$ scalar coset model:

$$\mathcal{L} = -\tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2}e^{2\phi}\, {*d\chi} \wedge d\chi$$

Defining

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the coset $\mathcal{K} = \mathcal{G}/\mathcal{H}$ with $\mathcal{G} = SL(2,\mathbb{R})$ and $\mathcal{H} = O(2)$ has generators as follows:

$\mathcal{K}:$	H and $(E_+ + E_-)$	(Non-Compact)
$\mathcal{H}:$	$(E_+ - E_-)$	(Compact)

It is convenient to use the Borel gauge for writing the coset representative:

$$\mathcal{V} = e^{\frac{1}{2}\phi H}\, e^{\chi E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & e^{\frac{1}{2}\phi}\chi \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}$$

in terms of which we find

$$d\mathcal{V}\mathcal{V}^{-1} = \tfrac{1}{2}H d\phi + E_+\, e^{\phi} d\chi$$
$$= \tfrac{1}{2}H d\phi + \tfrac{1}{2}(E_+ + E_-)\, e^{\phi} d\chi + \tfrac{1}{2}(E_+ - E_-)\, e^{\phi} d\chi$$

Since $d\mathcal{V}\mathcal{V}^{-1} = P + Q$, where P is the projection into the Lie algebra of the coset $\mathcal{K}$ and Q is the projection into the denominator algebra $\mathcal{H}$, we have

$$P_\phi = d\phi\,, \qquad P_\chi = e^{\phi}\, d\chi \qquad Q \to e^{\phi} d\chi$$

The Cartan-Maurer equation $d(d\mathcal{V}\mathcal{V}^{-1}) - (d\mathcal{V}\mathcal{V}^{-1}) \wedge (d\mathcal{V}\mathcal{V}^{-1}) = 0$ implies

$$dQ - Q \wedge Q - P \wedge P = 0\,, \qquad DP \equiv dP - Q \wedge P - P \wedge Q = 0$$

The Lagrangian can be written as $\mathcal{L} = -\frac{1}{2}(P_\phi)^2 - \frac{1}{2}(P_\chi)^2$, and the equations of motion are

$$D{*}P = 0$$

The (right-acting) $SL(2,\mathbb{R})$ global symmetry is $\mathcal{V} \longrightarrow \mathcal{O}\mathcal{V}\Lambda$, where $\mathcal{O}$ is a local $O(2)$ compensator that restores $\mathcal{V}$ to Borel gauge.

T^n-Reduced Supergravity Scalar Cosets

Introduce Cartan generators $\vec{H}$, and positive-root generators $(E_i{}^j, E^{ijk})$ corresponding to the axions $(\mathcal{A}^i_{(0)j}, A_{(0)ijk})$. They satisfy

$$
\begin{aligned}
[\vec{H}, E_i{}^j] &= \vec{b}_{ij}\, E_i{}^j\,, \qquad [\vec{H}, E^{ijk}] = \vec{a}_{ijk}\, E^{ijk} \\
[E_i{}^j, E_k{}^\ell] &= \delta^j_k\, E_i{}^\ell - \delta^\ell_i\, E_k{}^j\,, \qquad [E_\ell{}^m, E^{ijk}] = -3\delta^{[i}_\ell\, E^{jk]m} \\
[E^{ijk}, E^{\ell mn}] &= 0 \qquad (\text{for } D \geq 6)
\end{aligned}
$$

Defining $\mathcal{V} = \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3$ with

$$
\begin{aligned}
\mathcal{V}_1 &= e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}} \\
\mathcal{V}_2 &= \prod_{i<j} e^{\mathcal{A}^i_{(0)j} E_i{}^j} = \cdots e^{\mathcal{A}^2_{(0)4} E_2{}^4} e^{\mathcal{A}^2_{(0)3} E_2{}^3} \cdots e^{\mathcal{A}^1_{(0)4} E_1{}^4} e^{\mathcal{A}^1_{(0)3} E_1{}^3} e^{\mathcal{A}^1_{(0)2} E_1{}^2} \\
\mathcal{V}_3 &= \prod_{i<j<k} e^{A_{(0)ijk} E^{ijk}}
\end{aligned}
$$

we find that

$$
d\mathcal{V}\mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\phi}\cdot H + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}} \mathcal{F}^i_{(1)j}\, E_i{}^j + \sum_{ijk} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}} F_{(1)ijk}\, E^{ijk}
$$

Note that all the higher-order "transgression" terms in the 1-form field strengths are correctly produced. E.g.

$$\begin{aligned} \mathcal{F}^i_{(1)j} &= \gamma^k{}_j \, d\mathcal{A}^i_{(0)k} \\ \gamma^k{}_j &= [(1+\mathcal{A}_{(0)})^{-1}]^k{}_j = \delta^k_j - \mathcal{A}^k_{(0)j} + \mathcal{A}^k_{(0)\ell}\,\mathcal{A}^\ell_{(0)j} + \cdots \end{aligned}$$

In dimensions $3 \le D \le 5$ extra positive-root generators associated with the additional axions coming from dualisations are needed. These arise on the R.H.S. of $[E^{ijk}, E^{\ell mn}] = \cdots$. Adding the corresponding extra factors in the expression for the Borel-gauge coset representative $\mathcal{V}$, we again obtain the full set of 1-form field strengths for all the axions from $d\mathcal{V}\mathcal{V}^{-1}$.

This makes manifest the global symmetry under E_n, generated by $\Lambda \in E_n$, with $\mathcal{V} \longrightarrow \mathcal{O}\mathcal{V}\Lambda$, where $\mathcal{O}$ is a local compensating transformation in $K(E_n)$, the maximal compact subgroup of E_n. For example, the coset is $E_8/O(16)$ in $D = 3$.

Reduction to Two Dimensions

Two new features arise upon further reduction to $D = 2$:

- Can no longer reduce to the Einstein frame ($\mathcal{L} \sim \sqrt{-g}R + \cdots$).
- Dual of an axion is an axion. The dualisation of the scalar Lagrangian gives a non-locally related scalar Lagrangian with a (non-commuting) global symmetry. Intertwining of the symmetries gives an infinite-dimensional algebra.

Example: Reduction of pure gravity in $D = 4$ to $D = 2$. This would give an $SL(2,\mathbb{R})/O(2)$ scalar coset in $D = 3$ after dualising the KK vector to an axion:

$$
\begin{aligned}
ds_4^2 &= e^{\phi} ds_3^2 + e^{-\phi}\,(dz_1 + \mathcal{A}_{(1)})^2 \qquad \Rightarrow \\
\mathcal{L}_3 &= \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2\phi}(\mathcal{F}_{(2)})^2\Big) \qquad \Rightarrow \\
\mathcal{L}_3 &= \sqrt{-g}\Big(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}(\partial\chi)^2\Big)
\end{aligned}
\tag{2}
$$

where $e^{-2\phi} {*\mathcal{F}_{(2)}} = d\chi$. This axion reduces to an axion in $D = 2$.

We can instead leave the KK vector undualised in $D = 3$, giving just an axion after the further reduction to $D = 2$. This is the dual of the axion that would come from reduction of (2).

Misner-Matzner, Ehlers and Kac-Moody

The direct reduction to $D = 2$ with no dualisation in $D = 3$ is

$$ds_4^2 = e^{\varphi}\left[e^{\tilde{\psi}-\frac{3}{2}\varphi}\, ds_2^2 + e^{\tilde{\phi}}(dz_1 + \tilde{\chi} dz_2)^2 + e^{-\tilde{\phi}} dz_2^2\right]$$

which leads to the two-dimensional Lagrangian

$$\mathcal{L}_2 = e^{\varphi}\sqrt{-g}\left[R + \partial\varphi\cdot\partial\tilde{\psi} - \tfrac{1}{2}(\partial\tilde{\phi})^2 - \tfrac{1}{2}e^{2\tilde{\phi}}\,(\partial\tilde{\chi})^2\right]$$

Has an $SL(2,\mathbb{R})_A$ global symmetry (Misner-Matzner), for fractional linear transformations of $\tilde{\tau} = \tilde{\chi} + i\, e^{-\tilde{\phi}}$, wth φ and $\tilde{\psi}$ inert.

Dualise the axion $\tilde{\chi}$ according to $\tilde{\phi} = -\phi - \varphi$, $\tilde{\psi} = \psi + \phi + \frac{1}{2}\varphi$, and $e^{2\tilde{\phi}+\varphi} {*d\tilde{\chi}} = d\chi$ (equivalent to full dualisation in $D = 3$). Gives

$$\mathcal{L} = e^{\varphi}\sqrt{-g}\left[R + \partial\varphi\cdot\partial\psi - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}\,(\partial\chi)^2\right]$$

which has an $SL(2,\mathbb{R})_B$ global symmetry (Ehlers) on (ϕ, χ), with φ and ψ inert.

The $SL(2,\mathbb{R})_A$ and $SL(2,\mathbb{R})_B$ symmetries do not commute, and in fact successive A and B transformations generate an infinite sequence of conserved currents (Geroch), closing on affine $SL(2,\mathbb{R})$.

E_9 Symmetry from $D = 11$ Supergravity

If a theory reduced to $D = 3$ (and fully dualised) has a $\mathcal{K} = \mathcal{G}/\mathcal{H}$ scalar coset with $d\mathcal{V}\mathcal{V}^{-1} = P + Q$ then in $D = 2$ we get

$$\mathcal{L}_2 = e^{\varphi}\sqrt{-g}\Big[R + \partial\varphi\cdot\partial\psi - \tfrac{1}{2}\sum_A (P_A)^2\Big]$$

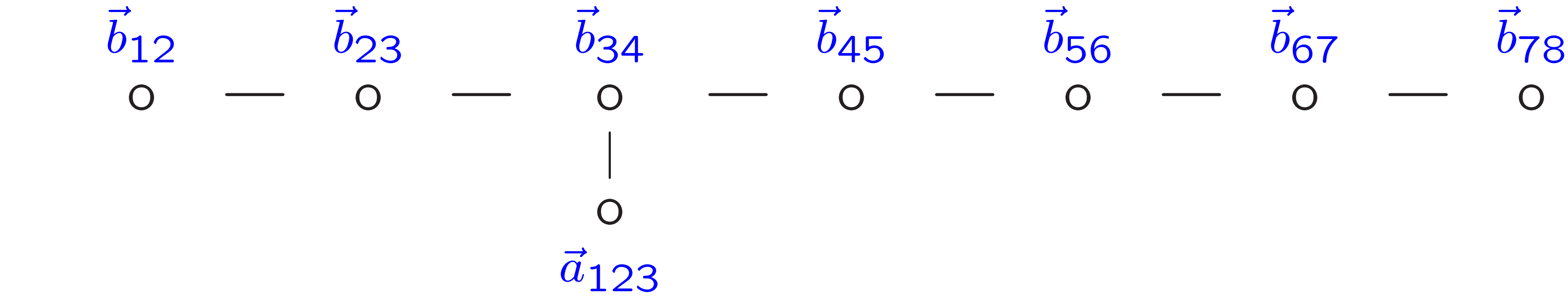
Thus reduction of the fully-dualised E_8-invariant supergravity Lagrangian in $D = 3$ gives an E_8-invariant Lagrangian in $D = 2$. The simple roots are $\vec{a}_{123}$ and $\vec{b}_{i,i+1}$ for $1 \le i \le 7$, as in $D = 3$. This is the analogue of the Ehlers $SL(2,\mathbb{R})$ of the $D = 4$ gravity reduction.

Now instead leave $\mathcal{A}^i_{(1)}$ and $A_{(1)ij}$ undualised in $D = 3$, and reduce them directly to axions in $D = 2$ (with dilaton vectors $+\vec{b}_i$ and $+\vec{a}_{ij}$). Splitting $i = (1, \alpha)$, for $2 \le \alpha \le 8$ we find that

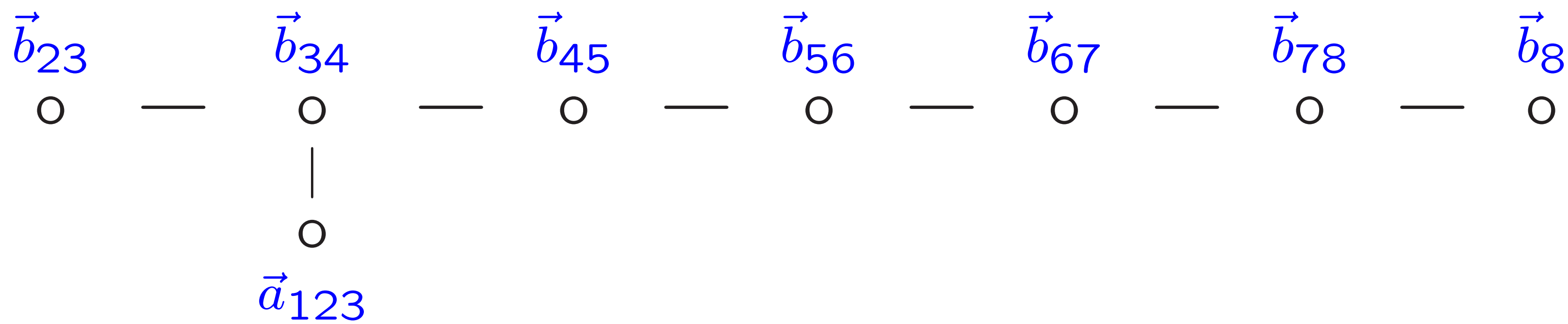
$$\vec{b}_\alpha\,, \qquad \vec{b}_{\alpha\beta}\,, \qquad \vec{a}_{1\alpha\beta}\,, \qquad \vec{a}_{1\alpha}$$

form the positive roots of $D_8 = O(16)$, with $\vec{a}_{123}$, $\vec{b}_{\alpha,\alpha+1}$ and $\vec{b}_8$ as the simple roots. (The remaining axions form a linear representation under D_8.) This D_8 is the analogue of the Misner-Matzner $SL(2,\mathbb{R})$ of the $D = 4$ gravity reduction.

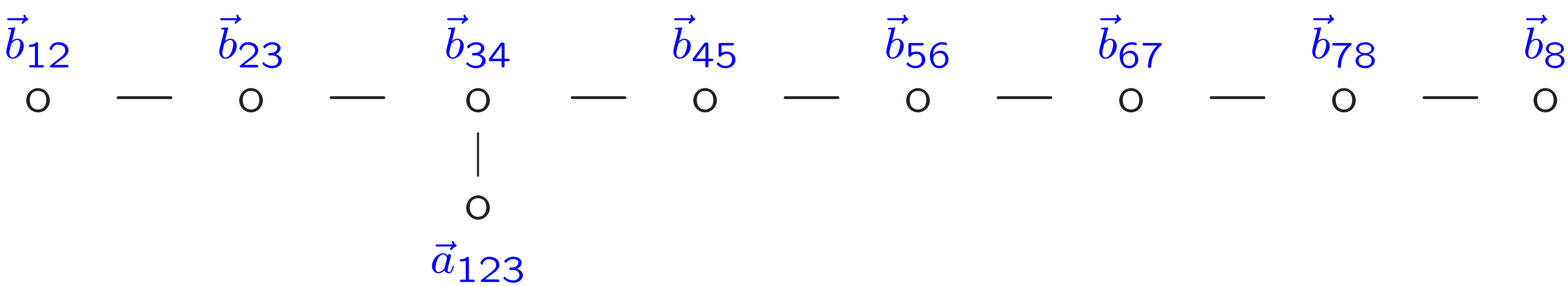
Thus we have the "Ehlers" E_8:



and the "Misner-Matzner" D_8:



whose intertwining gives the affine Kac-Moody E_9:



Intertwining in Flat-Space $SL(2,\mathbb{R})/O(2)$ Coset

Consider first a flat-space $D=2$ scalar coset model $SL(2,\mathbb{R})/O(2)$. For this model, $\mathcal{L}_2 = -\frac{1}{2}{*d\phi}\wedge d\phi - \frac{1}{2}e^{2\phi}{*d\chi}\wedge d\chi$, with equations of motion

$$d{*d\phi} - e^{2\phi}\,{*d\chi}\wedge d\chi = 0\,,\qquad d(e^{2\phi}\,{*d\chi}) = 0$$

We can introduce a doubled formalism by first taking the d off the second equation, and which then allows taking d off the first:

$$e^{2\phi}\,{*d\chi} = du_+\,,\qquad {*d\phi} - \chi\,du_+ = du_0$$

The new fields u_+ and u_0 form two members of a triplet that transforms linearly under the manifest $SL(2,\mathbb{R})$ symmetry of the Lagrangian above. The triplet is completed by defining

$$du_- = 2\chi\,du_0 + (\chi^2 + e^{-2\phi})du_+$$

The conserved currents $(J_+, J_0, J_-) = ({*du_+}, {*du_0}, {*du_-})$ transform linearly under infinitesimal $SL(2,\mathbb{R})$ transformations as

$$\delta J_+ = -\epsilon_0 J_+ - \epsilon_+ J_0\,,\quad \delta J_0 = \epsilon_+ J_- - \epsilon_- J_+\,,\quad \delta J_- = \epsilon_0 J_- + \epsilon_- J_0$$

We can write down a tilded set of currents, transforming linearly under $SL(2,\mathbb{R})$ of the dualised variables, which are related by

$$\phi = -\tilde{\phi}\,, \quad \chi = \tilde{u}_+\,, \quad u_+ = \tilde{\chi}\,, \quad u_0 = -\tilde{u}_0 - \chi u_+$$

We also read off that in terms of the untilded variables

$$d\tilde{u}_- = e^{2\phi} d\chi - 2u_+\, du_0 - d(u_+^2\, d\chi)$$

This is indeed integrable ($dd\tilde{u}_- = 0$), but to solve it locally requires introducing a new field, v_+; then $\tilde{u}_- = v_+ - u_+(u_0 + \chi\, u_+)$. This forms the $+$ component of a new triplet transforming linearly under the original $SL(2,\mathbb{R})$:

$$\begin{aligned} dv_+ &= e^{2\phi} d\chi - u_+\, du_0 + u_0\, du_+ \\ dv_0 &= -d\phi + \chi\, e^{2\phi} d\chi + \tfrac{1}{2} u_-\, du_+ - \tfrac{1}{2} u_+\, du_- \\ dv_- &= -d\chi + \chi^2\, e^{2\phi} d\chi - 2\chi\, d\phi + u_0\, du_- - u_-\, du_0 \end{aligned}$$

The intertwining can be continued *ad infinitum*, yielding a new triplet of $SL(2,\mathbb{R})$ currents at each step. These constitute the currents of the affine $SL(2,R)$ symmetry of the theory.

The generation of the Kac-Moody currents can be systematised, and applied to a general coset model, using a “linearisation” described by Breitenlöhner, Maison, Nicolai, ...

Construction of the Linear System

The idea is to introduce an arbitrary constant *spectral parameter* $t = \tanh \frac{1}{2}\theta$, and a coset representative $\widehat{\mathcal{V}}(x;t)$ such that $\widehat{\mathcal{V}}(x;0) = \mathcal{V}(x)$, with the relation

$$d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1} = Q + P\cosh\theta + {*P}\sinh\theta \tag{3}$$

(All t-dependence on the R.H.S. is made manifest here.) A simple calculation shows that the Cartan-Maurer equation implies

$$DP = 0\,, \qquad D{*P} = 0\,, \qquad dQ - Q\wedge Q - P\wedge P = 0$$

So we recover not only the content of the original (unhatted) Cartan-Maurer equation but also the field equation $D{*P} = 0$.

Expanding out (3) in powers of the spectral parameter t, we can read off an infinity of relations that imply an infinity of conserved currents. This gives a systematic construction of the Kac-Moody currents, whose few terms we constructed previously in the $SL(2,\mathbb{R})/O(2)$ example.

The $SL(2,\mathbb{R})/O(2)$ Example

Write $\widehat{\mathcal{V}}(x;t) = e^{\frac{1}{2}\widehat{\phi} H}\, e^{\widehat{\chi} E_+}\, e^{\widehat{\psi} E_-}$ and expand the fields as

$$\begin{aligned}\widehat{\phi} &= \phi_0 + t\,\phi_1 + t^2\,\phi_2 + \cdots\,, \qquad \widehat{\chi} = \chi_0 + t\,\chi_1 + t^2\,\chi_2 + \cdots \\ \widehat{\psi} &= t\,\psi_1 + t^2\,\psi_2 + \cdots\end{aligned}$$

Note that at order t^0 this reduces to the original $\mathcal{V}$ which is in Borel gauge, with ϕ_0 and χ_0 as the dilaton and axion.

Expanding to the first couple of orders in t we find at t^0

$$P_\phi = d\phi_0\,, \qquad P_\chi = e^{\phi_0} d\chi_0$$

and at t^1

$$\begin{aligned}{*d\phi_0} &= \tfrac{1}{2} d\phi_1 + \chi_0\, d\psi_1 \\ e^{2\phi_0}\, {*d\chi_0} &= d\psi_1 \\ 0 &= d\chi_1 + \phi_1 d\chi_0 - (\chi_0^2 + e^{-2\phi_0}) d\psi_1\end{aligned}$$

These three equations are precisely equivalent to the first-level triplet of $SL(2,\mathbb{R})$ currents we constructed previously, with

$$u_+ \longrightarrow \psi_1\,, \qquad u_0 \longrightarrow \tfrac{1}{2}\phi_1\,, \qquad u_- \longrightarrow \chi_1 + \chi_0\,\phi_1$$

We obtain higher triplets at each order in t.

Linear System Including Gravity

In the actual 2-dimensional theories coming from dimensional reduction there is an additional dilaton φ coming from the $D=3$ to $D=2$ metric reduction, and in $D=2$ we had

$$\mathcal{L}_2 = e^{\varphi}\sqrt{-g}\left[R + \partial\varphi\cdot\partial\psi - \tfrac{1}{2}\sum_A (P_A)^2\right]$$

The previous construction $d\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1} = Q + P\,\cosh\theta + {*P}\,\sinh\theta$ requires modification, with θ no longer constant. Instead set

$$d\theta = \sinh\theta\cosh\theta\, d\varphi + \sinh^2\theta\, {*d\varphi} \tag{4}$$

The Cartan-Maurer equation then implies

$$DP = 0\,, \qquad D(e^{\varphi}{*P}) = 0\,, \qquad dQ - Q\wedge Q = P\wedge P = 0$$

We can choose $ds_2^2 = 2dx^+dx^-$ (since the redundant field ψ was included in the reduction as the $D=2$ conformal factor). This implies $\partial_+\partial_- e^{\varphi} = 0$ and hence $e^{\varphi} = \rho_+(x^+) + \rho_-(x^-)$. Equation (4) can then be solved, giving

$$e^{2\theta} = \frac{w + \rho_-(x^-)}{w - \rho_+(x^+)}$$

The constant w can now be viewed as the spectral parameter.

Further Remarks

- The linear system again provides a systematic way of constructing the infinity of conserved currents of the Kac-Moody symmetries in $D = 2$.

- The symmetries can be used to generate new solutions from old ones.

- The Borel-gauge coset description, which arises naturally in the step-by-step Kaluza-Klein reduction scheme, provides a simple way of understanding the global symmetries in supergravity compactifications to $D \geq 3$.

- We have seen indications that this approach continues to provide a simple understanding of the symmetries in $D = 2$.